Final Exam Solutions–Spring 2001

Problem 1 - Zero Sound

a)

$$\delta f_{\vec{k}}(\vec{r},t) = \sum_q \int d\omega e^{i\vec{q}\vec{r}-i\omega t}\ \delta f_{\vec{k}}(\vec{q},\omega)$$

Substitute in $(\vec{r},t)$ transport equation and then multiply by $e^{-i\vec{q}\vec{r}+i\omega t}$, $i\int d^3r\ \int dt$

$$\Rightarrow \quad (\omega - \vec{q}\cdot\vec{v}_k)\delta f_{\vec{k}}(\vec{q},\omega) + \vec{q}\cdot\vec{v}_k\left(\frac{\partial f^{(0)}}{\partial \epsilon_{\vec{k}}}\right)\delta\epsilon_{\vec{k}} = 0$$

in collisionless limit.
Specialize to $T=0 \Rightarrow$
$\left(\frac{\partial f^{(0)}}{\partial \epsilon_k}\right) = -\delta(\epsilon_k - \epsilon_F)$ where $\delta\epsilon_k = \sum_{k'} f_{kk'}\delta f_{k'}$ (Landau interaction)

$$\Rightarrow \quad 0 = (\omega - \vec{q}\cdot\vec{v}_k)\delta f_k - \delta(\epsilon_k - \epsilon_F)\vec{q}\cdot\vec{v}_k\sum_{k'} f_{kk'}\delta f_{k'}$$

where all quantities are now F.T.

b) Assume $f_{kk'} = F_0^S/N_0$ isotropic.
Sum both sides of kinetic eqn. over ϵ_k

$$0 = \int d\epsilon_k(\omega - \vec{q}\cdot\vec{v}_k)\delta f_k - \int d\epsilon_k \delta(\epsilon_k - \epsilon_F)\vec{q}\cdot\vec{v}_k\frac{F_0^S}{N_0}\sum_{k'}\delta f_{k'}$$

Since we expect δf_k to be sharply peaked at Fermi surface because of $\delta(\epsilon_k - \epsilon_F)$, put

$$\delta f_{\vec{k}} = \delta(\epsilon_k - \epsilon_F)\delta f_{\hat{k}}$$

$$\Rightarrow \quad 0 = (\omega - \vec{q}\cdot\vec{v}_k)\delta f_{\hat{k}} - \vec{q}\cdot\vec{v}_k F_0^S\int\frac{d\Omega'}{4\pi}\delta f_{\hat{k}'}$$

(note $\vec{v}_k = \frac{\partial\epsilon_k}{\partial k}\Big|_{\epsilon_F}$ now!)

$$\Rightarrow \quad \boxed{\delta f_{\hat{k}} = \frac{\vec{q}\cdot\vec{v}_k}{\omega - \vec{q}\cdot\vec{v}_k}F_0^S\int\frac{d\Omega'}{4\pi}\delta f_{\hat{k}'}}$$

c) Solution with isotropic symmetry - quasiparticles on all parts of FS participate equally $\Rightarrow$ $\delta f_{\hat{k}} = \delta f$ independent of $\hat{k}$. Expand in Legendre polynomials

$$\delta f_{\hat{k}} = \sum_l \delta f_l\, P_l(\cos\theta)\ ;\ \cos\theta = \frac{\vec{q}\cdot\vec{v}_k}{q\,v_F}$$

Project out δf_0 by integrating $\int\frac{d\Omega}{4\pi}$

$$\int\frac{d\Omega}{4\pi}\delta f_{\hat{k}} = \delta f_0$$

since $P_0 = 1$.

$$\delta f_0 = \int\frac{d\Omega}{4\pi}\frac{\vec{q}\cdot\vec{v}_k}{\omega - \vec{q}\cdot\vec{v}_k}F_0^S\delta f_0$$

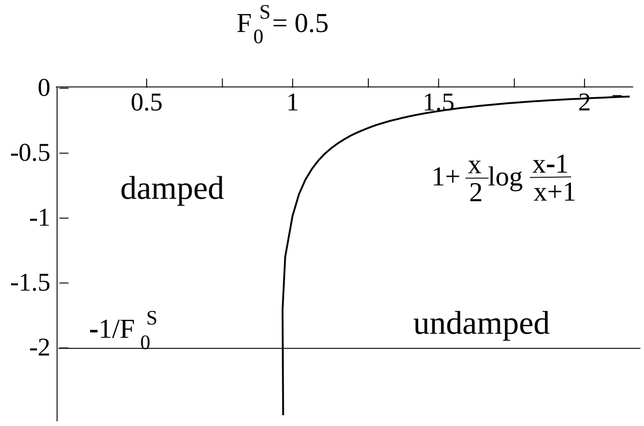
Solution only if

$$1 - F_0^S\int_{-1}^{1}\frac{dx}{2}\frac{qv_F x}{\omega - qv_F x} = 1 + F_0^S\left(1 + \frac{1}{2}\frac{\omega}{qv_F}log\frac{\omega - qv_F}{\omega + qv_F}\right) = 0$$

d) For any F_0^S, if there is a solution it will be of the form

$$\frac{\omega}{qv_F} = const \;\Rightarrow\; \underline{acoustic\ mode}$$

If $\omega < qv_F$ Log has Im part $\Rightarrow$ mode is damped. If $\omega > qv_F$ mode is undamped. Look for solns $\omega > qv_F$



Solution for any $F_0^S > 0$ - zero sound. For F_0^S small, speed $c \equiv \omega/q \simeq v_F$. For $F_0^S >> 1$, speed $c >> v_F$

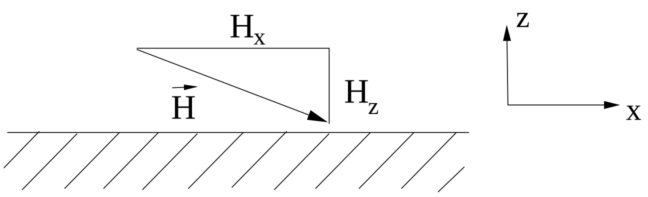
N.B. this approximation neglecting F_1^s, etc, is in general very unrealistic!

Problem 2

No "right answer"!.

Problem 3 - Superconducting sphere

a) Outside sphere must have $\vec{\nabla} \cdot \vec{H} = 0$, $\vec{\nabla} \times \vec{H} = 0$
Near surface, picture like this locally



Since on these scales system is symmetric, $\vec{H}$ can only depend on z. Therefore

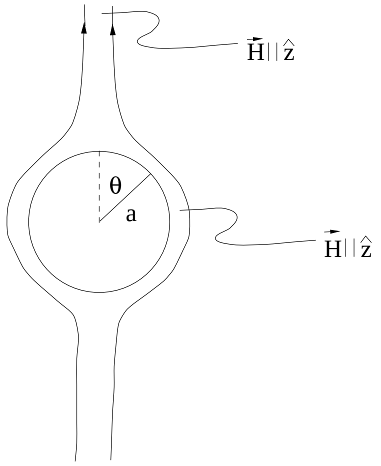
$$0 = \vec{\nabla} \cdot \vec{H} = \nabla_z\, H_z(z) + \nabla_x\, H_x(z) = \nabla_z\, H_z$$

So H_z is at most a constant ind. of z everywhere. This implies $\vec{\nabla} \times \vec{H} = 0$ inside sphere too $\Rightarrow$ no currents flowing $\Rightarrow$ trivial solution.

The above shows that we must have $\vec{H} \parallel$ surface of sphere everywhere for Meissner effect.

b) Look for soln. to $\vec{\nabla}\cdot\vec{H}=0$, $\vec{\nabla}\times\vec{H}=0$ outside sphere with BC. $H_{\perp}|_{r=a}=0$, $\vec{H}=H_0\hat{z}$ for $r>>a$, $a=$ radius of sphere.

Guess solution with form



In exterior this is standard magnetostatics problem, with sphere of uniform magnetization. Magnetostatic potential is dipolar:

$$\Phi_H=\frac{4}{3}\pi a^3\,M\frac{cos\,\theta}{r^2}$$

$$\vec{H}=\vec{H}_0-\vec{\nabla}\frac{4}{3}\pi a^3\,M\frac{cos\,\theta}{r^2}=\vec{H}_0+\frac{4}{3}\frac{\pi a^3\,M}{r^3}(2\,cos\,\theta\,\hat{r}+sin\,\theta\,\hat{\theta})$$

Require $H_r|_{r=a}=0$ B.C.

$$=H_0\,cos\,\theta+\frac{4\pi}{3}M\;2cos\,\theta=0$$

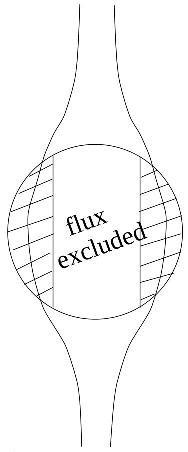
effective mag. "M" $=\frac{3}{8\pi}H_0$

$$\vec{H}=H_0\left(\hat{z}+\frac{a^3}{2r^3}(2cos\,\theta\,\hat{r}+sin\,\theta\,\hat{\theta})\right)$$

Component $\parallel$ to surface is

$$H_\theta|_{r=a}=H_0 sin\,\theta(1+\frac{1}{2})=\frac{3}{2}H_0 sin\,\theta$$

$(z=cos\,\theta\,\hat{r},sin\,\theta\,\hat{\theta})$.
So when $H_0=\frac{2}{3}H_c$, field at equator $H(\theta=\pi/2,r=a)=H_c\;\Rightarrow$ SC at that point is supressed. Energetically favorable for part of sample near equator to become normal:



(N.B. when $H_0 > H_c$, excluding flux is no longer energetically favorable, and solution above no longer applies).

Problem 4 - Localization in 1D

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \end{pmatrix},$$

where Ψ_i is amplitude to find electron on site i (neglect spin).

$$\mathcal{H}\Psi = \begin{pmatrix} u_1 & -t & 0 & 0 & \dots \\ -t & u_2 & -t & 0 & \dots \\ 0 & -t & u_3 & -t & \dots \\ 0 & 0 & -t & u_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \vdots \end{pmatrix} = E\Psi$$

For general site i, Schrodinger:

$$-t\Psi_{i-1} + u_i\Psi_i - t\Psi_{i+1} = E\Psi_i$$

a) Definition of transfer matrix:

$$\Psi_{i+1} = T_{11}\Psi_i + T_{12}\Psi_{i-1}$$

$$\Psi_i = T_{21}\Psi_i + T_{22}\Psi_{i-1}$$

agrees with S-eqn <u>if</u>

$$T_{12} = -1 \qquad T_{11} = (U_i - E)/t$$

$$T_{21} = 1 \qquad T_{22} = 0$$

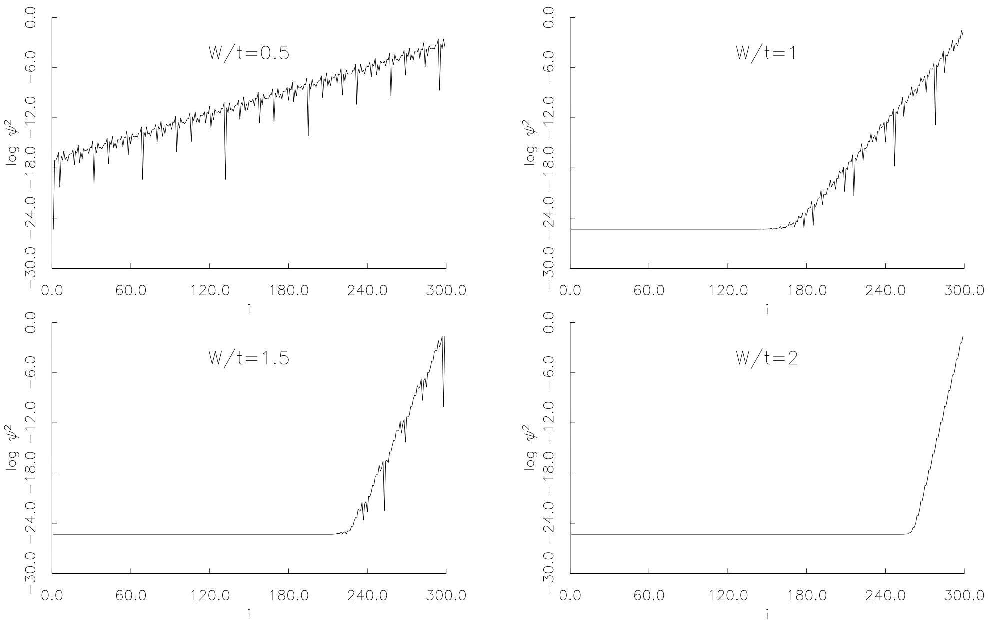
b) Take U_i random over interval $[-W/2, W/2]$.

The attcahed Fortran program implements transfer matrix on chain:

```
implicit double precision (a-h,o-z)
dimension T0(2,2),T1(2,2),T(2,2)
```

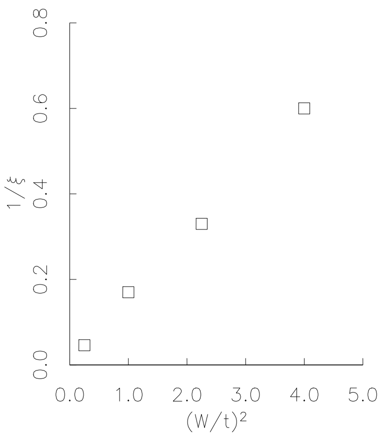
```
      dimension psi(100000)
      open(file="loc.d",unit=10,status="unknown")
      open(file="input.d",unit=11,status="unknown")
      read(11,*)N
      read(11,*)W
      read(11,*)E
      U=(ran(x)-.5)*W
      T0(1,1)=U-E
      T0(1,2)=-1.
      T0(2,1)=1.
      T0(2,2)=0.
      psi(1)=0.
      psi(2)=1.
      sum=0.
c     write(*,*)"T0"
c     write(*,*)T0(1,1),T0(1,2)
c      write(*,*)T0(2,1),T0(2,2)
c      write(*,*)" "
      do 10 i=1,N-2
      psi(i+2)=T0(1,1)*psi(i)+T0(1,2)*psi(i+1)
      U=(ran(x)-.5)*W
c     write(*,*)"check"
      T1(1,1)=U-E
      T1(1,2)=-1.
      T1(2,1)=1.
      T1(2,2)=0.
      T(1,1)=T0(1,1)*T1(1,1)+T0(1,2)*T1(2,1)
      T(1,2)=T0(1,1)*T1(1,2)+T0(1,2)*T1(2,2)
      T(2,1)=T0(2,1)*T1(1,1)+T0(2,2)*T1(2,1)
      T(2,2)=T0(2,1)*T1(1,2)+T0(2,2)*T1(2,2)
      T0(1,1)=T(1,1)
      T0(1,2)=T(1,2)
      T0(2,1)=T(2,1)
      T0(2,2)=T(2,2)
      if i=N goto 10
      sum=sum+psi(i)**2
c     write(10,*) i, psi(i)**2
c     write(*,*)T(1,1),T(1,2)
c     write(*,*)T(2,1),T(2,2)
c     write(*,*)" "
10     continue
      sum=sum+psi(N-1)**2+psi(N)**2
c     c=1./(real(N))
      c=1.
      write(*,*)T(1,1)*c,T(1,2)*c
      write(*,*)T(2,1)*c,T(2,2)*c
      write(*,*)
      write(*,*)"sum=",sum
      do 20 i=1,N-1
      write(10,*)i,psi(i)**2/sum
20     continue
      end
```

Fix eigenvalue $E = 0 \Rightarrow$ won't necessarily pick up exponentially decaying solutions which satisfy boundary conditions for given length chain. Generically get exponentially increasing solutions, which may be interpreted as exp. decreasing from other end of chain! Here are some examples for 4 different disorder values $W/t = 0.5, 1, 1.5, 2$, $N = 300$



The slopes for the above four graphs are in order: $slope_{0.5} = 11/240 = 0.046$; $slope_{1.} = 23/130 = 0.17$; $slope_{1.5} = 24/70 = 0.33$; $slope_{2.} = 24/40 = 0.6$.

c) Localization length decreases with increasing disorder, of course. Since we expect $|\Psi_i|^2 \sim exp - x_i/\xi$, plotting log$|\Psi_i|^2$ vs x_i gives a quasilinear plot whose slope is $1/\xi$. Here's a plot of the dependence of the slope on disorder.



So find roughly $\boxed{1/\xi \sim 0.18 \left(\frac{W}{t}\right)^2}$.