1 Josephson Effect

In 1962 Brian Josephson[1], then a 22–year old graduate student, made a remarkable prediction that two superconductors separated by a thin insulating barrier should give rise to a spontaneous (zero voltage) DC current, $I_s = I_c \sin \Delta\phi$, where $\Delta\phi$ is the difference in phase across the junction. And that if a finite (DC) voltage were applied, an AC current with frequency $\omega = 2eV/\hbar$ would flow. I_c is called the Josephson critical current.

There is a myth that Brian Josephson did his calculation (1962) and won the Nobel prize (1973) as part of the solution to a homework problem of Phil Anderson's. The truth is that Anderson was a lecturer on sabbatical at Cambrige in 1961-62, and he gave a series of lectures in which he mentioned the problem of tunneling between two superconductors, which Josephson then promptly solved. The idea was opposed at first by John Bardeen, who felt that pairing could not exist in the barrier region[2]. Thus much of the early debate centered on the nature of the tunneling process, whereas in fact today we know that the Josephson effect occurs in a variety of situations whenever two superconductors are separated by a "weak link", which can be an insulating region, normal metal, or short, narrow constriction.[3]

Let's first consider the last example as the conceptually simplest. The Ginzburg–Landau equation appropriate for this situation may be written

$$\xi^2 \frac{d^2 f}{dx^2} + f - f^3 = 0 \tag{1}$$

where $\xi = \sqrt{\frac{\hbar}{2m^* a(T)}}$ is the GL coherence length and $f(x) \equiv \Psi(x)/\Psi_\infty$. Take $L \ll \xi$, so the deviations of Ψ coming from the bulk value Ψ_1 of the first SC is small, and vice versa for the second SC. Changes of Ψ in the constriction occur over a length scale of L, so that the first term is of $O((\xi/L)^2) \gg f - f^3$. So we must solve a Laplace equation for f, ($\frac{d^2 f}{dx^2} = 0$)

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[1]Phys. Lett. 1, 251 (1962)

[2]Physics Today, July 2001

[3]**In Je. 2009 I received an email from Brian Josephson correcting this version of the history:**

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Date:  Wed, 10 Jun 2009 09:43:54 +0100
From:  Brian Josephson <bdj10@cam.ac.uk>
To:  pjh@phys.ufl.edu
Subject:  the Josephson myth
Dear Peter,

While browsing I came across your mention of the 'myth' that I discovered the effect
because of a problem set by Anderson.  Your correction is not completely correct
either!  It was Pippard, my supervisor, who drew my attention to Giaevar's tunnelling
expts.  and his theory, which started me thinking (especially as to how one could
get away without using coherence factors).  Anderson on the other hand told me of the
Cohen/Falicov/Phillps calculation involving a single superconductor when it came our
in PRL, which gave me the idea of how to do the two-sc.  case.  Previously I had got
the broken symmetry idea which was crucial from a number of papers including Anderson's
pseudospin model, and also expounded in his lecture course which I went to.

Best regards, Brian J.
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with B.C. $f(0) = 1$, $f(L) = e^{i\Delta\Phi}$. The solution will be

$$f = \left(1 - \frac{x}{L}\right) + \frac{x}{L} e^{i\Delta\phi}. \tag{2}$$

The solution can be thought of as two terms, the first Ψ_1, beginning to "leak" into the constriction from the left, the second Ψ_2 leaking into the constriction from the right. The GL expression for the current will be

$$\begin{aligned}
\mathbf{j} &= \frac{e^* \hbar}{2m^* i} \left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^* \right) - \underbrace{\frac{e^{*2}}{m^* c} \Psi^* \Psi \mathbf{A}}_{\text{zero}} \\
&= \frac{e^* \hbar}{2m^* i} \Psi_\infty^2 \left[\left(1 - \frac{x}{L} + \frac{x}{L} e^{-i\Delta\phi} \right) \left(-\frac{1}{L} + \frac{1}{L} e^{i\Delta\phi} \right) - c.c \right] \\
&= \frac{e^* \hbar \Psi_\infty^2}{m^* L} \sin \Delta\phi, \qquad (3)
\end{aligned}$$

which means that the current will be

$$I = I_c \sin \Delta\phi, \tag{4}$$

$$I_c = \frac{e^* \hbar \Psi_\infty^2}{m^* L} A, \tag{5}$$

where A is the cross–section.

Given that we have two weakly coupled QM systems, it is "not unreasonable" (justified by microscopic theory) to write down coupled Schrödinger equations

$$i \frac{\partial \Psi_1}{\partial t} = E_1 \Psi_1 + \alpha \Psi_2 \tag{6}$$

$$i \frac{\partial \Psi_2}{\partial t} = E_2 \Psi_2 + \alpha \Psi_1 \tag{7}$$

where $H_0^{(i)} \Psi_i = E_i \Psi_i$ and $E_1 = E_2 = E_0$ if the superconductors are identical. Take $|\Psi_i|^2$ to be the density of pairs in SC_i

$$\Psi_i = \sqrt{n_i} e^{i\phi_i} \Rightarrow \dot{\Psi}_i = \frac{1}{2\sqrt{n_i}} \dot{n}_i e^{i\phi_i} + i \sqrt{n_i} \dot{\phi}_i e^{i\phi_i} \Rightarrow$$

$$\frac{\dot{n}_1}{2\sqrt{n_1}} + i \sqrt{n_1} \dot{\phi}_1 = -i E_1 \sqrt{n_1} - i \alpha \sqrt{n_2} e^{i(\phi_2 - \phi_1)} \tag{8}$$

$$\frac{\dot{n}_2}{2\sqrt{n_2}} + i \sqrt{n_2} \dot{\phi}_2 = -i E_2 \sqrt{n_2} - i \alpha \sqrt{n_1} e^{i(\phi_1 - \phi_2)}. \tag{9}$$

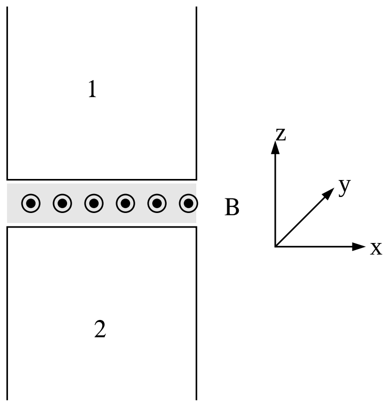

Figure 1:

If we take the real parts and use $\dot{n}_1 = -\dot{n}_2$ we get

$$\begin{aligned} \frac{\dot{n}_1}{2\sqrt{n_1}} &= \alpha\sqrt{n_2}\sin(\phi_2 - \phi_1) \\ -\frac{\dot{n}_1}{2\sqrt{n_2}} &= \alpha\sqrt{n_1}\sin(\phi_1 - \phi_2) \Rightarrow \\ \dot{n}_1 &= 2\alpha\sqrt{n_1 n_2}\sin(\phi_2 - \phi_1), \end{aligned} \tag{10}$$

Note I've put V=A=1. Then the current is just $j = 2e\dot{n}_1$. If we take the imaginary parts we have

$$\sqrt{n_1}\dot{\phi}_1 = -E_1\sqrt{n_1} - \alpha\sqrt{n_2}\cos(\phi_2 - \phi_1) \tag{11}$$

$$\sqrt{n_2}\dot{\phi}_2 = -E_2\sqrt{n_2} - \alpha\sqrt{n_1}\cos(\phi_1 - \phi_2), \tag{12}$$

and by subtracting and assuming $n_1 \simeq n_2$ (let's couple 2 identical superconductors at first) we get

$$\dot{\phi}_1 - \dot{\phi}_2 = E_2 - E_1 = 2e(V_1 - V_2) \tag{13}$$

where for the second equality we used the fact that the potential difference between the superconductors shifts the pair energies by $-2eV$. So we see that a finite voltage difference leads to a time changing phase difference $\Delta\phi$ which means an AC current via Eq. (10).

Magnetic fields. Now put a flux through the junction where the **B** field is along the $-\hat{y}$ direction and $A = -Bx\hat{z}$. The phase of the wave function Ψ must change by

$$\phi \rightarrow \phi - \frac{2e}{c}\int d\mathbf{S}\cdot\mathbf{A} \tag{14}$$

for the theory to be gauge invariant. Notice that ϕ is now space–dependent. So the Josephson

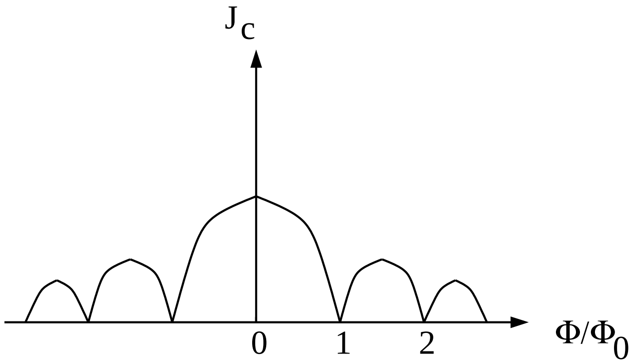

Figure 2:

equations will read

$$j = \underbrace{4e\alpha\sqrt{n_1 n_2}}_{j_c} \sin\left(\Delta\phi - \frac{2e}{c}\int_1^2 d\mathbf{S}\cdot\mathbf{A}\right), \tag{15}$$

$$2e(V_1 - V_2) = \frac{\partial}{\partial t}\left(\phi_2 - \phi_1 - \frac{2e}{c}\int_1^2 d\mathbf{S}\cdot\mathbf{A}\right), \tag{16}$$

and since

$$\int_1^2 \mathbf{S}\cdot\mathbf{A} = \int_0^d dz(-Bx) = -Bxd, \tag{17}$$

we will have

$$\begin{aligned} J &= \int_0^L dx j(x) = \int_0^L dx j_c \sin\left(\Delta\Phi - \frac{2e}{c}Bxd\right) \\ &= \frac{Lj_c}{2\pi\Phi/\Phi_0}\left[\cos\Delta\Phi - \cos\left(\Delta\phi + \frac{2\pi\Phi}{\Phi_0}\right)\right], \end{aligned} \tag{18}$$

where $\Phi_0 = \frac{2\pi c}{2e}$. What is the maximum current through the junction for all possible $\Delta\phi$?

We have to calculate $\frac{dJ}{d\Delta\Phi} = 0$ which leads to the relation

$$\tan\Delta\Phi = \cot\left(\pi\Phi/\Phi_0\right) \tag{19}$$

and with a bit of tedious trigonometry to

$$J_c = Lj_c\left|\frac{\sin\left(\pi\Phi/\Phi_0\right)}{\pi\Phi/\Phi_0}\right|. \tag{20}$$

This formula produces the Josephson–Fraunhofer interference pattern. DC SQUID (Superconducting Quantum Interference Device)

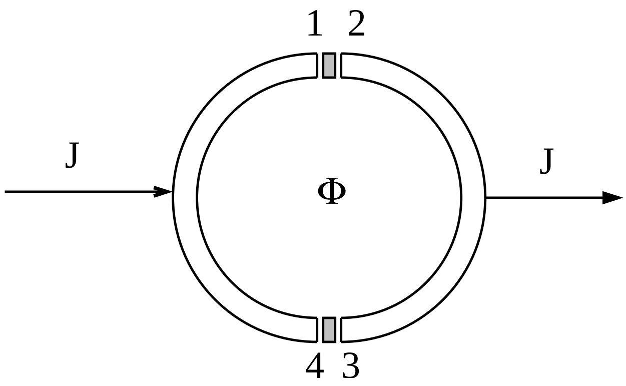

Figure 3:

We ignore resistance and capacitance for now. The inside SC thickness is assumed much greater than λ and since $v_s = 0$ we have $\nabla\phi = 2\mathbf{A}/\Phi_0$. The flux will be

$$\begin{aligned}\Phi = \oint d\mathbf{s}\cdot\mathbf{A} &= \int_1^2 d\mathbf{s}\cdot A + \frac{\Phi_0}{2}\int_2^3 d\mathbf{s}\cdot\nabla\phi + \int_3^4 d\mathbf{s}\cdot\mathbf{A} + \frac{\Phi_0}{2}\int_4^1 d\mathbf{s}\cdot\nabla\phi \\ &= \frac{\Phi_0}{2}(\phi_3-\phi_2) + \frac{\Phi_0}{2}(\phi_1-\phi_4) + \int_1^2 d\mathbf{s}\cdot\mathbf{A} + \int_3^4 d\mathbf{s}\cdot\mathbf{A} \\ &= \underbrace{\frac{\Phi_0}{2}(\phi_1-\phi_2) + \int_1^2 d\mathbf{s}\cdot\mathbf{A}}_{\equiv -\gamma_{12}} + \underbrace{\frac{\Phi_0}{2}(\phi_3-\phi_4) + \int_3^4 d\mathbf{s}\cdot\mathbf{A}}_{\equiv -\gamma_{34}}\end{aligned}$$

$$\Phi = \gamma_{43} - \gamma_{12}. \tag{21}$$

The Josephson current <u>through</u> the SQUID will be

$$\begin{aligned}J &= J_c\big(\sin\gamma_{12} + \sin\gamma_{43}\big) \\ &= J_c\big(\sin\gamma_{12} + \sin(\gamma_{12}+\Phi)\big).\end{aligned} \tag{22}$$

Which means that the current oscillates with the flux. And as a result of that the SQUID can be a sensitive measure of magnetic fields. In practice we include the capacitance and resistance of the device.