Final Exam Solutions–Spring 2023

Problem 1

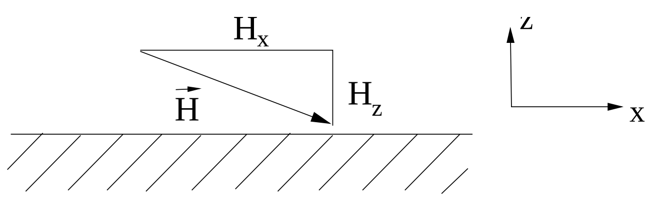
No "right answer"!.

Problem 2- one-phonon neutron scattering

Ashcroft-Mermin Problem 24-2 Allowed emitted phonon solutions follows from Figure 24. 6 but taking $\omega(\mathbf{k}) \to -\omega(\mathbf{k})$.

Problem 3 - Superconducting sphere

1. Outside sphere must have $\vec{\nabla} \cdot \vec{H} = 0$, $\vec{\nabla} \times \vec{H} = 0$
   Near surface, picture like this locally

   Since on these scales system is symmetric, $\vec{H}$ can only depend on z. Therefore

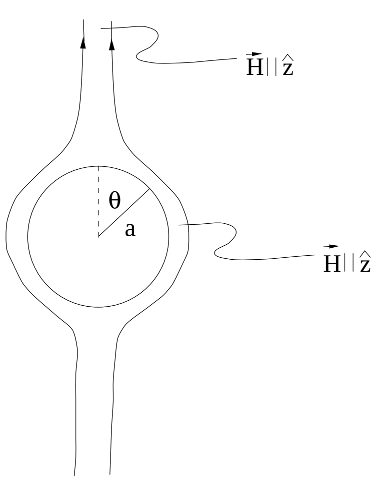
   $$0 = \vec{\nabla} \cdot \vec{H} = \nabla_z H_z(z) + \nabla_x H_x(z) = \nabla_z H_z$$

   So H_z is at most a constant ind. of z everywhere. This implies $\vec{\nabla} \times \vec{H} = 0$ inside sphere too $\Rightarrow$ no currents flowing $\Rightarrow$ trivial solution.

   The above shows that we must have $\vec{H} \parallel$ surface of sphere everywhere for Meissner effect.

2. Look for soln. to $\vec{\nabla} \cdot \vec{H} = 0$, $\vec{\nabla} \times \vec{H} = 0$ outside sphere with BC. $H_\perp|_{r=a} = 0$, $\vec{H} = H_0 \hat{z}$ for $r >> a$, $a =$ radius of sphere.

   Guess solution with form

In exterior this is standard magnetostatics problem, with sphere of uniform magnetization. Magnetostatic potential is dipolar:

$$\Phi_H = \frac{4}{3}\pi a^3 \, M \frac{\cos\theta}{r^2}$$

$$\vec{H} = \vec{H}_0 - \vec{\nabla}\frac{4}{3}\pi a^3 \, M \frac{\cos\theta}{r^2} = \vec{H}_0 + \frac{4}{3}\frac{\pi a^3 \, M}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

Require $H_r|_{r=a} = 0$ B.C.

$$= H_0 \cos\theta + \frac{4\pi}{3} M \, 2\cos\theta = 0$$

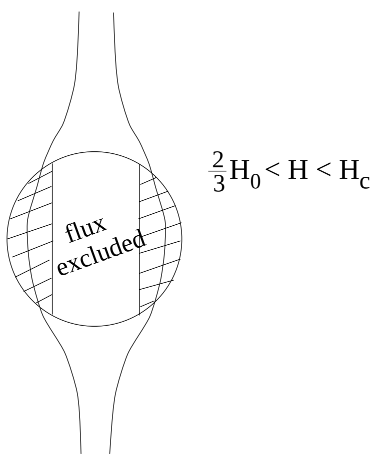
effective mag. "M" $= \frac{3}{8\pi} H_0$

$$\vec{H} = H_0 \left(\hat{z} + \frac{a^3}{2r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) \right)$$

Component $\|$ to surface is

$$H_\theta|_{r=a} = H_0 \sin\theta (1 + \frac{1}{2}) = \frac{3}{2} H_0 \sin\theta$$

$(z = \cos\theta\,\hat{r}, \sin\theta\,\hat{\theta})$.
So when $H_0 = \frac{2}{3}H_c$, field at equator $H(\theta = \pi/2, r = a) = H_c \Rightarrow$ SC at that point is supressed. Energetically favorable for part of sample near equator to become normal:



(N.B. when $H_0 > H_c$, excluding flux is no longer energetically favorable, and solution above no longer applies).

Problem 4 - Density of states of p-wave superconductors.

1. Suppose that a superconductor is described by an order parameter whose momentum dependence is $\Delta_{\mathbf{k}} = \Delta_0 f(\hat{k})$, where $f(\hat{k})$ is either $\sin\theta$ (axial state) or $\cos\theta$ (polar state). Note the two f functions are $\ell = 1$ spherical harmonics or linear combinations thereof, so to satisfy the Pauli principle these must represent spin triplet ($S = 1$) pair states. Nevertheless the quasiparticle spectrum may be taken to be independent of spin, $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}$.

Use the fact that, for any given angle θ of $\mathbf{k}$ on the Fermi surface, the quasiparticle excitations have the same form as that for an s-wave superconductor, but with gap $\Delta(\theta)$ to find an expression for the density of states for the two types of order parameters.

In class we showed that the density of quasiparticle states of a superconductor is $N(\omega) = N_0 \text{ Re } \omega/\sqrt{\omega^2 - \Delta^2}$. We are told this holds now for each angle,

$$N(\omega;\theta) = N_0 \text{ Re } \frac{\omega}{\sqrt{\omega^2 - \Delta^2(\theta)}}$$

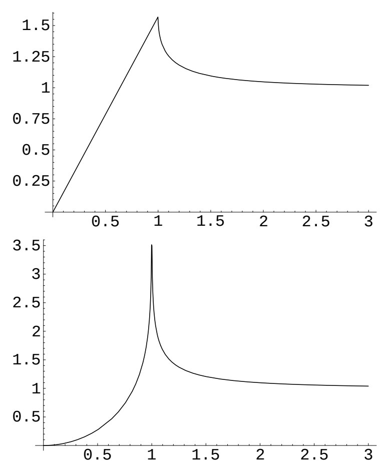
So for the total density of states, we need the two integrals

$$I_{pol} = \int_0^{\pi/2} d\theta \sin\theta \frac{\omega}{\sqrt{\omega^2 - \Delta_0^2 \cos^2\theta}} = \frac{\omega}{\Delta_0} \cot^{-1}\sqrt{\omega^2 - \Delta_0^2} \qquad \text{polar}$$

$$I_{ax} = \int_0^{\Pi/2} d\theta \sin\theta \frac{\omega}{\sqrt{\omega^2 - \Delta_0^2 \sin^2\theta}} = \omega \log\sqrt{\frac{\omega + \Delta_0}{\omega - \Delta_0}} \qquad \text{axial,}$$

The density of states is now just N_0 ReI.

2. Plot as a function of ω the full frequency dependence of the density of states in the two cases.



Density of states $N(\omega)/N_0$ vs. ω for polar (top) and axial (bottom) p-wave states. Note in both cases as $\omega \to \infty$ the functions approach 1, i.e. the normal state DOS.

3. Estimate the low-ω asymptotic behavior in both cases.

Expanding the functions $I_{ax,pol}$ for small ω gives

$$I_{pol} \simeq \frac{\omega}{\Delta_0}\left(\pi/2 + i\log\omega^2/4\Delta_0^2\right)$$

$$I_{ax} = \frac{\omega}{\Delta_0}\left(\frac{\omega}{\Delta_0} - i\frac{\pi}{2}\right)$$

so the desired asymptotic forms are $N_{pol} \simeq \pi\omega/(2\Delta_0)$ and $N_{ax} \simeq \omega^2/\Delta_0^2$. These two power laws are reflected directly in the temperature dependence of the specific heat of the two states.

Problem 5: 1D Ising Model.

Consider a chain of N sites governed by the Hamiltonian

$$H_I = J\sum_{i=1}^{N-1} S_i^z S_{i+1}^z. \tag{1}$$

Consider a chain of N sites governed by the Hamiltonian

$$H_I = J\sum_{i=1}^{N-1} S_i^z S_{i+1}^z. \tag{2}$$

Show that the partition function is given by

$$Z = \sum_{S_1^z=\pm\frac{1}{2}} \ldots \sum_{S_N^z=\pm\frac{1}{2}} \exp\left(-\beta J\sum_{i=1}^{N-1} S_i^z S_{i+1}^z\right) = 2\left(2\cosh[\beta J/4]\right)^{N-1} \tag{3}$$

Start from the end of the chain and show that $\sum_{S_N^z=\pm\frac{1}{2}} \exp\left(-\beta J S_{N-1}^z S_N^z\right) = 2\cosh(\beta J/4)$ independent of S_{N-1}^z. Thus $Z_N = 2\cosh(\beta J/4) Z_{N-1}$. Iterate the procedure, and remember to be careful at the other end of the chain to get the result.

The thermodynamic limit is defined by $N \to \infty$.

Partition function is

$$\begin{aligned}
Z &= \sum_{S_1^z=\pm1/2}\sum_{S_2^z=\pm1/2}\cdots\sum_{S_N^z=\pm1/2} \exp\left[-\beta J\sum_{i=1}^{N-1} S_i^z S_{i+1}^z\right] \\
&= \sum_{S_2^z=\pm1/2}\cdots\sum_{S_N^z=\pm1/2}\left(e^{\beta\frac{J}{2}S_2^z} + e^{-\beta\frac{J}{2}S_2^z}\right) e^{-\beta J S_2^z S_3^z}\ldots e^{-\beta J S_{N-1}^z S_N^z} \\
&= \sum_{S_3^z=\pm1/2}\cdots\sum_{S_N^z=\pm1/2}\left[2\cosh(\beta J/4)e^{-\beta\frac{J}{2}S_3^z} + 2\cosh(\beta J/4)e^{\beta\frac{J}{2}S_3^z}\right] e^{-\beta J S_3^z S_4^z}\ldots e^{-\beta J S_{N-1}^z S_N^z} \\
&= [2\cosh\beta J/4]^2 \sum_{S_4^z=\pm1/2}\cdots\sum_{S_N^z=\pm1/2}\left(e^{\beta\frac{J}{2}S_4^z} + e^{-\beta\frac{J}{2}S_4^z}\right) e^{-\beta J S_4^z S_5^z}\ldots e^{-\beta J S_{N-1}^z S_N^z} \\
&\vdots \\
&= 2[2\cosh\beta J/4]^{N-1}
\end{aligned} \tag{4}$$

Show that in the thermodynamic limit the free energy is given by

$$F = -NkT\ln(2\cosh(\beta J/4)) \tag{5}$$

$$\begin{aligned} F &= -T\ln Z = -T\ln\left[\ 2(2\cosh(\beta J/4))^{N-1}\right] \\ &= -T\ln 2 - T(N-1)\ln\left[2\cosh(\beta J/4)\right] \\ &= -T\left[(\ln 2 - \ln(2\cosh(\beta J/4)))\right] - TN\ln\left[2\cosh(\beta J/4)\right] \\ &\to_{N\to\infty} -TN\ln\left[2\cosh(\beta J/4)\right] \end{aligned} \tag{6}$$

Let us calculate the magnetization per site $M = -\frac{1}{N}\frac{\partial F}{\partial H}|_{H=0}$. We'll need to include the coupling to the magnetic field. Thus we need to know the free energy in the presence of a magnetic field H where

$$H_I = J\sum_{i=1}^{N} S_i^z S_{i+1}^z - H\sum_{i=1}^{N} S_i^z. \tag{7}$$

For convenience we now sum from $i = 1, ..., N$ assuming periodic boundary conditions meaning that $N+1 = 1$

Convince yourself that the partition function can be written as

$$Z = \sum_{S_i^z=\pm\frac{1}{2}}\prod_{i=1}^{N}\exp\left(-\beta\left[JS_i^zS_{i+1}^z - \frac{H}{2}(S_i^z + S_{i+1}^z)\right]\right) \tag{8}$$

Consider the Ising chain with N sites in magnetic field H with boundary conditions $S_{N+1} = S_1$.

$$\begin{aligned} \mathcal{H} &= J\sum_{i=1}^{N} S_i^zS_{i+1}^z + H\sum_{i=1}^{N} S_i^z \\ Z &= \sum_{S_1^z=\pm1/2}\sum_{S_2^z=\pm1/2}\cdots\sum_{S_N^z=\pm1/2}\exp\left[-\beta J\sum_{i=1}^{N-1}S_i^zS_{i+1}^z - \beta JS_N^zS_1^z - H\sum_{i=1}^{N}S_i^z\right] \\ &= \sum_{S_1^z=\pm1/2}\cdots\sum_{S_N^z=\pm1/2}\exp\left[-\beta J\left(S_1^zS_2^z + S_2^zS_3^z + ...S_N^zS_1^z\right) - \beta H\left(S_1^z + S_2^z + ... + S_N^z\right)\right] \\ &= \sum_{S_1^z=\pm1/2}\cdots\sum_{S_N^z=\pm1/2}\exp\left(-\beta JS_1^zS_2^z - \beta HS_1^z\right)\exp\left(JS_2^zS_3^z - \beta HS_2^z\right)\cdots\exp\left(-\beta JS_N^zS_1^z - \beta HS_N^z\right) \\ &= \sum_{S_1^z=\pm1/2}\cdots\sum_{S_N^z=\pm1/2}\exp\left(-\beta JS_1^zS_2^z - \beta\frac{H}{2}(S_1^z + S_2^z)\right)\cdots\exp\left(-\beta JS_N^zS_1^z - \beta\frac{H}{2}(S_N^z + S_1^z)\right) \\ &= \sum_{S_1^z=\pm1/2}\cdots\sum_{S_N^z=\pm1/2}\prod_{i=1}^{N}\exp\left(-\beta JS_i^zS_{i+1}^z - \beta\frac{H}{2}(S_i^z + S_{i+1}^z)\right) \end{aligned} \tag{9}$$

This partition function can be rewritten in terms of a *transfer matrix* P given by

$$P = \begin{pmatrix} \mathrm{P}_{11} & \mathrm{P}_{1-1} \\ \mathrm{P}_{-11} & \mathrm{P}_{-1-1} \end{pmatrix} \tag{10}$$

where

$$P_{11} = \exp(-\beta(\tilde{J} + \tilde{H})), \tag{11}$$
$$P_{-1-1} = \exp(-\beta(\tilde{J} - \tilde{H})), \tag{12}$$
$$P_{1-1} = P_{-11} = \exp(\beta\tilde{J}), \tag{13}$$

with $\tilde{J} = J/4$ and $\tilde{H} = H/2$.

> Show that
> $$Z = \mathrm{Tr}P^N. \tag{14}$$

First note, e.g. that 1 and -1 stand for spin up or down, so, e.g. $\langle\uparrow|P|\uparrow\rangle = \exp(-J/4 - H/2)$, which is just the first exponential factor in the framed equation (8) above, for the term with both spins up. Check all the matrix elements $\uparrow\downarrow$, $\downarrow\downarrow$, and $\downarrow\uparrow$, to see that the expression derived above for Z is just

$$Z = \sum_{S_1^z} \cdots \sum_{S_N^z} \langle S_1^z|P|S_2^z\rangle\langle S_2^z|P|S_3^z\rangle \cdots \langle S_{N-1}^z|P|S_N^z\rangle\langle S_N^z|P|S_1^z\rangle \tag{15}$$

But each factor of $\sum_{S_i^z} |S_i^z\rangle\langle S_i^z| = 1$, so one can remove all the intermediate bras and kets except 1:

$$Z = \sum_{S_1^z} \langle S_1^z|P \cdot P \cdots P|S_1^z\rangle = \sum_{S_1^z} \langle S_1^z|P^N|S_1^z\rangle = \mathrm{Tr}P^N \tag{16}$$

The idea is now to use that the trace Tr is basis independent. Thus we can diagonalise P and use this to obtain $\mathrm{Tr}P^N$.

> Find the eigenvalues λ_1 and λ_2 of P. Once you have them $Z = \mathrm{Tr}P^N = \lambda_1^N + \lambda_2^N$.

Find eigenvalues λ_1 and λ_2 of a 2D matrix:

$$\begin{vmatrix} e^{-\beta(\tilde{J}+\tilde{H})} - \lambda & e^{\beta\tilde{J}} \\ e^{\beta\tilde{J}} & e^{-\beta(\tilde{J}-\tilde{H})} - \lambda \end{vmatrix} = 0 \tag{17}$$

Solve

$$0 = \lambda^2 - 2e^{-\beta\tilde{J}}\cosh(\beta\tilde{H})\lambda - 2\sinh 2\beta\tilde{J} \tag{18}$$
$$\tag{19}$$
$$\lambda_\pm = e^{-\beta\tilde{J}}\left[\cosh(\beta\tilde{H}) \pm \sqrt{\sinh^2(\beta\tilde{H}) + e^{4\beta\tilde{J}}}\right] \tag{20}$$

The partition function is now given by the Tr of P^N is now $\lambda_+^N + \lambda_-^N$.

> Use the fact that $\ln(\lambda_1^N + \lambda_2^N) = N\ln\lambda_1 + \ln\left(1 + \left[\frac{\lambda_2}{\lambda_1}\right]^N\right)$ to show that in the thermodynamic limit
> $$F = -NkT\ln\left[e^{-\beta\tilde{J}}\cosh\beta\tilde{H} + \sqrt{e^{-2\beta\tilde{J}}\sinh^2\beta\tilde{H} + e^{2\beta\tilde{J}}}\right] \tag{21}$$

$$\begin{aligned} F &= -T \ln Z = -T \ln(\lambda_+^N + \lambda_-^N) \\ &= -TN \ln \lambda_+ - T \ln \left(1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right) \end{aligned} \tag{22}$$

Now note that $\lambda_-/\lambda_+ < 1$, so in the thermodynamic limit $N \to \infty$, this vanishes.

$$F \underset{N \to \infty}{\to} - TN \ln \lambda_+ \tag{23}$$

For $H = 0$ this result reduces to Eq. (5) as it should. Now we are finally ready to obtain the magnetisation $M(\tilde{H})$ per site, i.e. $M(\tilde{H}) = -\frac{1}{N} \frac{\partial F}{\partial \tilde{H}}$.

Show that

$$M(\tilde{H}) = \frac{\sinh \beta \tilde{H}}{\sqrt{\sinh^2 \beta \tilde{H} + e^{4\beta \tilde{J}}}}. \tag{24}$$

$$M(\tilde{H}) = \frac{\partial F}{\partial \beta \tilde{H}} = \frac{\sinh \beta \tilde{H}}{\sqrt{\sinh^2 \beta \tilde{H} + e^{4\beta \tilde{J}}}} \tag{25}$$