## Contents

# 6 Ginzburg-Landau Theory

## 6.1 GL Free Energy

While the BCS weak-coupling theory we looked at the last two weeks is very powerful, and provides at least a qualitatively correct description of most aspects of classic superconductors,[1] there is a complementary theory which a) is simpler and more physically transparent, although valid only near the transition; and b) provides exact results under certain circumstances. This is the Ginzburg-Landau theory [ V.L. Ginzburg and L.D. Landau, Zh. Eksp. Teor. Fiz. 20, 1064 (1950)], which received remarkably little attention in the west until Gor'kov showed it was derivable from the BCS theory. [L.P. Gor'kov, Zh. Eksp. Teor Fiz. 36, 1918 (1959)]. The theory simply postulated the existence of a macrosopic quantum wave function $\psi(r)$ which was equivalent to an order parameter, and proposed that on symmetry grounds alone, the free energy density of a superconductor should be expressible in terms of an expansion in this quantity:

$$\frac{f_s - f_n}{V} = a|\psi|^2 + b|\psi|^4 + \frac{1}{2m^*}|(\nabla + \frac{ie^*}{c}\vec{A})\psi|^2, \tag{1}$$

where the subscripts n and s refer to the normal and superconducting states, respectively.

Let's see why GL might have been led to make such a "guess". The superconducting-normal transition was empirically known to be second order in zero field, so it was natural to write down a theory analogous to the Landau theory of a ferromagnet, which is an expansion in powers of the magnetization, **M**. The choice of order parameter for the superconductor corresponding to **M** for the ferromagnet was not obvious, but a complex scalar field $\psi$ was a natural choice because of the analogy with liquid $He$, where $|\psi|^2$ is known to represent the superfluid density $n_s$;[2] a

[1] In fact one could make a case that the BCS theory is the most quantitatively accurate theory in all of condensed matter physics

[2] $\psi$ in the $He$ case has the microscopic interpretation as the Bose condensate amplitude.

quantum mechanical density should be a complex wave function squared. The biggest leap of GL was to specify correctly how electromagnetic fields (which had no analog in superfluid $He$) would couple to the system. They exploited in this case the similarity of the formalism to ordinary quantum mechanics, and coupled the fields in the usual way to "charges" $e^*$ associated with "particles" of mass $m^*$. Recall for a real charge in a magnetic field, the kinetic energy is:

$$< \Psi|\mathcal{H}_{kin}|\Psi > \; = \; -\frac{1}{2m}\int d^3r \Psi^*(\nabla + \frac{ie}{c}\vec{A})^2\Psi \tag{2}$$

$$= \; \frac{1}{2m}\int d^3r |(\nabla + \frac{ie}{c}\vec{A})\Psi|^2, \tag{3}$$

after an integration by parts in the second step. GL just replaced $e$, $m$ with $e^*$, $m^*$ to obtain the kinetic part of Eq. (1); they expected that $e^*$ and $m^*$ were the elementary electron charge and mass, respectively, but did not assume so.

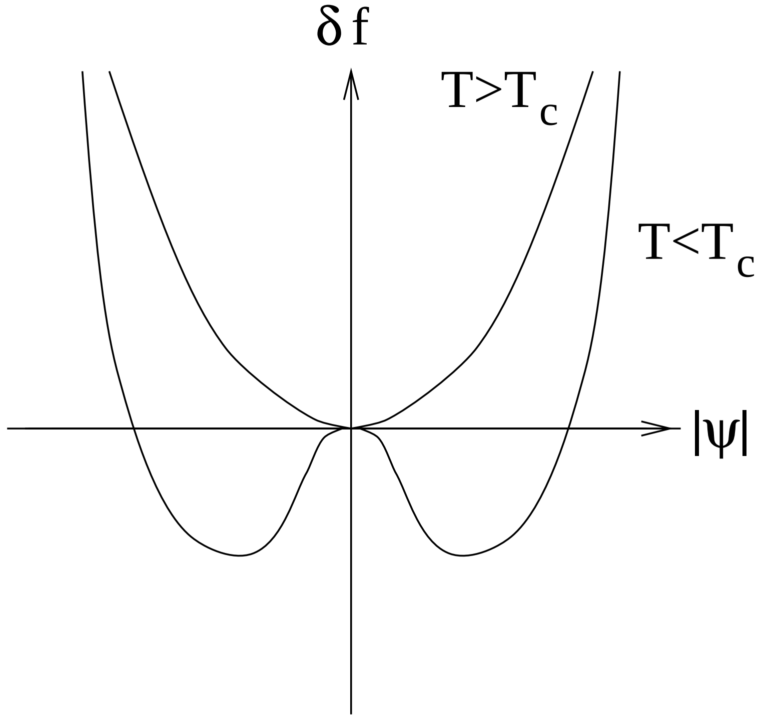


Figure 1: Mexican hat potential for superconductor.

A system described by this free energy will undergo a second-order phase transition in zero field when $a = 0$: clearly when $a$ is positive, the system can minimize $\delta f$ by having $\psi = 0$ (no superconductivity), whereas if $a$ is negative, $\delta f$ has a minimum with $\psi \neq 0$. The free energy (1) is a *functional* of the order parameter $\psi$, meaning the actual value of the order parameter realized in equilibrium satisfies $\delta f/\delta\psi = 0$.[3] Notice $f$ is independent of the phase $\phi$ of the order parameter, $\psi \equiv |\psi|e^{i\phi}$, and so the ground state for $a < 0$ is equivalent to any state $\psi$ related to it by multiplication by a pure phase. This is the $U(1)$ gauge invariance of which we spoke earlier. This symmetry is broken when the system chooses one of the ground states (phases) upon condensation (Fig 1.).

For a uniform system in zero field, the total free energy $F = \int d^3r f$ is minimized when $f$ is, so one find for the order parameter at the minimum,

$$|\psi|_{eq} = [\tfrac{-a}{2b}]^{1/2}, \quad a < 0 \tag{4}$$

$$|\psi|_{eq} = 0, \quad a > 0. \tag{5}$$

When $a$ changes sign, a minimum with a nonzero value becomes possible. For a second order transition as one lowers the temperature, we assume that $a$ and $b$ are smooth functions of $T$ near $T_c$. Since we are only interested in the region near $T_c$, we take only the leading terms in the Taylor series expansions in this region: $a(T,H) = a_0(T - T_c)$ and $b =$ constant. Eqs. (4) and (5) take the form:

$$|\psi(T)|_{eq} = [\tfrac{a_0(T_c-T)}{2b}]^{1/2}, \quad T < T_c \tag{6}$$

$$|\psi(T)|_{eq} = 0, \quad T > T_c. \tag{7}$$

Substituting back into Eqs.xxx, we find:

$$f_s(T) - f_n(T) = -\tfrac{a_0 2}{4b}(T_c - T)^2, \quad T < T_c \tag{8}$$

$$f_s(T) - f_n(T) = 0, \quad T > T_c. \tag{9}$$

The idea now is to calculate various observables, and determine the GL coefficients for a given system. Once they are determined, the theory acquires predictive power due to its extreme simplicity. It should be noted

[3]Thus you should not be perturbed by the fact that $f$ apparently depends on $\psi$ even for $a > 0$. The value of $f$ in equilibrium will be $f_n = f[\psi = 0]$.

that GL theory is applied to many systems, but it is in classic superconductors that it is most accurate, since the critical region, where deviations from mean field theory are large, is of order $10^{-4}$ or less. Near the transition it may be taken to be exact for all practical purposes. This is not the case for the HTSC, where the size of the critical region has been claimed to be as much as 10-20K in some samples.

**Supercurrents.** Let's now focus our attention on the term in the GL free energy which leads to supercurrents, the kinetic energy part:

$$F_{\mathrm{kin}} = \int d^3r \frac{1}{2m^*}|(\nabla + \frac{ie^*}{c}\vec{A})\psi|^2 \tag{10}$$

$$= \int d^3r \frac{1}{2m^*}[(\nabla|\psi|)^2 + (\nabla\phi - e^*/c\mathbf{A})^2|\psi|^2]. \tag{11}$$

These expressions deserve several remarks. First, note that the free energy is gauge invariant, if we make the transformation $\vec{A} \to \vec{A} + \nabla\Lambda$, where $\Lambda$ is any scalar function of position, while at the same time changing $\psi \to \psi \exp(-ie^*\Lambda/c)$ . Second, note that in the last step above I have split the kinetic part of $f$ into a term dependent on the gradients of the order parameter magnitude $|\psi|$ and on the gradients of the phase $\phi$. Let us use a little intuition to guess what these terms mean. The energy of the superconducting state below $T_c$ is lower than that of the normal state by an amount called the *condensation energy.*[4] From Eq. (1) in zero field this is of order $|\psi|^2$ very close to the transition. To make spatial variations of the *magnitude* of $\psi$ must cost a significant fraction of the condensation energy in the region of space in which it occurs.[5] On the other hand, the zero-field free energy is actually invariant with respect to changes in $\phi$, so fluctuations of $\phi$ alone actually cost no energy.

With this in mind, let's ask what will happen if we apply a weak magnetic field described by **A** to the system. Since it is a small perturbation, we don't expect it to couple to $|\psi|$ but rather to the phase $\phi$. The kinetic

[4] We will see below from the Gorkov derivation of GL from BCS that it is of order $N(0)\Delta^2$.

[5] We can make an analogy with a ferromagnet, where if we have a *domain wall* the magnetization must go to zero at the domain boundary, costing lots of surface energy.

energy density should then reduce to the second term in Eq. (11), and furthermore we expect that it should reduce to the intuitive two-fluid expression for the kinetic energy due to supercurrents, $\frac{1}{2}mn_s v_s^2$. Recall from the superfluid $He$ analogy, we expect $|\psi|^2 \equiv n_s^*$ to be a kind of density of superconducting electrons, but that we aren't certain of the charge or mass of the "particles". So let's put

$$f_{\rm kin} \simeq \frac{1}{2m^*}|(\nabla + \frac{ie^*}{c}\vec{A})\psi|^2. = \int d^3r \frac{1}{2m^*}(\nabla\phi - e^*/c\mathbf{A})^2|\psi|^2 \equiv \frac{1}{2}m^* n_s^* v_s^2. \tag{12}$$

Comparing with Eq. (xx), we find that the *superfluid velocity* must be

$$\vec{v_s} = \frac{1}{m^*}(\nabla\phi + \frac{e^*}{c}\vec{A}). \tag{13}$$

Thus the gradient of the phase is related to the superfluid velocity, but the vector potential also appears to keep the entire formalism gauge-invariant.

**Meissner effect.** The Meissner effect now follows immediately from the two-fluid identifications we have made. The supercurrent density will certainly be just

$$\vec{j_s} = -e^* n_s^* \vec{v_s} = -\frac{e^* n_s^*}{m^*}(\nabla\phi + \frac{e^*}{c}\vec{A}). \tag{14}$$

Taking the curl of this equation, the phase drops out, and we find the magnetic field:

$$\nabla \times \vec{j_s} = -\frac{{e^*}^2 n_s^*}{m^* c}\vec{B}. \tag{15}$$

Now recall the Maxwell equation

$$\vec{j_s} = \frac{c}{4\pi}\nabla \times \vec{B}, \tag{16}$$

which, when combined with (14), gives

$$\frac{c}{4\pi}\nabla \times \nabla \times \vec{B} = -\frac{c}{4\pi}\nabla^2\vec{B} = -\frac{{e^*}^2 n_s}{m^* c}\vec{B}, \tag{17}$$

or

$$\lambda^2\nabla^2\vec{B} = \vec{B}, \tag{18}$$

where

$$\lambda = \left(\frac{m^* c^2}{4\pi e^{*2} n_s^*}\right)^{1/2}. \tag{19}$$

Notice now that if we use what we know about Cooper pairs, this expression reduces to the BCS/London penetration depth. We assume $e^*$ is the charge of the pair, namely $e^* = 2e$, and similarly $m^* = 2m$, and $|\psi|^2 = n_s^* = n_s/2$ since $n_s^*$ is the density of pairs.

**Flux quantization.** If we look at the flux quantization described in Part 1 of these notes, it is clear from our subsequent discussion of the Meissner effect, that the currents which lead to flux quantization will only flow in a small part of the cross section, a layer of thickness $\lambda$. This layer encloses the flux passing through the toroid. Draw a contour $C$ in the interior of the toroid, as shown in Figure 2. Then $v_s = 0$ everywhere on $C$. It follows that

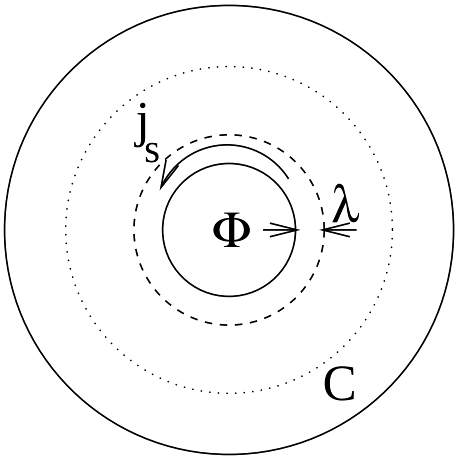


Figure 2: Quantization of flux in a toroid.

$$0 = \oint_C d\vec{\ell} \cdot \vec{v_s} = \frac{1}{m^*} \oint_C d\vec{\ell} \cdot (\nabla\phi + \frac{e^*}{c}\vec{A}). \tag{20}$$

The last integral may be evaluated using

$$\oint_C d\vec{\ell} \cdot \nabla\phi = 2\pi \times integer, \tag{21}$$

and

$$\frac{e^*}{c}\oint_C d\vec{\ell} \cdot \vec{A} \;=\; \frac{e^*}{c}\int_S d\vec{S} \cdot \nabla \times \vec{A} \tag{22}$$

$$= \frac{e^*}{c}\int_S d\vec{S}\cdot\vec{B} \quad (23)$$

$$= \frac{e^*}{c}\Phi. \quad (24)$$

Here $S$ is a surface spanning the hole and $\Phi$ the flux through the hole. Combining these results,

$$\Phi = 2\pi\frac{\hbar c}{2e}n = n\frac{hc}{2e} = n\Phi_0, \quad (25)$$

where $n$ is a integer, $\Phi_0$ is the flux quantum, and I've reinserted the correct factor of $\hbar$ in the first step to make the units right. Flux quantization indeed follows from the fact that the current is the result of a phase gradient.[6]

**Derivation from Microscopic Theory.** One of the reasons the GL theory did not enjoy much success at first was the fact that it is purely phenomenological, in the sense that the parameters $a_0, b, m^*$ are not given within any microscopic framework. The BCS theory is such a framework, and gives values for these coefficients which set the scale for all quantities calculable from the GL free energy. The GL theory is more general, however, since, e.g. for strong coupling superconductors the weak coupling values of the coefficients are simply replaced by different ones of the same order of magnitude, without changing the *form* of the GL free energy. In consequence, the dependence of observables on temperature, field, etc, will obey the same universal forms.

To see how the two theories are connected, I sketch the Gor'kov calculation, starting with the gap equation which we have looked at several times already. One expects that near the transition $\Delta$ will be small, since empirically the tranistion is of second order. So I'll expand around $\Delta = 0$:

$$\Delta = VT\sum_{\omega_n}\sum_k{}' \frac{\Delta}{\omega_n^2 + \xi_k^2 + |\Delta|^2} \quad (26)$$

[6] It is important to note, however, that a phase gradient doesn't guarantee that a current is flowing. For example, in the interior of the system depicted in Fig. 2, both $\nabla\phi$ and $\vec{A}$ are nonzero in the most convenient gauge, and cancel each other!

$$
\begin{aligned}
&\simeq \Delta N_0 VT\sum_{\omega_n}\int_{-\omega_D}^{\omega_D} d\xi_k\Big[\frac{1}{\omega_n^2+\xi_k^2}-\frac{|\Delta|^2}{(\omega_n^2+\xi_k^2)^2}+\frac{|\Delta|^4}{(\omega_n^2+\xi_k^2)^3}\Big]+... &(27)\\
&= 2\Delta N_0 V\Big[\int_0^{\omega_D}\frac{d\xi_k}{2\xi_k}\tanh\frac{\xi_k}{2T}+T\sum_{n=0}^{\infty}\{(-\frac{\pi}{2})\frac{|\Delta|^2}{\omega_n^3}+(-\frac{3\pi}{8})\frac{|\Delta|^4}{\omega_n^5}\}\Big]\\
&= \Delta N_0 V\Big[\log(\frac{2\omega_D e^\gamma}{\pi T})-\frac{|\Delta|^2}{(\pi T)^2}T\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}+\frac{3}{4}\frac{|\Delta|^4}{(\pi T)^4}T\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\Big].
\end{aligned}
$$

The left side and first term on the right side may now be combined using $T_c=(2e^\gamma\omega_D/\pi)e^{-1/N_0V}$ to find

$$
[\alpha+\beta|\Delta|^2+\gamma|\Delta|^4]\Delta=0, \tag{28}
$$

where I have defined

$$
\begin{aligned}
\alpha &= N_0\log(\frac{T}{T_c})\simeq -N_0(1-T/T_c) &(29)\\
\beta &= \frac{7}{8}\zeta(3)\frac{N_0}{(\pi T)^2} &(30)\\
\gamma &= -\frac{93}{128}\zeta(5)\frac{N_0}{(\pi T)^4}, &(31)
\end{aligned}
$$

and

$$
\zeta(z)=\frac{2^z}{2^z-1}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^z}. \tag{32}
$$

Someday you'll want to know what $\zeta(3)$ is, so I looked it up: 1.20206; you will almost certainly never want to know what $\zeta(5)$ is, but I looked it up anyway: 1.03693.

Now recall our object was to have a functional which when minimized yielded the equilibrium order parameter, $\delta f/\delta\Delta^*$. But the equilibrium order parameter is determined by the gap equation. determined by the gap equation. So in the region where $\Delta$ is small, we may perform a functional integration of the gap equation with respect to $\Delta^*$ to obtain the free energy.[7] The result is

$$
\begin{aligned}
f_s-f_n &= \int\delta\Delta^*(\alpha+\beta|\Delta|^2+\gamma|\Delta|^4)\Delta &(33)\\
&= \alpha|\Delta|^2+\frac{\beta}{2}|\Delta|^4+\frac{\gamma}{3}|\Delta|^6. &(34)
\end{aligned}
$$

[7]Recall $\Delta$ and $\Delta'$ are independent quantities when doing the functional integration or differentiation.

Thus it is clear that $\psi$ does indeed represent an order parameter, $\psi \propto \Delta$. The constant of proportionality can be fixed by comparing the GL and BCS expressions for the superfluid density, the second of which is obtained by expanding xxx for small $\Delta$, $n_s = xxxn|\Delta|^2$. One finds $\psi$=xxx.

## 6.2 Type I and Type II superconductivity

Now let's look at the problem of the instability of the normal state to superconductivity in finite magnetic field $\mathbf{H}$. A what magnetic field to we expect superconductivity to be destroyed, for a given $T < T_c$.[8] Well, overall energy is conserved, so the total condensation energy of the system in zero field,$f_s - f_n(T)$ of the system must be equal to the magnetic field energy $\int d^3r H^2/8\pi$ the system *would have* contained at the critical field $H_c$ in the absence of the Meissner effect. For a completely homogeneous system I then have

$$f_s(T) - f_n(T) = -H_c^2/8\pi, \tag{35}$$

and from Eq. (8) this means that, near $T_c$,

$$H_c = \sqrt{\frac{2\pi a_0^2}{b}}(T_c - T). \tag{36}$$

Whether this *thermodynamic critical field* $H_c$ actually represents the applied field at which flux penetrates the sample depends on geometry. We assumed in the simplified treatment above that the field at the sample surface was the same as the applied field. Clearly for any realistic sample placed in a field, the lines of field will have to follow the contour of the sample if it excludes the field from its interior. This means the value of $\mathbf{H}$ at different points on the surface will be different: the homogeneity assumption we made will not quite hold. If we imagine ramping up the applied field from zero, there will inevitably come a point $\mathbf{H}_{appl} = \mathbf{H}_{appl,c}$ where

[8] Clearly it *will* destroy superconductivity since it breaks the degenerace of between the two componenets of a Cooper pair.

the field at certain points on the sample surface exceeds the critical field, but at other points does not. For applied fields $H_{appl,c} < H_{appl} < H_c$, part of the sample will then be normal, with local field penetration, and other parts will still exclude field and be superconducting. This is the *intermediate state* of a type I superconductor. The structure of this state for a real sample is typically a complicated "striped" pattern of superconducting and normal phases. Even for small fields, edges and corners of samples typically go normal because the field lines bunch up there; these are called "demagnetizing effects", and must be accounted for in a quantitatively accurate measurement of, say, the penetration depth. It is important to note that these patterns are completely geometry dependent, and have no intrinsic length scale associated with them.

In the 50's, there were several materials known, however, in which the flux in sufficiently large fields penetrated the sample in a manner which did *not* appear to be geometry dependent. For example, samples of these so-called "type II" superconductors with nearly zero demagnetizing factors (long thin plates placed parallel to the field) also showed flux penetration in the superconducting state. The type-II materials exhibit a second-order transition at finite field and the flux $\mathbf{B}$ through the sample varies continuously in the superconducting state. Therefore the mixed state must have currents flowing, and yet the Meissner effect is not realized, so that the London equation somehow does not hold.

The answer was provided by Abrikosov in 1957 [A.A.A., Sov. Phys. JETP 5, 1174 (1957).] in a paper which Landau apparently held up for several years because he did not believe it. Let us neglect the effects of geometry again, and go back to our theorist's sample with zero demagnetizing factor. Can we relax any of the assumptions that led to the London equation (14)? Only one is potentially problematic, that $n_s^*(r) = |\psi(r)|^2$ = constant independent of position. Let's examine–as Abrikosov did–the energy cost of making spatial variations of the order parameter. The free energy in zero field is

$$F = \int d^3r[a|\psi|^2 + \frac{1}{2m^*}|\nabla\psi|^2 + b|\psi|^4], \tag{37}$$

or

$$\frac{1}{-a}F = \int d^3r[-|\psi|^2 + \xi^2|\nabla\psi|^2 + \frac{b}{-a}|\psi|^4], \tag{38}$$

where I've put

$$\xi = [\frac{1}{-2m^*a}]^{1/2} = [\frac{1}{-2m^*a_0(T_c - T)}]^{1/2}. \tag{39}$$

Clearly the length $\xi$ represents some kind of *stiffness* of the quantitiy $|\psi|^2$, the superfluid density. [Check that it does indeed have dimensions of length!] If $\xi$, the so-called *coherence length*, is small, the energy cost of $n_s$ varying from place to place will be small. If the order parameter is somehow changed from its homogeneous equilibrium value at one point in space by an external force, $\xi$ specifies the length scale over which it "heals". We can then investigate the possibility that, as the kinetic energy of superfluid flow increases with increasing field, if $\xi$ is small enough it might eventually become favorable to "bend" $|\psi|^2$ instead. In typical type I materials, $\xi(T = 0)$ is of order several hundreds or even thousands of Angstrom, but in heavy fermion superconductors, for example, coherence lengths are of order 50-100A. The smallest coherence lengths are attained in the HTSC, where $\xi_{ab}$ is of order 12-15A, whereas $\xi_c$ is only 2-3A.

The general problem of minimizing F when $\psi$ depends on position is extremely difficult. However, we are mainly interested in the phase boundary where $\psi$ is small, so life is a bit simpler. Let's recall our quantum mechanics analogy once more so as to write $F$ in the form:

$$F = \int d^3r[a|\psi|^2 + b|\psi|^4] + < \psi|\hat{H}_{kin}|\psi >, \tag{40}$$

where $\hat{H}_{kin}$ is the operator

$$-\frac{1}{2m^*}(\nabla + \frac{ie^*}{c}\vec{A})^2. \tag{41}$$

Now note 1) sufficiently close to the transition, we may always neglect the 4th-order term, which is much smaller; 2) to minimize $F$, it suffices to minimize $< \hat{H}_{kin} >$, since the $|\psi|^2$ term will simply fix the overall normalization. The variational principle of quantum mechanics states that the minimum value of $< H >$ over all possible $\psi$ is achieved when $\psi$ is the ground state (for a given normalization of $\psi$). So we need only solve the eigenvalue problem

$$\hat{H}_{kin}\psi_j = E_j\psi_j \tag{42}$$

for the lowest eigenvalue, $E_j$, and corresponding eigenfunction $\psi_j$. For the given form of $\hat{H}_{kin}$, this reduces to the classic quantum mechanics problem of a charged particle moving in an applied magnetic field. The applied field $H$ is essentially the same as the microscopic field $B$ since $\psi$ is so small (at the phase boundary only!). I'll remind you of the solution, due to to Landau, in order to fix notation. We choose a convenient gauge,

$$\mathbf{A} = -Hy\hat{x}, \tag{43}$$

in which Eq. 44 becomes

$$\frac{1}{2m^*}[(-i\frac{\partial}{\partial x} + \frac{y}{\ell_M^2})^2 - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}]\psi_j = E_j\psi_j, \tag{44}$$

where $\ell_M = (c/e^*H)^{1/2}$ is the *magnetic length.* Since the coordinates $x$ and $z$ don't appear explicitly, we make the ansatz of a plane wave along those directions:

$$\psi = \eta(y)e^{ik_x x + ik_z z}, \tag{45}$$

yielding

$$\frac{1}{2m^*}[(k_x + \frac{y}{\ell_M^2})^2 - \frac{\partial^2}{\partial y^2} + k_z^2]\eta(y) = E\eta(y). \tag{46}$$

But this you will recognize as just the equation for a one-dimensional harmonic oscillator centered at the point $y = -k_x\ell_M^2$ with an additional additive constant $k_z^2/2m^*$ in the energy. Recall the standard harmonic oscillator equation

$$(-\frac{1}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}kx^2)\Psi = E\Psi, \tag{47}$$

with ground state energy

$$E_0 = \frac{\omega_0}{2} = \frac{1}{2}(k/m)^{1/2}, \tag{48}$$

where k is the spring constant, and ground state wavefunction corresponding to the lowest order Hermite polynomial,

$$\Psi_0 \approx \exp[-(mk/4)^{1/2}x^2]. \tag{49}$$

Let's just take over these results, identifying

$$\hat{H}_{kin}\psi_{k_x,k_z} = \frac{e^*H}{2m^*c}\psi_{k_x,k_z}. \tag{50}$$

The ground state eigenfunction may then be chosen as

$$\psi_{k_x,k_z} = \psi_0(\frac{\pi\ell_M^2}{L_y^2})^{-1/4}e^{ik_xx+ik_zz}\exp[-(y+k_x\ell_M^2)^2/2\ell_M^2)], \tag{51}$$

where $L_y$ is the size of the sample in the $y$ direction ($L_xL_yL_z = V = 1$). The wave functions are normalized such that

$$\int d^3r|\psi_{k_x,k_z}|^2 = \psi_0^2 \tag{52}$$

(since I set the volume of the system to 1). The prefactors are chosen such that $\psi_0^2$ represents the average superfluid denstity. One important aspect of the particle in a field problem seen from the above solution is the large degeneracy of the ground state: the energy is independent of the variable $k_x$, for example, corresponding to many possible orbit centers.

We have now found the wavefunction which minimizes $<\hat{H}_{kin}>$. Substituting back into (42), we find using (52)

$$F = [a_0(T-T_c)+\frac{e^*H}{2m^*c}]\int d^3r|\psi|^2 + b\int d^3r|\psi|^4. \tag{53}$$

When the coefficient of the quadratic term changes sign, we have a transition. The field at which this occurs is called the *upper critical field* $H_{c2}$,

$$H_{c2}(T) = \frac{2m^* c a_0}{e^*}(T_c - T). \tag{54}$$

What is the criterion which distinguishes type-I and type II materials? Start in the normal state for $T < T_c$ as shown in Figure 3, and reduce the field. Eventually one crosses either $H_c$ or $H_{c2}$ first. Whichever is crossed first determines the nature of the instability in finite field, i.e. whether the sample expels all the field, or allows flux (vortex) penetration (see section C).

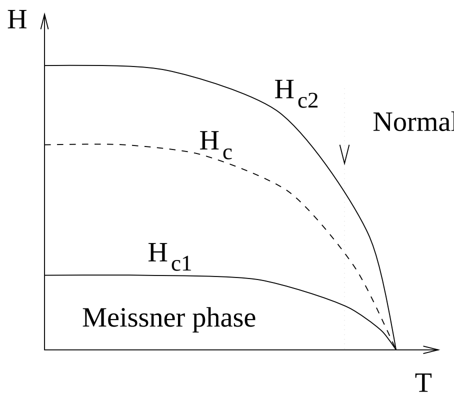


Figure 3: Phase boundaries for classic type II superconductor.

In the figure, I have displayed the situation where $H_{c2}$ is higher, meaning it is encountered first. The criterion for the dividing line between type 1 and type II is simply

$$|\frac{dH_c}{dT}| = |\frac{dH_{c2}}{dT}| \tag{55}$$

or, using the results (38) and (56),

$$\frac{(m^*)^2 c^2 b}{\pi(e^*)^2} = \frac{1}{2}. \tag{56}$$

This criterion is a bit difficult to extract information from in its current form. Let's define the *GL parameter* $\kappa$ to be the ratio of the two fundamental lengths we have identified so far, the penetration depth and the coherence length:

$$\kappa = \frac{\lambda}{\xi}. \tag{57}$$

Recalling that

$$\lambda^2 = -\frac{m^* c^2}{4\pi e^{*2} n_s^*} = -\frac{m^* c^2 b}{2\pi e^{*2} a} \tag{58}$$

and

$$\xi^2 = -\frac{1}{2m^* a}. \tag{59}$$

The criterion (58) now becomes

$$\kappa^2 = \frac{m^* c^2 b/2\pi e^{*2} a}{1/2m^* a} = \frac{(m^*)^2 c^2 b}{\pi e^{*2}} = \frac{1}{2}. \tag{60}$$

Therefore a material is type I (II) if $\kappa$ is less than (greater than) $\frac{1}{\sqrt{2}}$. In type-I superconductors, the coherence length is large compared to the penetration depth, and the system is stiff with respect to changes in the superfluid density. This gives rise to the Meissner effect, where $n_s$ is nearly constant over the screened part of the sample. Type-II systems can't screen out the field close to $H_{c2}$ since their stiffness is too small. The screening is incomplete, and the system must decide what pattern of spatial variation and flux penetration is most favorable from an energetic point of view. The result is the *vortex lattice* first described by Abrikosov. vskip .2cm

## 6.3 Vortex Lattice

I commented above on the huge degeneracy of the wave functions (53). In particular, for fixed $k_z = 0$, there are as many ground states as allowed values of the parameter $k_x$. At $H_{c2}$ it didn't matter, since we could use the eigenvalue alone to determine the phase boundary. Below $H_{c2}$ the fourth order term becomes important, and the minimization of $f$ is no longer an eigenvalue problem. Let's make the plausible assumption that if some spatially varying order parameter structure is going to form below $H_{c2}$, it will be periodic with period $2\pi/q$, i.e. the system *selects* some wave vector $\mathbf{q}$ for energetic reasons. The $x$-dependence of the wave functions

$$\psi_{k_x,k_z=0} = \psi_0(\frac{\pi\ell_M^2}{L_y^2})^{-1/4} e^{ik_x x} \exp[-(y + k_x\ell_M^2)^2/2\ell_M^2)]. \qquad (61)$$

is given through plane wave phases, $e^{ik_x x}$. If we choose $k_x = qn_x$, with $n_x$ =integer, all such functions will be invariant under $x \to x + 2\pi/q$. Not all $n_x$'s are allowed, however: the center of the "orbit", $k_x\ell_M^2$ should be inside the sample:

$$-L_y/2 < k_x\ell_M^2 = q\ell_M^2 n_x < L_y/2, \qquad (62)$$

Thus $n_x$ is restricted such that

$$-\frac{L_y}{2q\ell^2} = -n_{max}/2 < n_x < n_{max}/2 = \frac{L_y}{2q\ell^2} \qquad (63)$$

and the total number of degenerate functions is $L_y/(q\ell_M^2)$.

Clearly we need to build up a periodic spatially varying structure out of wave functions of this type, with "centers" distributed somehow. What is the criterion which determines this structure? All the wave functions (63) minimize $< \hat{H}_{kin} >$, and are normalized to $\int d^3r|\psi|^2 = |\psi_0|^2$. They are all therefore degenerate at the level of the quadratic GL free energy, $F = \int d^3r|\psi|^2 + < \hat{H}_{kin} >$. The fourth order term must stabilize some linear combination of them. We therefore write

$$\psi(r) = \sum_{n_x} C_{n_x}\psi_{n_x}(r), \qquad (64)$$

with the normalization $\Sigma_{n_x} |C_{n_x}|^2 = 1$, which enforces $\int d^3r|\psi(r)|^2 = \psi_0^2$. Note this must minimize $< \hat{H}_{kin} >$. Let's therefore choose the $C_{n_x}$ and $q$ to minimize the remaining terms in $F$, $\int d^3r[a|\psi|^2 + b|\psi|^4]$. Substituting and using the normalization condition and orthogonality of the different $\psi_{k_z,k_x}$, we find

$$f \ = \ \tilde{a}\psi_0^2 + \tilde{b}\psi_0^4. \qquad (65)$$

with

$$\tilde{a}(H,T) = a_0(T-T_c) + \frac{e^*H}{2m^*c} = \frac{e^*}{2m^*c}(H - H_{c2}(T)), \quad (66)$$
$$\tilde{b}(H) = sb, \quad (67)$$

and

$$s = (\frac{\pi\ell_M^2}{L_y^2})^{-1} \sum_{n_{x1},n_{x2},n_{x3},n_{x4}}^{n_{max}} C^*_{n_{x1}} C^*_{n_{x2}} C_{n_{x3}} C_{n_{x4}} \quad (68)$$
$$\int dz \int dx\ e^{iq(-n_{x1}-n_{x2}+n_{x3}+n_{x4})x} \times \quad (69)$$
$$\int dy\ e^{\{-\frac{1}{2\ell_M^2}[(y+qn_{x1}\ell_M^2)^2+(y+qn_{x2}\ell_M^2)^2+(y+qn_{x3}\ell_M^2)^2+(y+qn_{x4}\ell_M^2)^2]\}}. \quad (70)$$

The *form* of $f[\psi_0]$ is now the same as in zero field, so we immediately find that in equilibrium,

$$\psi_0|_{eq} = (\frac{-\tilde{a}}{2\tilde{b}})^{1/2}. \quad (71)$$

and

$$f = \frac{-\tilde{a}^2}{4\tilde{b}}. \quad (72)$$

This expression depends on the variational parameters $C_{n_x}, q$ only through the quantity $s$ appearing in $\tilde{b}$. Thus if we minimize s, we will minimize f (remember $b > 0$, so $f < 0$). The minimization of the complicated expression (xx) with constraint $\Sigma_{n_x} |C_{n_x}|^2 = 1$ is difficult enough that A. Abrikosov made a mistake the first time he did it, and I won't inflict the full solution on you. To get an idea what it might look like, however, let's look at a very symmetric linear combination, one where all the $C_{n_x}$'s are equal:

$$C_n = n_{max}^{-1/2}. \quad (73)$$

Then

$$\psi(r) \sim \sum_n e^{inqx} \exp[-(y+nq\ell^2)^2/2\ell_M^2], \quad (74)$$

which is periodic in x with period $2\pi/q$,

$$\psi(x+2\pi/q, y) = \psi(x,y), \quad (75)$$

and periodic in $q$ with period $q\ell_M^2$, up to a phase factor

$$\psi(x, y + q\ell_M^2) = e^{-iqx}\psi(x, y). \tag{76}$$

Note if $q = \sqrt{2\pi}/\ell_M$, $|\psi|^2$ forms a square lattice! The area of a unit cell is $(2\pi/q) \times (q\ell_M^2) = 2\pi\ell_M^2$, and the flux through each one is therefore

$$\Phi_{cell} = 2\pi\ell_M^2 H = 2\pi\frac{c}{e^* H}H = \frac{hc}{2e} = \Phi_0 \tag{77}$$

where I inserted a factor of $\hbar$ in the last step. We haven't performed the minimization explicitly, but this is a characteristic of the solution, that each cell contains just one flux quantum. The picture is crudely depicted in Fig. 4a). Note by symmetry that the currents must cancel on the

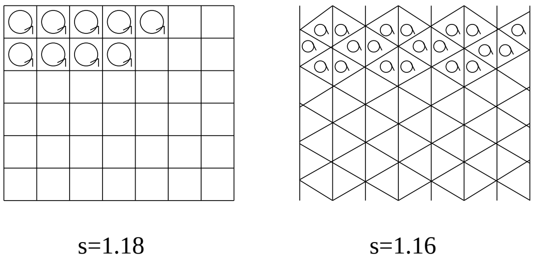


Figure 4: a) Square vortex lattice; b) triangular vortex lattice.

boundaries of the cells. Since $\vec{j_s} = -en_s\vec{v_s}$, integrating $\nabla\phi + \frac{2e}{\hbar c}\vec{A} = 0$ around each square must give, as in our previous discussion of flux quantization in a toroid,

$$\Phi_{cell} = n\Phi_0, n = integer. \tag{78}$$

Somehow the vortex lattice consists of many such rings. The problem with this idea is that the only way $\oint \nabla\phi \cdot d\vec{\ell} = \delta\phi$ around the boundary can be nonzero and the usual argument about single-valuedness of the wave function carried through is if there is a "hole" in the wave function. If there is no hole, or region from which the wave function is excluded, the path can be shrunk to a point, but the value of the integral must remain the same since the integrand is the gradient of a scalar field. This is unphysical because it would imply a finite phase change along an infinitesimal path (and a divergence of the kinetic energy!) The only way out of the paradox is to have the system introduce its own "hole" in itself, i.e. have the amplitude of the order parameter density $|\psi|^2$ go to zero at the center of each cell. Intuitively, the magnetic field will have an accompanying maximum here, since the screening tendency will be minimized. This reduction in order parameter amplitude, magnetic flux bundle, and winding of the phase once by $2\pi$ constitute a magnetic "vortex", which I'll discuss in more detail next time.

Assuming $C_n =$ constant, which leads to the square lattice does give a relatively good (small) value for the dimensionless quantity $s$, which turns out to be 1.18. This was Abrikosov's claim for the absolute minimum of $f[|\psi|^2]$. But his paper contained a (now famous) numerical error, and it turns out that the actual minimum $s = 1.16$ is attained for another set of the $C_n$'s, to wit

$$C_n = n_{max}^{-1/2}, \quad n = even \tag{79}$$
$$C_n = i n_{max}^{-1/2}, \quad n = odd. \tag{80}$$

This turns out to be a *triangular* lattice (Fig. 4b), for which the optimal value of $q$ is found to be

$$q = \frac{3^{1/4}\pi^{1/2}}{\ell}, \tag{81}$$

Again the area of the unit cell is $2\pi\ell^2$, and there is one flux quantum per unit cell.

## 6.4 Properties of Single Vortex. Lower critical field $H_{c1}$

Given that the flux per unit cell is quantized, it is very easy to see that the lattice spacing $d$ is actually of order the coherence length near $H_{c2}$. Using (56) and (41) we have

$$H_{c2} = \frac{c}{e^*}\frac{1}{\xi^2} = \frac{\Phi_0}{2\pi\xi^2}. \tag{82}$$

On the other hand, as $H$ is reduced, $d$ must increase. To see this, note that the area of the triangular lattice unit cell is $A = \sqrt{3}d^2/2$, and that there is one quantum of flux per cell, $A = \Phi_0/H$. Then the lattice constant may be expressed as

$$d = \frac{4\pi}{\sqrt{3}}\xi(\frac{H_{c2}}{H})^{1/2}. \tag{83}$$

Since $\lambda \gg \xi$ is the length scale on which supercurrents and magnetic fields vary, we expect the size of a magnetic vortex to be about $\lambda$. This means at $H_{c2}$ vortices are strongly overlapping, but as the field is lowered, the vortices separate, according to (85), and may eventually be shown to influence each other only weakly. To find the structure of an isolated vortex is a straightforward but tedious exercise in minimizing the GL free energy, and in fact can only be done numerically in full detail. But let's exploit the fact that we are interested primarily in strongly type-II systems, and therefore go back to the London equation we solved to find the penetration depth in the half-space geometry for weak fields, allow $n_s$ to vary spatially, and look for vortex-like solutions. For example, equation (17) may be written

$$-\lambda^2\nabla \times \nabla \times \vec{B} = \vec{B}. \tag{84}$$

Let's integrate this equation over a surface perpendicular to the field $\vec{B} = \vec{B}(x,y)\hat{z}$ spanning one cell of the vortex lattice:

$$-\lambda^2 \int \nabla \times (\nabla \times \vec{B}) \cdot d\vec{S} \;=\; \int \vec{B} \cdot d\vec{S}, \tag{85}$$

$$-\lambda^2 \frac{4\pi}{c} \oint \vec{j}_s \cdot d\vec{\ell} \;=\; \Phi_0. \tag{86}$$

But we have already argued that $\vec{j}_s \cdot d\vec{\ell}$ should be zero on the boundary of a cell, so the left side is zero and there is a contradiction. What is wrong? The equation came from assuming a two-fluid hydrodynamics for the superconductor, with a nonzero $n_s$ everywhere. We derived it, in fact, from BCS theory, but *only for the case where $n_s$ was constant.* Clearly there must be another term in the equation when a vortex-type solution is present, one which can only contribute over the region where the superfluid density is strongly varying in space, i.e. the coherence length-sized region in the middle of the vortex where the order parameter goes to zero (vortex "core"). Let's simply add a term which enables us to get the right amount of flux to Eq. (86). In general we should probably assume something like

$$\lambda^2 \nabla \times \nabla \times \vec{B} + \vec{B} = \Phi_0 g(\vec{r})\hat{z} \tag{87}$$

where $g(r)$ is some function which is only nonzero in the core. The flux will then come out right if we demand $\int d^3r g(\vec{r}) = 1$. But let's simplify things even further, by using the fact that $\xi \ll \lambda$: let's treat the core as having negligible size, which means it is just a *line singularity.* We therefore put $g(\vec{r}) = \delta(\vec{r})$. Then the modified London equation with line singularity acting as an inhomogeneous "source term" reads

$$\begin{aligned} -\lambda^2 \nabla^2 \vec{B} + \vec{B} &= \Phi_0 \delta^2(\vec{r})\hat{z} && (88) \\ -\lambda^2 \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \frac{\partial B_z}{\partial \rho}) + B_z &= \Phi_0 \delta^2(\vec{r}), && (89) \end{aligned}$$

where $\rho$ is the radial cylindrical coordinate. Equation (91) has the form of a modified Bessel's equation with solution:

$$B_z = \frac{\Phi_0}{2\pi\lambda^2} K_0(\frac{\rho}{\lambda}). \tag{90}$$

The other components of $\vec{B}$ vanish. If you go to Abramowitz & Stegun you can look up the asymptotic limits:

$$\begin{aligned} B_z &= \frac{\Phi_0}{2\pi\lambda^2}[\log(\frac{\lambda}{\rho}) + 0.116], \quad \xi < \rho \ll \lambda && (91) \\ B_z &= \frac{\Phi_0}{2\pi\lambda^2}\sqrt{\frac{\pi\lambda}{2\rho}} e^{-\rho/\lambda}, \quad \rho \gg \lambda. && (92) \end{aligned}$$

Note the form (93) is actually the correct asymptotic solution to (91) all the way down to $\rho = 0$, but the fact that the solution diverges logarithmically merely reflects the fact that we didn't minimize the free energy properly, and allow the order parameter to relax as it liked within the core. So the domain of validity of the solution is only down to roughly the core size, $\rho \simeq \xi$, as stated. In Figure 5 I show schematically the structure of the magnetic and order parameter profiles in an isolated vortex. The

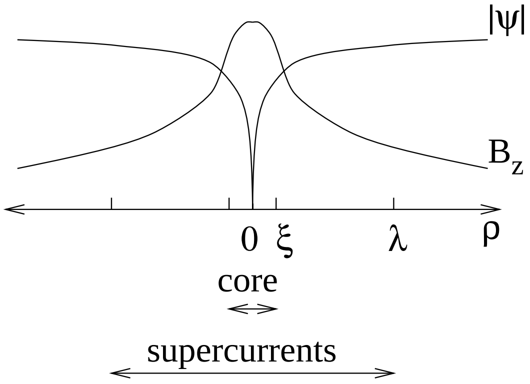


Figure 5: Isolated vortex

solution may now be inserted into the free energy and the spatial integrals performed, with some interesting results:

$$F = L_z \frac{\Phi_0^2}{16\pi^2\lambda^2} \log(\kappa). \tag{93}$$

It is easy to get an intuitive feel for what this means, since if we assume the field is uniform and just ask what is the magnetic energy, we get roughly

$$\begin{aligned} F_v &= \frac{1}{8\pi} \times \text{vortex volume} \times B^2 & (94)\\ &\simeq \frac{1}{8\pi} \times (\pi\lambda^2 L_z) \times (\Phi_0/\pi\lambda^2)^2 & (95)\\ &= L_z \frac{\Phi_0^2}{8\pi^2\lambda^2}, & (96) \end{aligned}$$

the same result up to a slowly varying factor. Now the *lower critical field* $H_{c1}$ is determined by requiring the Gibbs free energies of the Meissner phase with no vortex be equal to the Gibbs free energy of the phase with a vortex.[9] $G$ differs from $F$ through a term $-\int BH/4\pi$. In the Meissner

[9] We haven't talked about the Gibbs vs. Helmholtz free energy, but recall the Gibbs is the appropriate potential to use when the external field $H$ is held fixed, which is the situation we always have, with a generator supplying work to maintain $H$.

state $G = F$, so we may put

$$F = F + E_{line}L_z - \frac{1}{4\pi}H_{c1}\int Bd^3r \tag{97}$$
$$= F + E_{line}L_z - \frac{1}{4\pi}\Phi_0 L_z, \tag{98}$$

where $E_{line}$ is the free energy per unit length of the vortex itself. Therefore

$$H_{c1} = \frac{4\pi E_{line}}{\Phi_0} \tag{99}$$

is the upper critical field. But the line energy is given precisely by Eq. (95), $E_{line} = \frac{\Phi_0^2}{16\pi^2\lambda^2}\log(\kappa)$, so

$$H_{c1}(T) = \frac{\Phi_0}{4\pi\lambda^2}\log(\kappa). \tag{100}$$

### 6.5 Anisotropic GL theory and layered superconductors.

The GL theory is expected to be quantitatively accurate until extremely close to $T_c$ in the case of a 3D superconductor with long coherence length. When the material in question is layered, as in the case of the HTSC, a modification of the GL theory is necessary even to describe physics outside the critical regime, which may still be extremely small (although probably of order 10K in the optimally doped YBCO material). The basic physics can be understood by considering a stack of weakly coupled 2D superconducting layers. In the limit of zero coupling, the transition temperature corresponding to off-diagonal long-range order goes to zero, so superconductivity at finite T occurs only as a result of the weak interlayer interaction.[10] There will be a coherence length $\xi_{ab}$ describing order parameter variations within individual planes, and another one $\xi_c$ describing the stiffness with respect to changing $|\psi_n|$ with respect to $|\psi_{n+1}|$, where $n$ is a layer index.

[10]Here I take a rather strict definition of superconductivity, since off-diagonal *algebraic* order ("Kosterlitz-Thouless-Beresinskii transition") may occur at around the mean-field $T_c$

The idea of Lawrence and Doniach [W.E. Lawrence and S. Doniach, Proc. 12th Int. Conf. Low Temp. Phys. (Kyoto 1970).] was that two different regimes of physical behavior might be obtained according to whether $\xi_c(T)$ was greater than or less than the layer spacing $s$. Since $\xi_c(T)$ diverges at $T_c$ if it is really a superconducting coherence length, there will always be a range of temperatures sufficiently close to the transition where $\xi_c > s$. The behavior of the system in this region should be well described by an anisotropic $(\xi_{ab}, \xi_c)$ version of the GL theory. But as we go down in temperature it is possible that $\xi_c(T)$ becomes comparable to $s$ at some $T^*$, below which the system must behave differently, more like a set of Josephson-coupled 2D layers. The free energy which interpolates between these two limits is

$$F = \sum_n \int d^2r \{a|\psi_n|^2 + b|\psi_n|^4 + \frac{1}{2m_{ab}}|\nabla_{ab}\psi_n|^2 + \frac{1}{2m_c s^2}|\psi_n - \psi_{n-1}|^2\}. \tag{101}$$

The two masses are *phenomenological* parameters in a GL theory, not necessarily identical with the band masses, for example, although we expect them to be similar. If we want to include an external field, we replace $\nabla_{ab} \to \nabla_{ab} - (e^*/c)\mathbf{A}$ ($\mathbf{A}$ is in the plane) and $\psi_n \to \psi_n \exp ie^*(ns)A_z/c$.[11]

**3D Regime: anisotropic GL theory.** Clearly for $s \to 0$, the Lawrence-Doniach free energy will reduce to an anisotropic GL theory with masses $m_{ab}$ and $m_c$. In this limit we have an anisotropic coherence length and penetration depth.

$$\xi_i^2 = -\frac{1}{2m_i a}, \tag{102}$$

$$\lambda_i^2 = -\frac{m_i c^2 b}{2\pi e^2 a}, \tag{103}$$

where $i = ab, c$. Note the anisotropy in the penetration depth as defined is opposite that of the coherence length: in the HTSC $\xi_c \ll \xi_{ab}$ but $\lambda_c \gg \lambda_{ab}$ (recall $\lambda_i$ is the penetration length for magnetic field oriented such that

[11] This is the discrete analog of the continuum gauge-invariant minimal coupling prescription for the electromagnetic field. See, e.g., xx

supercurrents flow in the $i$ direction; the field itself is perpendicular to this direction). Also note the critical fields scale with these same lengths, $H_{c2} \sim 1/\xi^2$ and $H_{c1} \sim 1/\lambda^2$. But again recall, the mass which enters is for currents flowng in the perpendicular direction, e.g. $H_{c2\|c} = \Phi_0/2\pi\xi_{ab}^2$. Since for fields in the plane both masses enter, the relevant length scale is actually the geometric mean, e.g. $H_{c2\|ab} = \Phi_0/2\pi\xi_{ab}\xi_c$. It is it is useful to remember all of these scaling relations:

$$\gamma \equiv \frac{m_c}{m_{ab}}^{1/2} = \frac{\lambda_c}{\lambda_{ab}} = (\frac{\xi_{ab}}{\xi_c}) = \frac{H_{c2\|ab}}{H_{c2\|c}} = \frac{H_{c1\|c}}{H_{c1\|ab}}. \tag{104}$$

If, for example, we have a perfectly straight vortex line, it will look quite different depending on the direction of the field, as shown in Figure 6. One can of course place the field at an arbitrary angle with respect to

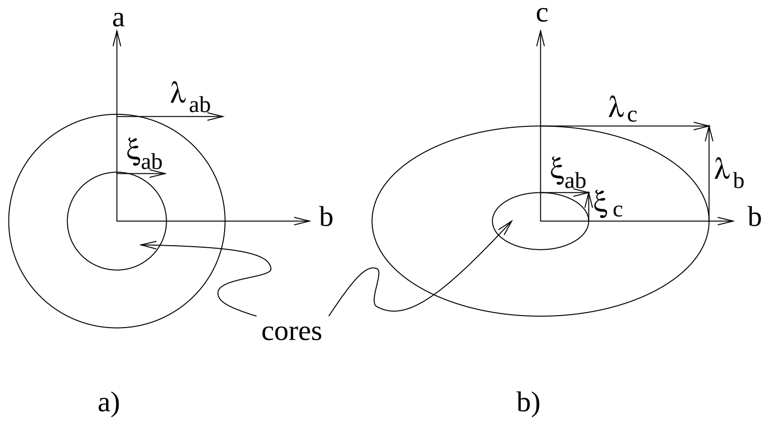


Figure 6: a) vortex for field along c axis; b) along a axis

the coordinate axes, in which case the problem becomes quite interesting. The upper critical field is a relatively simple calculation, however, since one doesn't care about individual vortices: one finds (anisotropic harmonic oscillator problem!):

$$H_{c2}(\theta) = \frac{H_{c2\|ab}}{(\cos^2\theta + \gamma^2 \sin^2\theta)^{1/2}} \tag{105}$$

Before leaving the subject of the anisotropic GL regime, I note that the HTSC materials themselves have quite different values of the anisotropy paramter $\gamma$. For optimally doped YBCO, $\gamma$ is about 7, for example, while it gets as large as $\sim 150$ for BSCCO.

**2D regime and crossover.** The simplest version of the LD theory gives a crossover to 2D behaior at $T^*$ when $\xi_c(T^*) = s/\sqrt(2)$. At lower temperatures, the vortex cores lie between the layers, and the parallel upper critical field is formally infinite (recall the layers are infinitesimally thin!). We can do a simple calculation to see how this arises. Take the field to lie along the y axis, and imagine first a single film of thickness $d \ll \lambda,\ \xi_c(0)$. Since this is a pure 2D system, no complete vortex lattice is possible, but the film does have a parallel critical field $H_{c,\|}$. One may make a crude estimate by noting that $H_{c2} \sim \Phi_0/\xi^2$ is determined by the requirement that one flux quantum go through an area $\xi^2$, one coherence length on a side. If $d < \xi$, this area is limited to $d\xi$, so $H_{c,\|}$ should be roughly $\Phi_0/d\xi$. This means its temperature dependence near the transition is different from that of the bulk critical fields:

$$\begin{aligned} H_{c\|} &\simeq \Phi_0/d\xi && (106)\\ &\sim (1-T/T_c)(1-T/T_c)^{-1/2} \sim (1-T/T_c)^{1/2}. && (107) \end{aligned}$$

So the critical field of a thin (2D) film varies as the *square root* of the reduced temperature $t = (1-T/T_c)$ at the transition, *not* linearly with $t$.

Now let's return to the LD model, which is supposed to provide a crude understanding of the $3D \to 2D$ crossover. For $H \parallel \hat{y}$, let's choose the gauge $A_z = Hx,\ \mathbf{A} = 0$. The GL differential equation corresponding to (103) in this field is then

$$0 = a\psi_n + 2b|\psi_n|^2\psi_n - \frac{1}{2m_{ab}}\nabla^2\psi_n - \frac{1}{2m_c s^2}(\psi_{n+1}e^{-ie*Hxs/c} - 2\psi_n + \psi_{n-1}e^{ie^*Hxs/c}). \tag{108}$$

For this gauge choice, we expect that the lowest energy solution to the linearized problem (drop $\psi_n^3$ ) will be independent of both y and z (i.e., n), so we drop the layer index to find

$$-\frac{\partial^2}{\partial x^2}\psi + \frac{2m_{ab}}{m_c s^2}[1 - \cos\frac{2\pi H s x}{\Phi_0}]\psi = \frac{1}{\psi_{ab}^2}\psi. \tag{109}$$

Note now that if $H \to \infty$, the cos term oscillates so rapidly it will not contribute to any integral of (xx). If we neglect this term, we see that a spatially uniform solution is possible at infinite field if $2m_{ab}/(m_c s^2) = 1/\xi_{ab}^2$, or

$$\xi_c = (\frac{m_{ab}}{m_c})^{1/2}\xi_{ab} = \frac{s}{\sqrt{2}}. \tag{110}$$

So we have an acceptable solution at $T^*$ defined by $\xi_c(T^*) = s/\sqrt{2}$ only if $H = \infty$! In reality what we expect is something like Fig. 7a), where the parallel critical field tries to diverge, at $T^*$, but remembers that it can't because of finite layer thickness, and coupling of the field to the spin degrees of freedom, both of which LD neglected. The full range of behavior, from fully 3D to fully 2D, can be explored by carefully building *superconducting heterostructures* where the layer separation and thickness can be controlled independently. Fig. 7b. shows data and LD fits for an $Nb/Ge$ heterostructure. Note in particular the $H_{c\|} \sim (1 - T/T_c)^{1/2}$ behavior in the true 2D case, and the upturn in the "crossover" case.