A Journal club presentation of Scaling and river networks: A Landau theory for erosion

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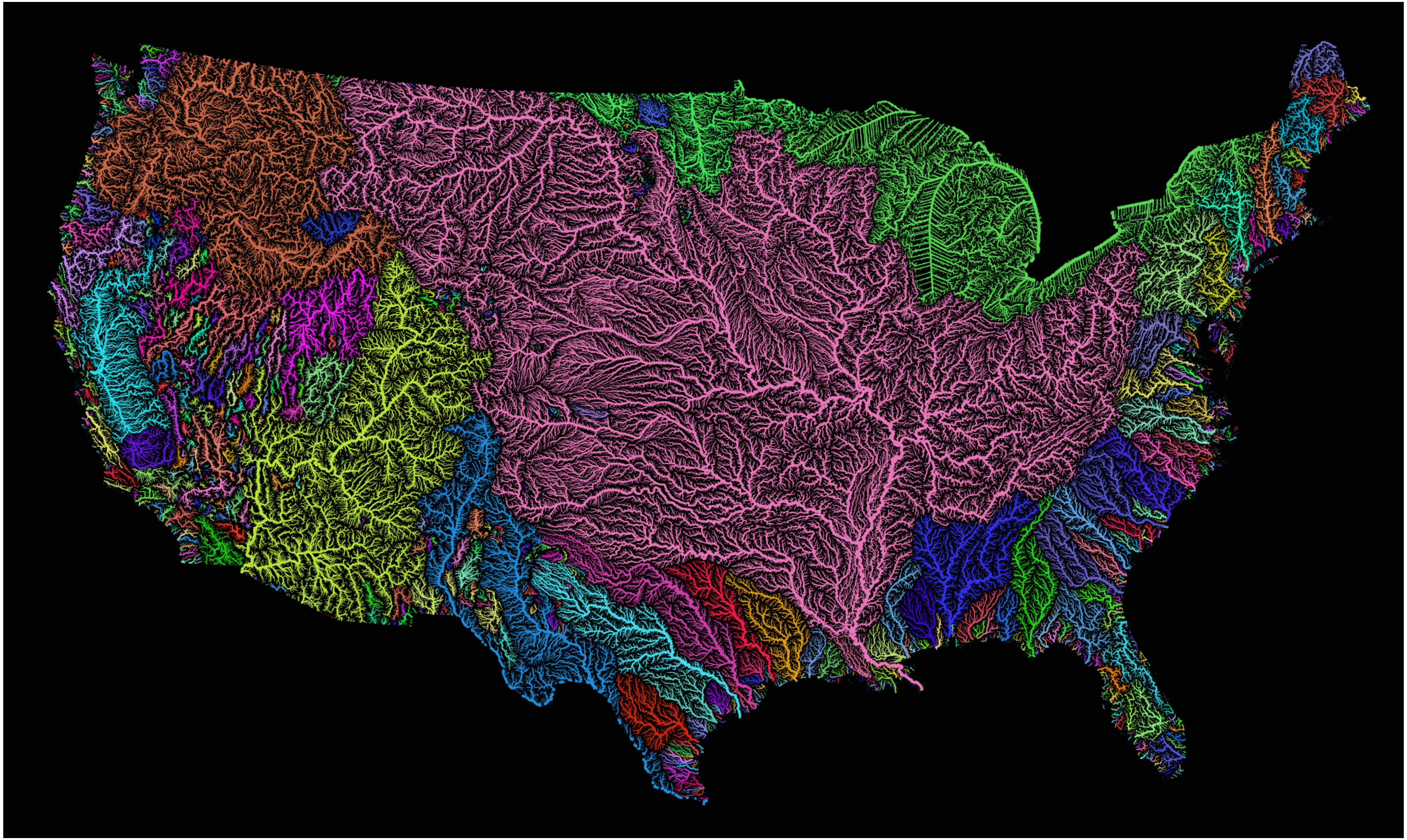
Image from Doñana national park in spain https://protectedareawatch.org/europe/donana-spain/



Source: https://www.etsy.com/shop/GrasshopperGeography

What is the goal of this model

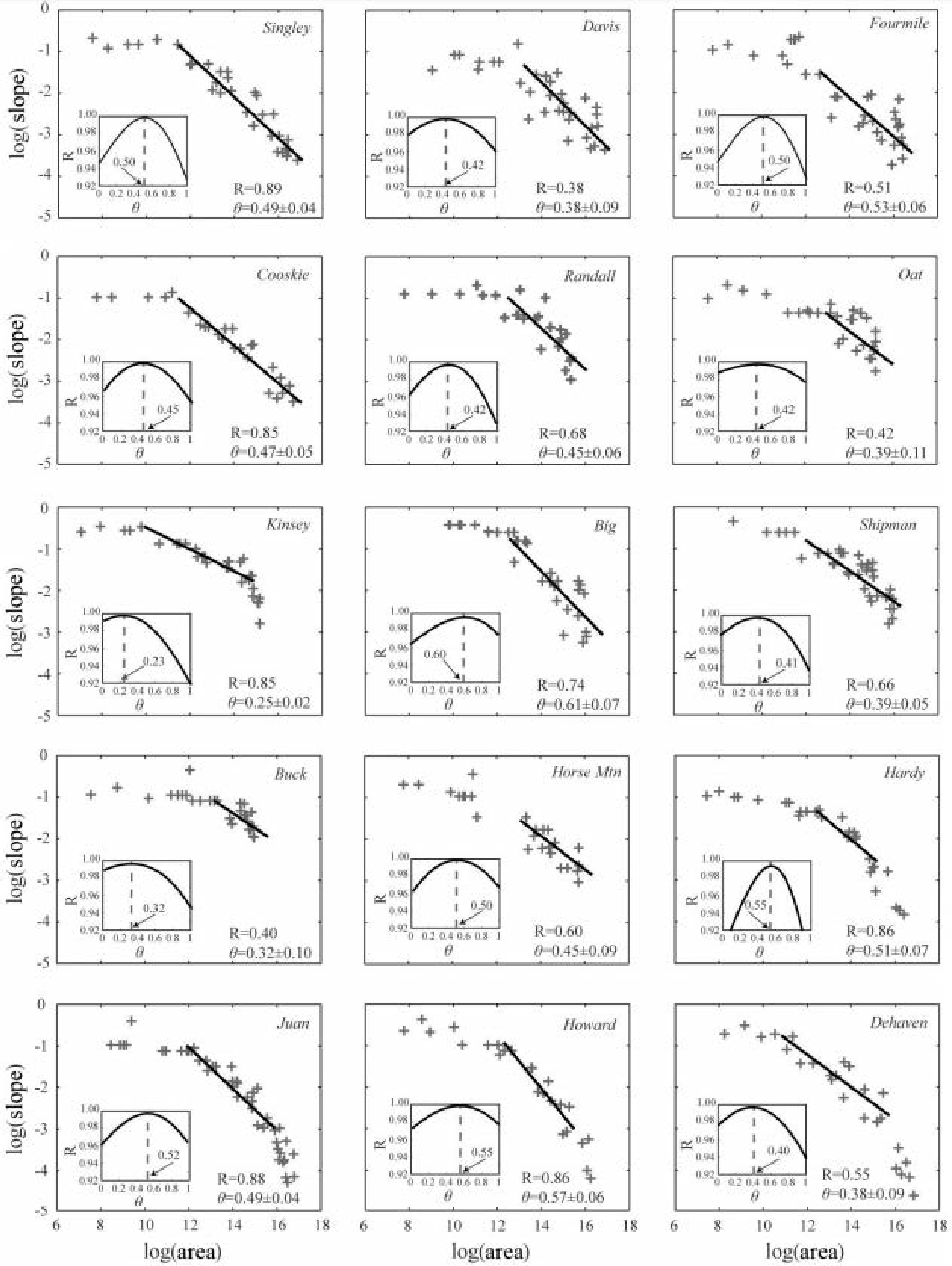
- How do rivers form?
  - Water cycle and erosion
- Can these fractal patterns and scaling laws be obtained from the laws of physics?

Universality in river networks

- Slope area law

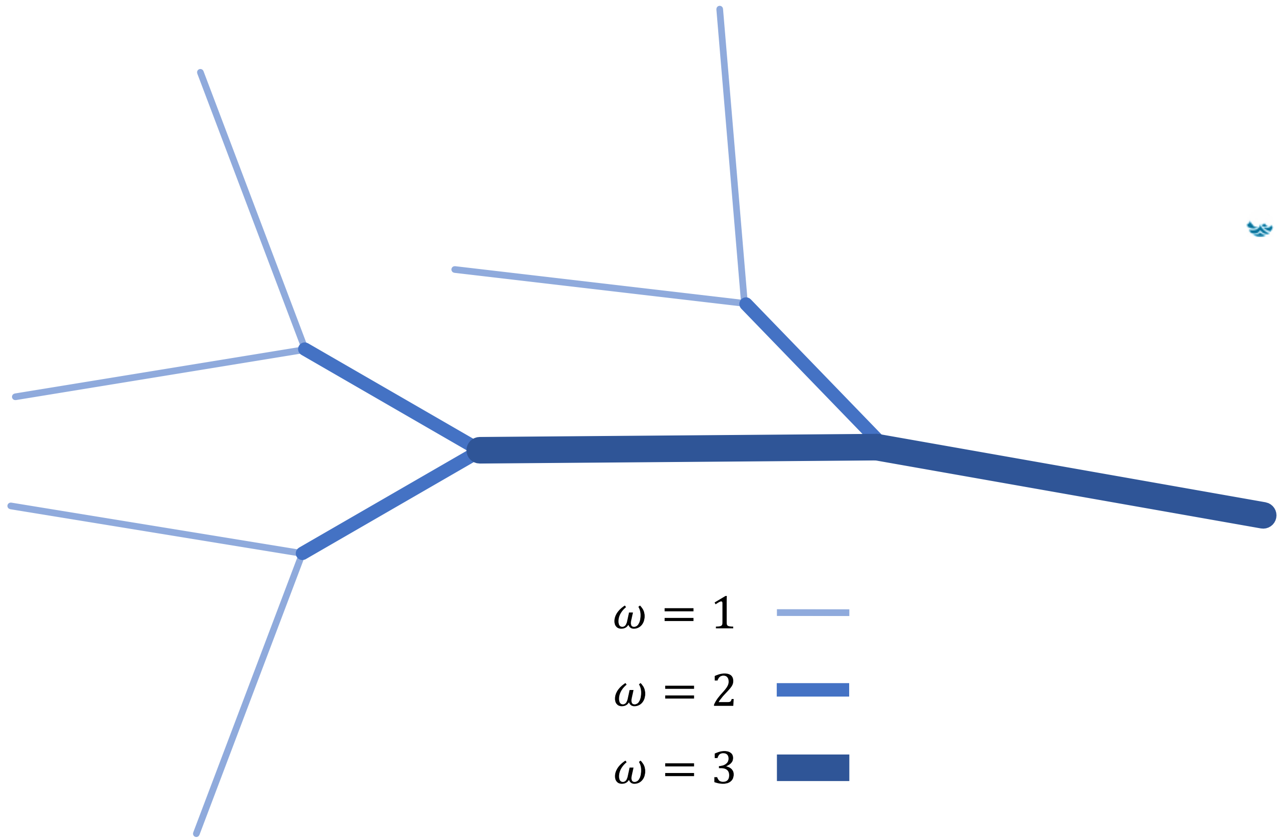
  $s = k_s\, \mathrm{Q}^{-\theta}$

  $\theta \approx 0.5$



Slope area data from various rivers. θ=0.45 on average
Taken by Wang, Y., Zhang, H., Zheng, D., Yu, J., Pang, J., and Ma, Y.

Horton's laws



- Horton's laws

$$R_b(\omega) = \frac{N_\omega}{N_{\omega+1}} \approx 4$$

$$R_L(\omega) = \frac{L_\omega}{L_{\omega+1}} \approx 2$$

- Fractal dimension

$$D = d_c \frac{\ln(R_b)}{\ln(R_L)}$$

$$d_c \approx 1.1 \; to \; 1.2$$

The Model

- Assumptions
  - Rainfall is uniform (no underground flow)
  - Eroded material is not redeposited
  - Water flows fast, soil flows slowly
  - Initial landscape is without features
- Langevin equation

$$\dot{\mathrm{h}} = \mathrm{F}\left(\mathrm{h}, \mathrm{h}^2, \dots, \nabla h, \nabla^2 h, \|\nabla h\|^2, \dots\right) + \eta(\boldsymbol{r}, t)$$

Simplifying.

- The change in the heights shouldn't depend on the height directly

$$\dot{\mathrm{h}} = \mathrm{F}\left(\nabla h, \nabla^2 h, \|\nabla h\|^2, \dots \right) + \eta(\boldsymbol{r}, t)$$

$$\mathrm{F} = A + \boldsymbol{B} \cdot \nabla h + C * \|\nabla h\|^2 + D * \nabla^2 h + \cdots$$

- A Represents a uniform motion of the entire system $(\mathrm{A} = 0)$
- $\boldsymbol{B}$ Represents a preferred direction of soil flow $(\mathbf{B} = 0)$

$$\dot{\mathrm{h}} = C * \|\nabla h\|^2 + D * \nabla^2 h + \eta(\boldsymbol{r}, t)$$

  - Kardar-Parisi-Zhang (KPZ) Equation
- C Is the erosion coefficient
- D Is a smoothing term

Just add water

- Uniform rainfall

  $\nabla \cdot \boldsymbol{q} = R$

- Water flows down hill (Dynamical system approximation)

  $\boldsymbol{q} = \nabla h * \sigma$

- No water means no erosion

  $C = -c * \|\boldsymbol{q}\|$

  $\dot{\mathrm{h}} = -c * \|\boldsymbol{q}\| \, \|\nabla h\|^2 + D * \nabla^2 h + \eta(\boldsymbol{r}, t)$

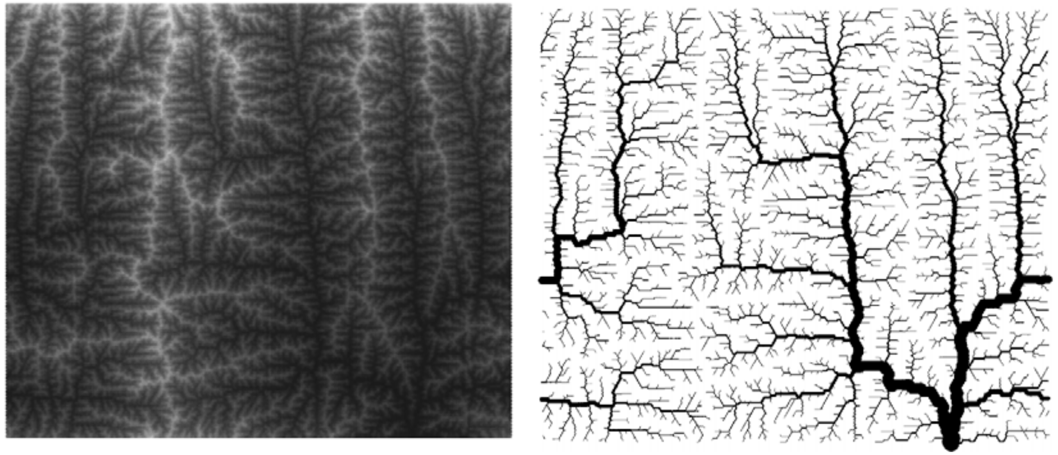
- A final state implies uniform erosion.

  $\mathrm{k} = -c * \|\boldsymbol{q}\| \, \|\nabla h\|^2 \; => \|\nabla h\| \propto \|\boldsymbol{q}\|^{-1/2}$

Simulation

- 256x256 Triangular lattice
- Water flows along edges between lattice points
- Initial landscape is a downward sloping hill with some noise
  - No local minima in the initial lattice
- Updates occur according to

$$\Delta h = -\|\nabla h\|^2 \|q\| \, \Delta t$$



Sample data from a single simulation, the left is a height map and right is the water flow

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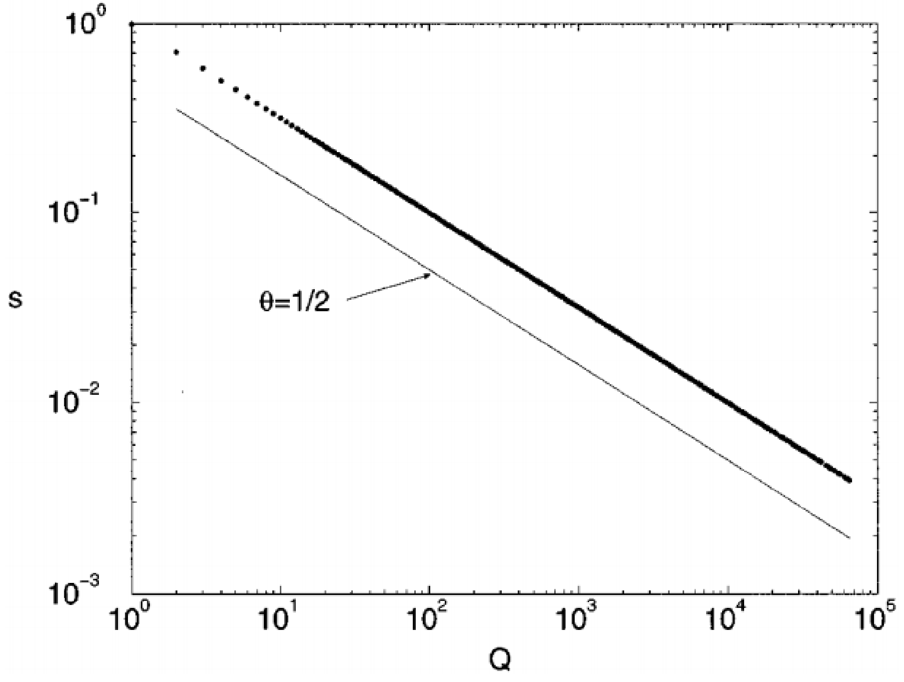
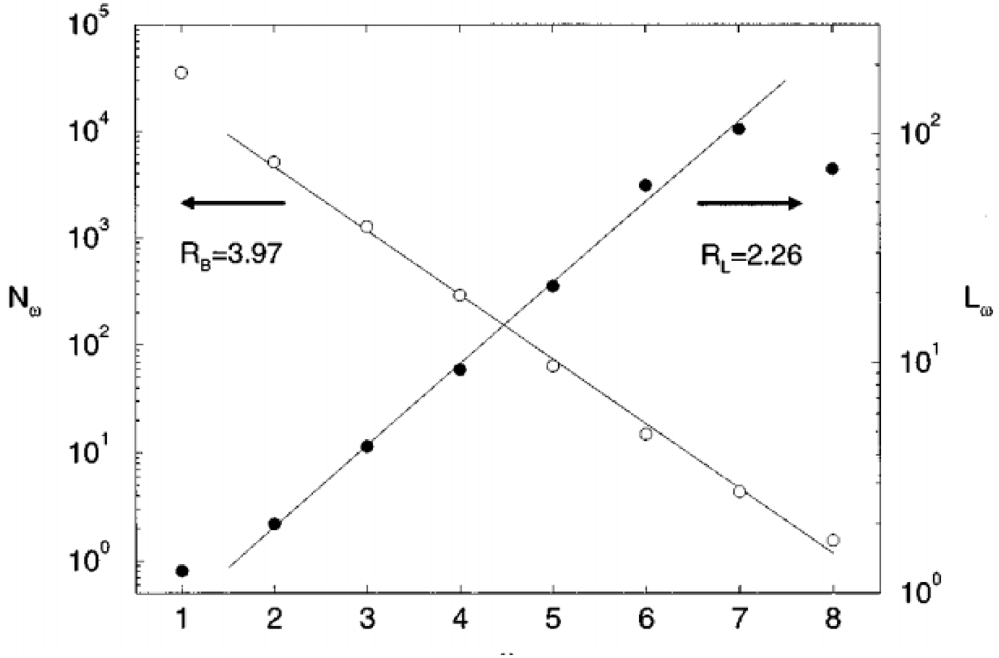
The results





Results from 20 independent simulations, The right suggests that θ = ½ is an attractive solution. The left shows that the simulation results are similar to that of real rivers With respect to Horton's laws.
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Citations

- Somfai, E. & Sander, L. M. (1997). Scaling and river networks: A Landau theory for erosion. *Phys. Rev. E, 56*, R5-R8.
- Wang, Y., Zhang, H., Zheng, D., Yu, J., Pang, J., and Ma, Y.: Coupling slope–area analysis, integral approach and statistic tests to steady-state bedrock river profile analysis, Earth Surf. Dynam., 5, 145-160, https://doi.org/10.5194/esurf-5-145-2017, 2017.