PHZ7427–Solid State II

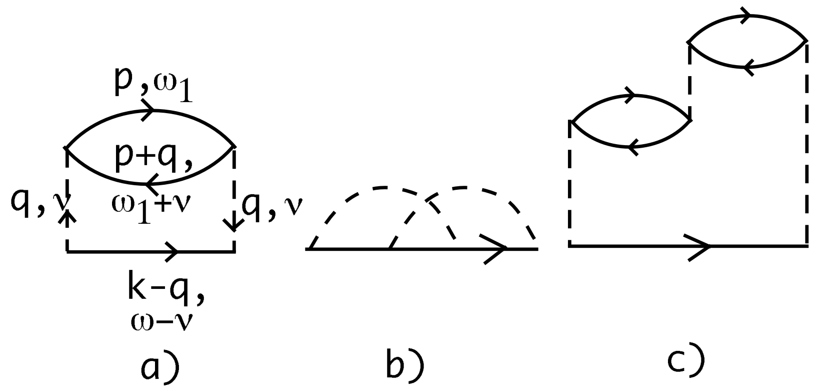
Spring 1999

Test 1

Feb. 22, 1999

Due: March 5, 1999
Take-home (may work with other students in class)

1. **Feynman diagrams**. Write down the contribution of the diagrams a), b), and c) shown to the self-energy $\Sigma(\mathbf{k}, \omega_n)$ of an interacting Fermi system, in terms of the bare propagator $G(\mathbf{k}, \omega_n)$. The dashed lines are Coulomb interactions $V(q) = \frac{4\pi e^2}{q^2}$. You need *not* evaluate momentum and frequency sums.



2. **Spin response functions.** Use the spin-density operator $\hat{\sigma}_z(\mathbf{r}) \equiv \hat{\psi}^\dagger_\sigma(\mathbf{r})\,(\sigma_z)_{\sigma\sigma'}\,\hat{\psi}_{\sigma'}(\mathbf{r})$ to construct the retarded spin-spin correlation function χ^R for spin-1/2 particles at real times, and the analogous time-ordered function χ for imaginary times:

$$i\chi^R(1,1') = \mathrm{Tr}\left\{\hat{\rho}_0\left[\hat{\sigma}_z(1), \hat{\sigma}_z(1')\right]\right\}\theta(t-t'), \qquad (1)$$
$$\chi(1,1') = -\mathrm{Tr}\left\{\hat{\rho}_0 T_\tau \hat{\sigma}_z(1)\,\hat{\sigma}_z(1')\right\} \qquad (2)$$

Show explicitly that the Fourier transforms of the two are related by a spectral weight function $\rho(\mathbf{q}, \omega)$:

$$i\chi^R(\mathbf{q},\nu) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\,\frac{\rho(\mathbf{q},\omega')}{\nu-\omega'+i0^+}, \qquad (3)$$

$$\chi(\mathbf{q},\nu_n) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\,\frac{\rho(\mathbf{q},\omega')}{i\nu_n-\omega'} \qquad (4)$$

3. **Spin susceptibility of a Fermi gas.** Using the function χ in problem 2, study the linear response of a uniform spin-1/2 Fermi system to a weak external magnetic field $\mathrm{H}\hat{z}$, with coupling

$$H' = -\mu_0 \int d^3r\;\hat{\sigma}_z(\mathbf{r})\,\mathrm{H}(\mathbf{r},t) \qquad (5)$$

(a) Show that the expectation value of the induced magnetization $\hat{M} = \mu_0 \hat{\sigma}_z$ is given by

$$\langle \hat{M}_z (\mathbf{q}, \nu) \rangle = \mu_0^2 \chi^R (\mathbf{q}, \nu) \, \mathrm{H} (\mathbf{q}, \nu) \quad . \tag{6}$$

(b) Use Wick's theorem to evaluate $\chi(\mathbf{q}, \nu_n)$ for a noninteracting Fermi system, and analytically continue to find $\chi^R(\mathbf{q}, \nu)$, using the results of problem 3. Show that

$$\chi_0^R (\mathbf{q}, \nu; T) = \sum_{\mathbf{k}} \frac{f (\varepsilon_{\mathbf{k}}) - f (\varepsilon_{\mathbf{k+q}})}{\varepsilon_{\mathbf{k+q}} - \varepsilon_{\mathbf{k}} - \nu - i0^+} \quad , \tag{7}$$

the *Lindhard function.*

(c) Find the static nonuniform magnetic susceptibility of the noninteracting system $\chi_0^R (\mathbf{q}, 0; T) \equiv \langle \hat{M}_z (\mathbf{q}, 0) \rangle / \mathrm{H} (\mathbf{q}, 0) = \mu_0^2 \chi_0^R (\mathbf{q}, 0; T)$, and verify that

$$\lim_{q \to 0} \chi_0^R (\mathbf{q}, 0; T) \simeq \begin{cases} \mu_0^2 N (\epsilon_F) & T = 0 \quad \text{(Pauli paramagnetism)} \\ \dfrac{\mu_0^2 n}{T} & T \to \infty \quad \text{(Curie law)} \end{cases} \tag{8}$$

where n is the density.

(d) Find an exact expression for the imaginary part of the dynamical spin susceptibility $\mathrm{Im}\chi^R(\mathbf{q}, \nu)$ at $T = 0$ and sketch it for some fixed $0 < q < k_F$.

4. **Fermi liquid quasiparticle relaxation time.** Evaluate the contribution of the diagram in Problem 1a) to the imaginary part of the retarded self energy, Im $\Sigma^R(\mathbf{k}, \omega)$ for $|\mathbf{k}| = k_F$. Note this is the leading order contribution to the quasiparticle relaxation time in FL theory, and corresponds roughly to the Golden rule calculation done in class. Hint: note further that the "bubble" connected to the single propagator line in the diagram is exactly the susceptibility you found in Eq. (7); use this result to simplify your calculation. You should be able to show that for $\omega \ll \varepsilon_F$, your result is proportional to $\omega^2 + (\pi T)^2$. In fact, this is the only thing I'm interested in; don't try to obtain an exact result valid for all ω, T.