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WILSON THEORY FOR 2-DIMENSIONAL ISING SPIN SYSTEMS

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Synopsis

A transformation is constructed for a triangular lattice in the space of Ising-spin hamiltonians with short-range forces. Evidence is presented that this transformation exhibits a fixed point corresponding to a hamiltonian with dominant nearest-neighbour interaction. In various approximations the location of the fixed point, the plane of criticality and the eigenvalue structure of the linearized transformation at the fixed point are calculated and found in reasonable agreement with known exact data for the 2-dimensional triangular Ising system.

1. *Introduction.* Wilson's introduction[1]) of the concept of renormalization in the discussion of critical phenomena has found most of its applications to systems with continuous (classical) spin variables[2,3]). In these theories the dimension $d = 4$ plays a crucial role and lower dimensions are treated through an expansion in $\varepsilon = 4 - d$ [2,4]). In this respect the 2-dimensional case is the most difficult one.

According to universality the critical singularities for classical spins are the same as for spin systems in which the variables can only assume discrete values as in the Ising model. By writing the sum over spin states as an integral over continuous variables, Hubbard[5]) has constructed support for this similarity.

This paper is concerned with a direct application of Wilson's theory to a 2-dimensional spin-$\frac{1}{2}$ system which is in our opinion simpler and which yields more relevant information than the procedure followed usually. It differs in two aspects. Firstly the introduction of continuous spins is avoided. The renormalization equations concern a transformation of a discrete spin system to an another discrete spin system on an isomorphic lattice in a space of the same dimensionality. Secondly, by sticking to the same dimension, the procedure bypasses the so-called ε-expansion. The idea underlying Wilson's theory is a realization of Kadanoff's derivation of the scaling laws by a subdivision of the spin system into cells, which supposedly interact in a similar way as the original spins.

In Kadanoff's picture one has to be close to the critical point in order that the cells can be made such that they contain many sites but that the linear dimensions are small with respect to the correlation length. In Wilson's theory there is no necessity for large cells, but the price to be paid is that one has to work with much more complicated hamiltonians than the nearest-neighbour Ising interaction.

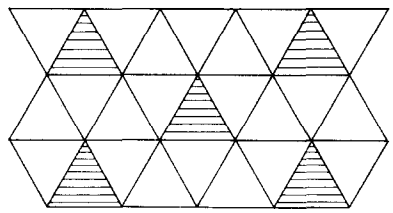

Fig. 1. Triangular lattice with cells shaded.

In this paper we study a 2-dimensional triangular lattice, being the most convenient lattice for our purpose. Triangular cells are drawn on the lattice as indicated in fig. 1. The cell system forms again a triangular lattice. With the cells we associate spins which can again assume only two values and which are determined by the site spins inside the cell. Then the partition function of the site spins is computed under the constraint of a given configuration of cell spins. The resulting free energy depends on the cell spin configuration and may be considered as a hamiltonian for the cell spins. Thus a transformation is achieved from the site-spin hamiltonian to a cell spin hamiltonian.

Generally we expect the following picture to result. Consider the space of hamiltonians with basically short-range forces (in which the usual factor $-\beta = -1/k_B T$ is absorbed). It can be divided in subcritical, critical and supercritical systems. The transformation from site hamiltonians to cell hamiltonians will map each group onto itself.

In the surface of critical systems the mapping will be contracting such that it exhibits a fixed point. The transformation properties around the fixed point determine the singularities of the free energy accross the surface of criticality.

Of course an exact evaluation of this program is not feasible but the approximate realizations of this transformation bear out this picture.

In section 2 we define the transformation and the notation to be used. The fixed-point properties are discussed in section 3 in general terms and in section 4 we treat the transformation on the basis of a simple perturbation theory. In section 5 a cluster-type approximation is developed of which a number of approximations is discussed in section 6. The paper closes with a review of the results and an outline of different possibilities on different lattices.

2. *Definition of the transformation.* In the lattice as shown in fig. 1 we shall denote the sites with the index i and the cells with index i'. Generally we give the quantities referring to the cell system a prime.

Thus the site spin variables are denoted by $s_{i'}$ and the cell spins by $s'_{i'}$. Inside the cell i' we may label the site spins $s^1_{i'}$, $s^2_{i'}$, $s^3_{i'}$ as indicated in fig. 2. Then the cell spin is defined as:

$$s'_{i'} = \text{sign}\,(s^1_{i'} + s^2_{i'} + s^3_{i'}). \tag{2.1}$$

We note that for the uniqueness of $s'_{i'}$ the cells must contain an *odd* number of sites. For a given $s'_{i'}$ the site spins $s^1_{i'}$, $s^2_{i'}$, $s^3_{i'}$ can still assume 4 configurations to be labelled as:

$$\begin{aligned}
&\sigma_{i'} = 0: \quad s^1_{i'} = s'_{i'}, \quad s^2_{i'} = s'_{i'}, \quad s^3_{i'} = s'_{i'};\\
&\sigma_{i'} = 1: \quad s^1_{i'} = -s'_{i'}, \quad s^2_{i'} = s'_{i'}, \quad s^3_{i'} = s'_{i'};\\
&\sigma_{i'} = 2: \quad s^1_{i'} = s'_{i'}, \quad s^2_{i'} = -s'_{i'}, \quad s^3_{i'} = s'_{i'};\\
&\sigma_{i'} = 3: \quad s^1_{i'} = s'_{i'}, \quad s^2_{i'} = s'_{i'}, \quad s^3_{i'} = -s'_{i'}.
\end{aligned} \tag{2.2}$$

The spin configurations of the sites can be either given by the s_i or alternatively by specifying the $s'_{i'}$ and $\sigma_{i'}$.

Let a be a subset of sites and define s_a as:

$$s_a = \prod_{i \in a} s_i. \tag{2.3}$$

The most general site-spin hamiltonian can then be written as:

$$H(s) = \sum_a K_a s_a, \tag{2.4}$$

where a runs through all subsets of sites of the lattice. We consider lattices with periodic boundary conditions and N sites in the limit of $N \to \infty$. The interaction parameter K_a follows formally from $H(s)$ *via*:

$$K_a = 2^{-N} \sum_{\{s\}} s_a H(s), \tag{2.5}$$

where $\{s\}$ runs through all configurations of site spins. By expressing the s_i in terms of $s'_{i'}$ and $\sigma_{i'}$ as given by (2.1) and (2.2) we may write $H(s)$ as a function $H(s', \sigma)$. Then the transformation is defined by:

$$\exp H'(s') \equiv \sum_{\{\sigma\}} \exp H(s', \sigma) \tag{2.6}$$

with $\{\sigma\}$ running through all internal configurations of the cells with fixed configuration $\{s'\}$ of the cell spins. (2.6) defines a transformation of the site-spin hamiltonian $H(s)$ to a cell-spin hamiltonian $H'(s')$, which may be decomposed as in (2.4)

$$H'(s') = \sum_{a'} K'_{a'} s'_{a'} \tag{2.7}$$

with a' running through all subsets of cells. In practice we consider the transition from the interaction parameters K_a to the $K'_{a'}$ as the renormalization equations. In both sums (2.4) and (2.7) we allow for the empty set ϕ. K_ϕ is an irrelevant parameter having no influence on the $K'_{a'}$ except as an additive constant included in K'_ϕ. K'_ϕ does depend on all the K_a and will be interpreted as a regular part of the free energy [see (3.5)].

Subsets a (or a') which can be identified by a symmetry operation of the lattice will be collected in a class α. α may stand for *e.g.* "single site (cell)", "nearest-neighbour pair", *etc.* We shall restrict ourselves to homogeneous systems where all K_a of $a \in \alpha$ will have the same value. As a consequence the $a \in \alpha$ will have the same K'_a. However, it will be seen later on, that one has to define carefully what will be understood by the symmetries of the lattice in this connection. Thus effectively the transformation takes place between interaction constants for classes of sets:

$$K'_\alpha = K'_\alpha(K), \qquad \alpha \neq \phi. \tag{2.8}$$

The problem is to find a nontrivial (*i.e.* not all $K_\alpha = 0$) fixed point of the transformation (2.8) *i.e.* a set K^*_α with

$$K^*_\alpha = K_\alpha(K^*), \qquad \alpha \neq \phi \tag{2.9}$$

and to analyse the properties of the transformation (2.8) in the neighbourhood of this fixed point.

3. *Properties of eigenvalues and eigenvectors at the fixed point.* Consider the free energy F (including the factor $-\beta$) of the site system as a function of the interaction parameters K_α:

$$F(K) = \ln \sum_{\{s\}} \exp H(s). \tag{3.1}$$

In order to take advantage of the transformation (2.6) we carry out the sum over the site spins s by first summing over internal configuration σ and then over cell spins s'. With (2.6) we obtain:

$$F(K) = \ln \sum_{\{s'\}} \exp H(s') \equiv F'(K'). \tag{3.2}$$

The functions F' and F must be similar. To be precise: subtract off the empty set contributions K_ϕ and K'_ϕ and divide by N *viz.* N' the number of sites *viz.* cells. The resulting free energy per site (cell) will be the same function in the thermodynamic limit

$$\begin{cases} \lim_{N\to\infty} (F - K_\phi)/N = f(K), \\ \lim_{N'\to\infty} (F' - K'_\phi)/N' = f(K'). \end{cases} \tag{3.3}$$

Thus with (3.2) the following transformation law for the free energy is obtained:

$$l^{-d} f(K') = f(K) - f_{\text{reg}}(K), \tag{3.4}$$

where

$$l^d = N/N', \qquad f_{\text{reg}} = (K'_\phi - K_\phi)/N, \tag{3.5}$$

l is the lattice spacing of the cells measured in the site spacing ($d = 2$ is the dimensionality of the lattice).

The basic assumption (borne out in our examples) is that the K'_α (and thus f_{reg}) are regular functions of the K_β in the neighbourhood of the fixed point which the transformation (2.8) (is hoped to) exhibit. Thus we define the matrix:

$$T_{\alpha\beta} = (\partial K'_\alpha/\partial K_\beta)_{K=K^*} \tag{3.6}$$

describing the linear response of the coefficients K'_α on variations of the K_β around the fixed point. $T_{\alpha\beta}$ will not be symmetric but rather has (as will be seen later) nearly a triangular form. We are particularly interested in the eigenvalues λ_i and the associated left eigenvectors $\varphi_\alpha^{(i)}$

$$\sum_\alpha \varphi_\alpha^{(i)} T_{\alpha\beta} = \lambda_i \varphi_\beta^{(i)}. \tag{3.7}$$

With the $\varphi_\alpha^{(i)}$ we construct the functions

$$u_i = \sum_\alpha \varphi_\alpha^{(i)} (K_\alpha - K_\alpha^*), \qquad u'_i = \sum_\alpha \varphi_\alpha^{(i)} (K'_\alpha - K_\alpha^*) \tag{3.8}$$

such that the u transform under (2.8) in linear approximation:

$$u'_i = \lambda_i u_i. \tag{3.9}$$

Consider now the u_i as new coordinates in the interaction parameter space and take the free energy as function of the u_i. Then (3.4) implies

$$f(u_1, u_2, \ldots) = f_{\text{reg}}(u_1, u_2, \ldots) + l^{-d} f(\lambda_1 u_1, \lambda_2 u_2, \ldots). \tag{3.10}$$

Thus if f has a singular piece it should fulfil

$$f_{\text{sing}}(u_1, u_2, \ldots) = l^{-d} f_{\text{sing}}(\lambda_1 u_1, \lambda_2 u_2, \ldots) \tag{3.11}$$

which is clearly a requirement for homogeneity. Let us for simplicity take $u_2' = u_3' = \cdots = 0$ and investigate which power law behaviour is consistent with (3.11):

$$f_{\text{sing}}(u_1, 0, \ldots) = A u_1^{a_1} + \cdots = l^{-d} f_{\text{sing}}(\lambda_1 u_1, 0 \cdots) = l^{-d} A (\lambda_1 u_1)^{a_1} + \cdots. \tag{3.12}$$

Equating coefficients left and right yields the relation:

$$a_1 = d \ln l / \ln \lambda_1. \tag{3.13}$$

We note that (3.12) determines the type of singularity and *not* its amplitude A. If no singular piece is present the free energy can be expressed with (3.10) in terms of f_{reg} in the neighbourhood of the fixed point.

From physical considerations we expect for a powerlike singularity an exponent $1 < a < \infty$ such that the free energy and its first derivatives are continuous but the higher derivatives possibly singular. Thus corresponding to such a singularity an eigenvalue $1 < \lambda < l^d$, should be present.

The 2-dimensional Ising model has two singularities, one temperature like and one magnetic field like. We denote the "thermal" exponent by a_{T} and the "magnetic" exponent by a_{H}. The connection between the exponents a_{T} and a_{H} and the usual critical exponents is well known (see *e.g.* ref. 6):

$$\alpha = 2 - a_{\text{T}}, \qquad \delta = 1/(a_{\text{H}} - 1) \tag{3.14}$$

with α and δ the critical exponents of the specific heat and the critical "isotherm". The other exponents follow from the scaling laws (we restrict ourselves here to the thermodynamic properties). As $\alpha = 0$ and $\delta = 15$ we should expect two eigenvalues:

$$\lambda_{\text{T}} = l, \qquad \lambda_{\text{H}} = l^{15/8} \tag{3.15}$$

with $l = \sqrt{3}$ for our transformation on the triangular lattice. The other eigenvalues are expected to be real and less than 1 in magnitude. Then no physical singularities (*i.e.* those keeping the free energy finite) can be associated with these eigenvalues.

The spin model exhibits an up-down symmetry which can be used to locate the fixed point in the following way. If we start out with a hamiltonian with all interaction parameters zero corresponding to interactions with an odd number of spins (the field terms), the transformed hamiltonian will have also only even interaction parameters. The zero-field space is invariant under the transformation. Thus a fixed point will lie at zero fields as it should, which leaves us with the problem of finding a fixed point in the subspace of even spin interaction parameters.

A second consequence of this symmetry is that the matrix $T_{\alpha\beta}$ will break up into an even–even block and an odd–odd block. Thus the eigenvectors ϕ_α^{T} have only components for α even and ϕ_α^{H} only for α odd. The coefficients ϕ_α^{T} are of particular interest as they give the direction of the plane $u_{\mathrm{T}} = 0$, which is tangent to the surface of criticality ("ridge line" in Wilson's terminology) at the fixed point:

$$0 = u_{\mathrm{T}} = \sum_\alpha \varphi_\alpha^{\mathrm{T}} (K_\alpha - K_\alpha^*). \tag{3.16}$$

(3.16) contains the variation of the critical temperature with the interaction parameters. If we disregard the curvature of the surface of criticality, we find the intersection K_{c} with the Ising axis (all $K_\alpha = 0$ except for α = nearest neighbours = n.n.):

$$\varphi_{\mathrm{n.n}}^{\mathrm{T}} K_{\mathrm{c}} = \sum_\alpha \varphi_\alpha^{\mathrm{T}} K_\alpha^*. \tag{3.17}$$

Writing $K_{\mathrm{c}} = J/k_{\mathrm{B}}T_{\mathrm{c}}$ and $K_\alpha^* = J_\alpha/k_{\mathrm{B}}T_{\mathrm{c}}(J)$ we find for the critical temperature $T_{\mathrm{c}}(J)$ of a general system:

$$T_{\mathrm{c}}(J) = T_{\mathrm{c}} \left\{1 + \sum_{\alpha \neq \mathrm{n.n}} (\varphi_\alpha^{\mathrm{T}}/\varphi_{\mathrm{n.n}}^{\mathrm{T}})\, (J_\alpha/J_{\mathrm{n.n}})\right\}. \tag{3.18}$$

Dalton and Wood[8]) have studied the variation of the critical temperature due to the admixture of a second-neighbour interaction in a triangular lattice from series expansions of the partition function above and below the critical temperature.

(3.16) shows also that u_{T} is a temperature-like parameter. The intercept of the plane $u_{\mathrm{T}} = \varepsilon$ with the Ising axis yields

$$\varphi_{\mathrm{n.n}}^{\mathrm{T}} K_{\mathrm{n.n}} = \varepsilon + \sum_\alpha \varphi_\alpha^{\mathrm{T}} K_\alpha^* = \varepsilon + \varphi_{\mathrm{n.n}}^{\mathrm{T}} K_{\mathrm{c}}$$

or

$$\varepsilon = \varphi_{\mathrm{n.n}}^{\mathrm{T}} (K_{\mathrm{n.n}} - K_{\mathrm{c}}) \sim T - T_{\mathrm{c}} \tag{3.19}$$

which shows that indeed ε measures for Ising systems the temperature distance to the plane of criticality.

4. *Simple perturbation theory.* An exact evaluation of the transformation (2.6) is more difficult than the solution of the Ising model due to the appearance of arbitrary interactions. In this section we discuss an approximate evaluation based on a separation of H in an unperturbed part H^0 which is manageable and a remainder V

$$H = H^0 + V. \tag{4.1}$$

Then we may employ for the transformation (2.6) the following convenient notation:

$$\exp H'(s') = \left[\sum_{\{\sigma\}} \exp H^0(s', \sigma)\right] \langle \exp V \rangle_0, \tag{4.2}$$

where the unperturbed (partial) average $\langle \ \rangle_0$ is defined as:

$$\langle A \rangle_0 = \left[\sum_{\{\sigma\}} e^{H^0(s', \sigma)} A(s', \sigma)\right] \Big/ \left[\sum_{\{\sigma\}} e^{H^0(s', \sigma)}\right]. \tag{4.3}$$

The use of the average notation suggests the cumulant method for finding succesive orders in V

$$\langle \exp V \rangle_0 = \exp\{\langle V \rangle_0 + \tfrac{1}{2}[\langle V^2 \rangle_0 - \langle V \rangle_0^2] + \cdots\}. \tag{4.4}$$

The point is now the choice of H^0 and V. The most straightforward choice is to collect in H^0 all intracell interactions and in V all intercell interactions. H^0 becomes (for zero magnetic field)

$$H^0 = \sum_{i'} H^0_{i'} = K \sum_{i'} (s^1_{i'} s^2_{i'} + s^2_{i'} s^3_{i'} + s^3_{i'} s^1_{i'}) = K \sum_{i'} (-1 + 4\delta_{\sigma_{i'}, 0}) \tag{4.5}$$

with K as nearest-neighbour interaction and $\sigma_{i'}$ defined in (2.2). Note that H^0 does not depend on the cell-spin values. Thus the evaluation of the first factor in (4.2) is trivial leading to:

$$\sum_{\{\sigma\}} \exp H^0(\sigma) = \prod_{i'} \left\{\sum_{\sigma_{i'}} \exp H^0_{i'}(\sigma_{i'})\right\} = (z^0)^N \tag{4.6}$$

with

$$z^0 = \exp(3K) + 3\exp(-K). \tag{4.7}$$

To illustrate the evaluation of $\langle V \rangle_0$ we restrict ourselves first to including only nearest-neighbour interactions.

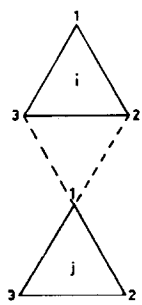

Fig. 2

A typical pair of neighbouring cells i' and j' is shown in fig. 2. The interaction between these cells is given by

$$V_{i'j'} = K(s_{i'}^2 + s_{i'}^3)\, s_{j'}^1. \tag{4.8}$$

For the calculation of $\langle V_{i'j'}\rangle_0$ we use the following averages:

$$\begin{aligned} f_1 &= s_{i'}^1 \langle s_{i'}^\alpha \rangle_0 = (e^{3K} + e^{-K})/z^0, \qquad \alpha = 1, 2, 3, \\ f_2 &= \langle s_{i'}^\alpha s_{i'}^\beta \rangle_0 = (e^{3K} - e^{-K})/z^0, \qquad \alpha \neq \beta = 1, 2, 3, \\ f_3 &= s_{i'}^1 \langle s_{i'}^1 s_{i'}^2 s_{i'}^3 \rangle_0 = (e^{3K} - 3e^{-K})/z^0 \end{aligned} \tag{4.9}$$

with these rules we find:

$$\langle V_{i'j'}\rangle_0 = 2f_1^2 K s_{i'}' s_{j'}'. \tag{4.10}$$

The same results hold for all nearest-neighbour cells. Thus to first order we find for the hamiltonian:

$$H'(s') = N \ln z^0 + 2f_1^2 K \sum_{\langle i', j'\rangle} s_{i'}' s_{j'}', \tag{4.11}$$

where the summation runs over all pairs of nearest-neighbour cells $\langle i', j'\rangle$. Clearly the cell hamiltonian $H'(s')$ contains only a (renormalized) nearest-neighbour coupling K' of strength:

$$K' = 2f_1^2 K. \tag{4.12}$$

This is the simplest form of a transformation. With f_1 given by (4.9) we find a non-trivial fixed point $K^* = 0.3356$ and an eigenvalue

$$\lambda_T = [\partial K'/\partial K]_{K=K^*} = 1.634. \tag{4.13}$$

One should compare these numbers with the Ising values $K_c = 0.2744 = (\ln 3)/4$ and $\lambda_T = \sqrt{3} = 1.737$ for the triangular lattice[9]).

For subcritical systems $K > K^*$ the new value K' is larger than the old K, whereas for supercritical systems $0 < K < K^*$ one has $K' < K$.

Turning back to the perturbation series we point out that if all the couplings different from the nearest neighbour are absent they also will not appear through the transformation in first order. Thus studying the second-order perturbation term we restrict ourselves again to nearest-neighbour coupling.

The problem concerns the evaluation of fluctuations

$$\left\langle \left(\sum_{\langle i',j' \rangle} V_{i'j'} \right)^2 \right\rangle_0 - \left\langle \sum_{\langle i',j' \rangle} V_{i'j'} \right\rangle_0^2$$
$$= \sum_{\langle i',j' \rangle} \sum_{\langle k',l' \rangle} \langle (V_{i'j'} - \langle V_{i'j'} \rangle_0)(V_{k'l'} - \langle V_{k'l'} \rangle_0) \rangle_0 . \tag{4.14}$$

If we take in (4.14) products of $\langle i',j' \rangle$ and $\langle k',l' \rangle$ with no cell in common the term will vanish since then:

$$\langle V_{i'j'} V_{k'l'} \rangle_0 = \langle V_{i'j'} \rangle_0 \langle V_{k'l'} \rangle_0 \tag{4.15}$$

because different cells are independent in the average $\langle \ \rangle_0$. Also a product of $\langle i',j' \rangle = \langle k',l' \rangle$ will not lead to an interesting contribution as it is independent of the cell spins. Thus remains the typical term:

$$\langle V_{i'j'} V_{j'k'} \rangle_0 - \langle V_{i'j'} \rangle_0 \langle V_{j'k'} \rangle_0 .$$

The possible arrangements of cells i', j' and k' can easily be counted and are shown in fig. 3. To give an example the contribution of the arrangement of fig. 3a is computed with (4.9) as:

$$\langle V_{i'j'} V_{j'k'} \rangle_0 = K^2 \langle s_{i'}^2 (s_{j'}^2 + s_{j'}^3) s_{j'}^3 (s_{k'}^1 + s_{k'}^2) \rangle_0 = K^2 (2f_1^2)(1 + f_2) s_{i'}' s_{k'}' ,$$
$$\langle V_{i'j'} \rangle_0 \langle V_{j'k'} \rangle_0 = 4 f_1^4 K^2 s_{i'}' s_{k'}' . \tag{4.16}$$

So a net nearest-neighbour interaction between the cells i' and k' results of strength $K^2 f_1^2 (1 + f_2 - 2f_1^2)$.

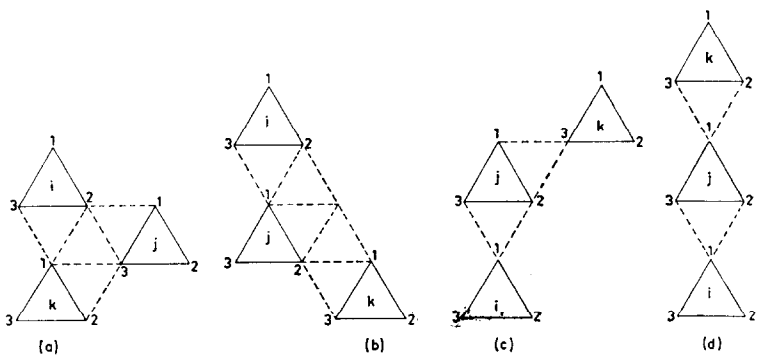

Fig. 3

The other arrangements are treated similarly. Figs. 3b and 3c lead to a next-nearest neighbour interaction (the strength of which is denoted by L') and fig. 3d to a third-neighbour coupling (M'). Apart from the computation of the individual

terms, we have to figure out how many times a particular interaction may result from different choices in the summations in (4.14). The results are the following contributions in second order

$$(K')_2 = 4(a + b) f_1^2 K^2,$$
$$(L')_2 = (a + 7b) f_1^2 K^2, \tag{4.17}$$
$$(M')_2 = 4b f_1^2 K^2$$

with a and b given by:

$$a = 1 - f_1^2, \qquad b = f_2 - f_1^2. \tag{4.18}$$

Thus in second order in V new interactions (L' and M') are generated. So, if we consider the nearest-neighbour coupling K' as a first-order quantity, than L' and M' are of second order and all other interactions of higher order.

To be complete in second order we must treat the effect of the L' and M' coupling also in the first order $\langle V' \rangle_0$. For the arrangement of fig. 2, we find for the potential: due to L and M:

$$V_{i'j'} = L(s^1_{i'}s^1_{j'} + s^2_{i'}s^2_{j'} + s^3_{i'}s^3_{j'}) + M(s^2_{i'}s^3_{j'} + s^3_{i'}s^2_{j'}). \tag{4.19}$$

The average $\langle \rangle_0$ of this potential leads to an additional coupling between the (nearest-neighbour) cells i' and j'. As can be seen in fig. 2b, this interaction leads also to a direct connection between second-neighbour cells i' and k' through the bond $Ms^2_{i'}s^1_{k'}$. Taking these terms into account we obtain additional contributions to K' and L'

$$\begin{aligned} (K')_1 &= 3f_1^2 L + 2f_1^2 M, \\ (L')_1 &= \qquad f_1^2 M. \end{aligned} \tag{4.20}$$

Adding the various contributions, the full second-order transformation equations read:

$$\begin{aligned} K' &= 2f_1^2 K + 4(a + b) f_1^2 K^2 + 3f_1^2 L + 2f_1^2 M, \\ L' &= \qquad (a + 7b) f_1^2 K^2 \qquad + f_1^2 M, \\ M' &= \qquad\qquad 4b f_1^2 K^2. \end{aligned} \tag{4.21}$$

This transformation has a nontrivial fixed point at

$$K^* = 0.2789, \qquad L^* = -0.0143, \qquad M^* = -0.0152. \tag{4.22}$$

The matrix $T_{\alpha\beta}$ follows for the even–even block from differentiation of (4.21) with respect to K, L and M and insertion of the fixed-point parameters. One finds:

$$T_{\alpha\beta} = \begin{pmatrix} 1.8966 & 1.3446 & 0.8964 \\ -0.0403 & 0.0 & 0.4482 \\ -0.0782 & 0.0 & 0.0 \end{pmatrix} \tag{4.23}$$

with eigenvalues

$$\lambda_T = \lambda_1 = 1.7835, \qquad \lambda_2 = 0.2286, \qquad \lambda_3 = -0.1156. \tag{4.24}$$

The left eigenvactor ϕ_α^T corresponding to the largest eigenvalue λ_T is ($\phi_{n.n}^T = 1$):

$$\phi_\alpha^T = (1, 0.7539, 1.0961). \tag{4.25}$$

Linear extrapolation yields with (3.17) $K_c = 0.2514$.

So far the magnetic (odd) potentials have been taken to be zero. For the odd–odd components of $T_{\alpha\beta}$ we study the response to a variation of a parameter K_β. From (2.6) we find:

$$\delta \exp H'(s') = [\exp H'(s')]\, \delta H'(s') = \Big[\sum_{\{\sigma\}} \Big(\sum_{b \in \beta} s_b\Big) \exp H(s', \sigma)\Big]\, \delta K_\beta. \tag{4.26}$$

Using (4.2) and expanding in powers of V leads to

$$\delta H'(s') = \sum_{b \in \beta} \{\langle s_b \rangle_0 + \langle (s_b - \langle s_b \rangle_0)\, V \rangle_0 + \cdots\}\, \delta K_\beta. \tag{4.27}$$

The summation in (4.26) and (4.27) runs over all the specific sets of sites b belonging to the class β having the interaction parameter K_β. From (4.27) the matrix $T_{\alpha\beta}$ can be formed by decomposing $\delta H'(s')$ with respect to the various cell interactions. The first term in (4.27) can be evaluated with the relations (4.9). Let us first take β corresponding to the class of single sites. Then we find:

$$\sum_{i'} \langle s_{i'} \rangle_0 = \sum_{i'} \sum_{\alpha=1}^{3} \langle s_{i'}^\alpha \rangle_0 = 3 f_1 \sum_{i'} s_{i'}' \tag{4.28}$$

showing that only single-cell contributions appear. All other classes of (odd) β will lead, in this approximation, also to single-cell contributions. The matrix $T_{\alpha\beta}$ consists therefore of a row for α, the single-cell class, with all other matrix elements zero. So a magnetic field of strength δH on the sites is transformed to a cell field $\delta H' = 3 f_1 \delta H$ and the magnetic eigenvalue λ_H becomes:

$$\lambda_H = 3 f_1. \tag{4.29}$$

In the next-order term we restrict V to the nearest-neighbour terms. For β the single-site class a typical term reads

$$\left\langle \sum_{\alpha=1}^{3} (s_{i'}^{\alpha} - f_1 s_{i'}') \, K \, (s_{j'}^{2} + s_{j'}^{3}) \, s_{l'}^{1} \right\rangle_0 .$$

Here again we must have i equal to j or l in order that the contribution does not vanish. For $i = j$ one has a single-cell contribution:

$$K \sum_{\alpha=1}^{3} \langle (s_{j'}^{\alpha} - f_1 s_{j'}') \, (s_{j'}^{2} + s_{j'}^{3}) \, s_{l'}^{1} \rangle_0 = 2Kf_1 \, (a + 2b) \, s_{l'}' . \tag{4.30}$$

Counting the number of times that a cell can be produced by different choices of i', j' and l' one finds for the matrix element $T_{\alpha\alpha}$ corresponding to single-cell–single-site sets

$$T_{\alpha\alpha} = 3f_1 \, (1 + 4K \, (a + 2b)) . \tag{4.31}$$

As a single-site potential produces only a single-cell potential, the first column in the odd–odd block of $T_{\alpha\beta}$ is zero except for β being the single-site class and the magnetic eigenvalue $\lambda_{\mathrm{H}} = T_{\alpha\alpha}$ is given by (4.31) to first order in V'. Inserting the first-order fixed-point value $K^* = 0.3356$ one finds $\lambda_{\mathrm{H}} = 3.03$.

The calculation of higher orders in this perturbation scheme is straight forward but becomes soon quite involved.

5. *Cluster approximation.* The perturbation theory uses K^*, the fixed-point nearest-neighbour coupling, as expansion parameter. The value of K^* is by no means sufficiently small to guarantee a fast convergence for this expansion. One of the results of the perturbation theory is, however, that the interactions like second (L) and third (M) neighbour are formed in higher orders in the perturbation and turn out to be small compared to the nearest-neighbour interaction. Thus the fixed-point hamiltonian is relatively close to a pure Ising (nearest-neighbour) hamiltonian. This would mean that the transformation essentially takes place between the short-range interaction parameters.

In this section we develop a cluster expansion for which the strength of the interactions in not necessarily small but which uses the fact that short-range interactions are most important. Let us illustrate the method on the simplest cluster possible: the pair of cells shown in fig. 2. Disregarding the remaining cells of the lattice we can compute the free energy of this cluster as:

$$\exp H' \, (s_{i'}', s_{j'}') = \sum_{\sigma_{i'}, \sigma_j} \exp H \, (s_{i'}' \sigma_{i'}, s_{j'}' \sigma_{j'}) = (z^0)^2 \, \langle \exp V_{i'j'} \rangle \ . \tag{5.1}$$

The last average follows by writing:

$$\exp V_{i'j'} = \exp K \, (s_{i'}^{2} + s_{i'}^{3}) \, s_{j'}^{1} = [\cosh K]^2 \, (1 + u s_{i'}^{2} s_{j'}^{1}) \, (1 + u s_{i'}^{3} s_{j'}^{1}) \tag{5.2}$$

with

$$u = \operatorname{tgh} K. \tag{5.3}$$

Using the rules (4.9) we find:

$$\exp H'(s'_{i'}, s'_{j'}) = [z^0 \cosh K]^2 (1 + 2uf_1^2 s'_{i'} s'_{j'} + u^2 f_2). \tag{5.4}$$

Writing the right-hand side of (5.4) again as an exponential, we obtain $H'(s'_{i'}, s'_{j'})$ with a nearest-neighbour coupling K' given by:

$$K' = \tfrac{1}{2} \ln \frac{1 + 2uf_1^2 + u^2 f_2}{1 - 2uf_1^2 + u^2 f_2} = \operatorname{arctgh} \frac{2f_1^2 \operatorname{tgh} K}{1 + f_2 \operatorname{tgh}^2 K}. \tag{5.5}$$

Expanding (5.5) for small K and retaining only the first term leads back to the perturbation result (4.12). From the viewpoint of perturbation theory (5.5) includes also the higher-order perturbation terms in which only cell i and j are involved.

When we want to extend the result (5.5) to larger clusters we encounter the combinatorial problem of keeping track to which cluster certain bonds belong. To investigate that problem, consider an arbitrary set of cells a. The partition function of set a is written as:

$$\sum_{\{\sigma\}} \exp H_a(s', \sigma) = \exp H'_a(s') = \exp \sum_{b \subseteq a} K'^a_b s'_b, \tag{5.6}$$

i.e., we decompose the energy of set a into its constituent sets b*. The interaction coefficients K^a_b do in general depend on the choice of a. One would like to take a as large as possible, but such a choice is restricted by computational difficulties. We expect, however, that if a is large enough any enlargement of a would have only a minor impact on K^a_b. Thus we represent K^a_b by an Ursell expansion:

$$K'^a_b = \sum_{b \subseteq c \subseteq a} U^c_b, \tag{5.7}$$

where the U^c_b are hoped to decrease rapidly with increasing c. The expansion (5.7) can be inverted with Moebius's exclusion – inclusion principle yielding:

$$U^c_b = \sum_{b \subseteq a \subseteq c} (-1)^{c-a} K'^a_b, \tag{5.8}$$

where c and a in $(-1)^{c-a}$ stand for the number of cells in the sets c and a. (5.8) enables us to compute the U^c_b from the partition functions of all subsets of c.

* From now on we shall omit in this section the primes on sets of cells.

Now in (5.7) we may let a go to infinity and obtain the transformation in terms of the Ursell coefficients:

$$K_b' = \sum_{b \subseteq c} U_b^c. \tag{5.9}$$

This still contains the whole problem and we shall have to approximate by selecting a suitable collection of sets c. Restricting c as in (5.7) by selecting it as a subset out of a set a would be unsystematic, because for every b we would have to decide how to locate the set a such that b is included. A much more systematic approach is to take all c out of a class γ of equivalent sets. Inserting (5.8) back into (5.9) the approximation becomes:

$$K_b' = \sum_{b \subseteq c \in \gamma} \left\{ \sum_{b \subseteq a \subseteq c} (-1)^{c-a} K_b'^a \right\}. \tag{5.10}$$

We want to interchange the summations over a and c, as the latter occurs only in connection with $(-1)^{c-a}$ and therefore involves only a combinatorical problem. For that purpose split the summation over a by first summing over all possible a out of a class α and then over the possible classes α.

The class α must be contained in γ and must contain the class β of b. (The inclusion relation is transferred from sets to classes by inspecting whether members can include each other.) Then (5.10) may be written as:

$$K_b' = \sum_{\beta \subseteq \alpha \subseteq \gamma} \sum_{b \subseteq a \in \alpha} \left[\sum_{a \subseteq c \in \gamma} (-1)^{c-a} \right] K_b'^a. \tag{5.11}$$

The expression between brackets in (5.11) is a sum over terms of the same sign $(-1)^{\gamma-\alpha}$ as the number of cells in c and a is the same for all members of the classes γ and α. The number of terms in this sum is denoted by $D_\gamma(\alpha)$ as it is also completely determined by the classes γ and α

$$D_\gamma(\alpha) = \sum_{a \subseteq c \in \gamma} 1 \qquad (a \in \alpha). \tag{5.12}$$

Thus (5.11) reduces to

$$K_b' = \sum_{\beta \subseteq \alpha \subseteq \gamma} (-1)^{\gamma-\alpha} D_\gamma(\alpha) \sum_{b \subseteq a \in \alpha} K_b'^a. \tag{5.13}$$

The last summation involves all sets a which include b and belong to class α. For all b out of class β this will lead to the same result. Thus we may write

$$K_\beta'^\alpha = \sum_{b \subseteq a \in \alpha} K_b'^a \tag{5.14}$$

which then shows that the resulting K_b' in (5.13) depends also only on the class β to which b belongs. So we obtain for the transformation

$$K_\beta' = \sum_{\beta \subseteq \alpha \subseteq \gamma} (-1)^{\gamma-\alpha} D_\gamma(\alpha) K_\beta'^{\alpha}. \tag{5.15}$$

This approximation is systematic in the sense that it concerns a transformation of interaction parameters K_δ of class δ (on which the $K_\beta'^{\alpha}$ depend) to parameters K_β' for class β and that no specific sets are involved any more. (5.15) is not optimal from the viewpoint of computational efficiency. For a given class γ we have to compute all the $K_\beta'^{\alpha}$ for subclasses of γ. Without additional computational work we may extend the approximation to (5.9) by not taking the sets out of one class γ but out of all subclasses of a class δ. With this extra summation (5.15) may be written as:

$$K_\beta' = \sum_{\beta \subseteq \alpha \subseteq \delta} C_\delta(\alpha) K_\beta'^{\alpha} \tag{5.16}$$

with $C_\delta(\alpha)$ defined by:

$$C_\delta(\alpha) = \sum_{\alpha \subseteq \gamma \subseteq \delta} (-1)^{\gamma-\alpha} D_\gamma(\alpha). \tag{5.17}$$

The approximation (5.16) is used in this paper for several δ's. Taking δ to be the class of nearest-neighbour cells reduces (5.16) to (5.5). The problem can be handled in three separate stages. The first one is the determination of the coefficients $D_\gamma(\alpha)$ and $C_\delta(\alpha)$ which is a relativily simple counting problem. The second is the determination of the parameters K_b^a from (5.6) (which for $\delta = 4$ and 5 has been done on a computer) and the third stage in the combination of the various interactions parameters as indicated in (5.14). This last step is in fact done easier by keeping a certain a fixed and collecting the various possibilities in which a b out of β can be fitted into a.

The values of $C_\delta(\alpha)$ are listed for a number of δ and $\alpha \in \delta$ in table I. Note that several $C_\delta(\alpha)$ are zero.

We point out that the approximation formula (5.16) can also be used for determining the matrix $T_{\alpha\beta}$. By differentrating (5.16) with respect to K_β we obtain

$$T_{\alpha\beta} = \partial K_\alpha'/\partial K_\beta = \sum_{\alpha \subseteq \gamma \subseteq \delta} C_\delta(\gamma) (\partial K_\alpha'^{\gamma}/\partial K_\beta). \tag{5.18}$$

The derivative on the right-hand side may be computed as in the perturbation calculation of the magnetic part of $T_{\alpha\beta}$. For a specific set c we have

$$\partial H_c(s)/\partial K_\beta = \left\langle \sum_{b' \in \beta} s_{b'}' \right\rangle_c, \tag{5.19}$$

TABLE I

$C_\delta(\alpha)$ as defined in (5.17) and (5.12)

δ \ α	1	2	2′	2″	3	3′	3″	3‴	4	4′	4″	5
2	−5	1	—	—	—	—	—	—	—	—	—	—
3	1	−1	—	—	1	—	—	—	—	—	—	—
4	1	0	0		−2	0	—	—	1	—	—	—
5	−2	2	0	0	1	0	−1	0	−3	0	0	1

where $\langle\ \rangle_c$ is the average over the internal configurations of the set of cells c:

$$\langle A\rangle_c = \sum_{\{\sigma\}} e^{H_c'(s',\sigma)} A(s',\sigma) \Big/ \sum_{\{\sigma\}} e^{H_c'(s',\sigma)}. \tag{5.20}$$

The value of $\partial K_a'^c/\partial K_\beta$ is selected out as in (2.5)

$$\partial K_a'^c/\partial K_\beta = 2^{-c} \sum_{\{s'\}} s_a' \left\langle \sum_{b\in\beta} s_b \right\rangle_c. \tag{5.21}$$

Then with (5.14) the derivative needed for (5.18) is found as

$$\partial K_\alpha'^\gamma/\partial K_\beta = \sum_{a\subseteq c\in\gamma} 2^{-c} \left[\sum_{s'} s_{a'}' \left\langle \sum_{b\in\beta} s_b \right\rangle_c\right]. \tag{5.22}$$

In practice it is easier to convert the sum over c at fixed a into a sum over a at fixed c, which can be done using the symmetries of the lattice.

6. *Approximations.* The approximation (5.16) has been worked out for a basic figure consisting out of 2, 3, 4 and 5 cells in the arrangements shown in the first entry of table I. For our convenience we have numbered the figures as indicated in table I. We discuss the various approximations separately.

6.1. Two-cell approximation. This case has been discussed partly as an illustration of the cluster expansion at the beginning of section 5. The two-cell approximation involves only the nearest-neighbour interaction as no other interactions between cell pairs can be generated by the transformation. The transformation equation (5.5) for zero field has a fixed point at $K^* = 0.365$ and an eigenvalue $\lambda_T = 1.544$.

As far as magnetic perturbations around the fixed point are concerned this figure contains only single cells as odd subsets. Thus the odd–odd part of the matrix $T_{\alpha\beta}$ is 1-dimensional. In this case it is not difficult to work out the formula for $T_{\alpha\beta}$ (and thus λ_H) analytically with result:

$$\lambda_H = 3f_1 + \frac{6u\,(2f_1 + u)\,(a + 2b)}{1 + 2f_1^2u^2 + f_2u^2} \tag{6.1}$$

with $u = \text{tgh}\, K^*$ and the f_i defined in (4.9) and a and b in (4.18).

6.2. Three-cell approximation. The three cells are located in a triangle, such that all pairs of cells are nearest neighbours. Thus the transform (in zero field) concerns again only the nearest-neighbour interaction parameter. With table I we have explicitly:

$$K_2' = K_2'^3 - K_2'^2. \tag{6.2}$$

Without too much difficulty the formula can be worked out analytically and the fixed point and eigenvalue can be determined numerically. The result is:

$$K^* = 0.255, \qquad \lambda_T = 1.501. \tag{6.3}$$

Magnetic perturbations give rise to a single and triple-cell potential. Thus the odd–odd block of $T_{\alpha\beta}$ is 2-dimensional and follows from differentiation of (*cf.* table I):

$$\begin{aligned} K_1' &= K_1'^3 - K_1'^2 + K_1'^1, \\ K_3' &= K_3'^3. \end{aligned} \tag{6.4}$$

The result yields at the fixed point $T_{\alpha\beta}$:

$$T_{\alpha\beta} = \begin{pmatrix} 2.707 & 7.031 \\ -0.063 & 0.364 \end{pmatrix} \tag{6.5}$$

which yields eigenvalues:

$$\lambda_H = \lambda_1 = 2.501, \qquad \lambda_2 = 0.571. \tag{6.6}$$

6.3. Four-cell approximation. The cells are arranged in a diamond shape as shown in table I. Now 3 even interaction parameters come into play: the nearest neighbour (K), the second neighbour (L) and the 4-spin (diamond) interaction (N). This case has been done completely numerically. The configurations of the site spins are systematically generated for every constraint given by the configurations of the cell spins. The resulting free energies are decomposed into their contributions from the three possibilities for cell interaction. This is done for the diamond and a subtriangle, which have to be combined according to table I. Once the transform is established (*i.e.*, a map of input values K, L, N to output values K', L', N') the fixed point has to be determined. An efficient method turns out to be the following. With only slightly more effort one obtains also the matrix $T_{\alpha\beta}$ defined as in section 3, but now taken at input values K, L, N. Assume that one is close enough to the fixed point that $T_{\alpha\beta}$ may be equated with the fixed point value. Then put:

$$\begin{aligned} K'_\alpha &= K^*_\alpha + \delta K'_\alpha = K^*_\alpha + \sum_\beta T_{\alpha\beta}\delta K_\beta, \\ K_\alpha &= K^*_\alpha + \delta K_\alpha = K^*_\alpha + \sum_\beta \delta_{\alpha\beta}\delta K_\beta. \end{aligned} \tag{6.7}$$

Elimination of K^*_α yields a set of equations for δK_β reading:

$$\sum_\beta (\delta_{\alpha\beta} - T_{\alpha\beta})\, \delta K_\beta = K_\alpha - K'_\alpha. \tag{6.8}$$

So for a set of input values K_α the differences δK_α with the fixed point can be estimated and thus a first location of the fixed point is found. A few steps are sufficient to have convergence. The result is:

$$K^* = 0.257, \qquad L^* = -0.0022, \qquad N^* = -0.00085. \tag{6.9}$$

The even–even block of the matrix $T_{\alpha\beta}$ is found to be:

$$T_{\alpha\beta} = \begin{pmatrix} 1.531 & 1.728 & 1.107 \\ 0.052 & 0.249 & 0.170 \\ -0.033 & -0.047 & 0.103 \end{pmatrix} \tag{6.10}$$

with eigenvalues:

$$\lambda_T = \lambda_1 = 1.567, \qquad \lambda_2 = 0.166, \qquad \lambda_3 = 0.150. \tag{6.11}$$

The left eigenvector associated with λ_T is ($\phi^T_{n.n} = 1$)

$$\varphi^T_\alpha = (1, 1.205, 2.990). \tag{6.12}$$

Using formula (4.16) one finds with these values of the components of ϕ_α^T for the pure Ising case $K_c = 0.253$ if one neglects the curvature of the plane of criticality. $\phi_\alpha^T = 1.205$ should be compared to the value 1.35 obtained by Dalton and Wood[8]) for the triangular lattice from series expansions for the free energy.

The magnetic perturbations involve three interactions: the single site (1) and two triple cells (3 and 3'). The odd–odd block of $T_{\alpha\beta}$ reads at the fixed point:

$$T_{\alpha\beta} = \begin{pmatrix} 2.532 & 3.003 & 7.482 \\ -0.048 & 0.416 & 1.335 \\ 0.009 & 0.044 & 0.218 \end{pmatrix} \tag{6.13}$$

with eigenvalues:

$$\lambda_H = \lambda_1 = 2.497, \qquad \lambda_2 = 0.600, \qquad \lambda_3 = 0.070. \tag{6.14}$$

A point of discussion emerges here with respect to the class 3'. In figs. 3b and 3c we have made a distinction between the two configurations, here both denoted by 3'. Let us call the two variants corresponding to 3b and 3c by $3'_b$ and $3'_c$ and see in what strength both are produced in a magnetic perturbation. It turns out that the cell configurations $3'_b$ and $3'_c$ have different coupling strength even if one has an input equal strengths on the site configurations $3'_b$ and $3'_c$. For a complete discussion of the magnetic perturbation one should extend the odd–odd block of $T_{\alpha\beta}$ to a 4×4 matrix. In doing so one recovers the same eigenvalues plus an eigenvalue zero because $T_{\alpha\beta}$ obtains the form:

$$T_{\alpha\beta} = \begin{pmatrix} 2.532 & 3.003 & 3.741 & 3.741 \\ -0.048 & 0.416 & 0.667 & 0.667 \\ 0.029 & 0.130 & 0.125 & 0.125 \\ -0.016 & -0.042 & 0.093 & 0.093 \end{pmatrix} \tag{6.15}$$

with the third and fourth row (corresponding to $3'_b$ and $3'_c$) equal. (The different strengths of $3'_b$ and $3'_c$ introduce also a distinction between the triangle 3 and the one obtained by reflection with respect to the vertical axis. Taking this distinction into account has only a minute effect on the largest eigenvalue and is therefore omitted.)

6.4. Five-cell approximation. The extension of the numerical method as to include 5 cells is straightforward. For the locations of the fixed point we need 7 parameters: nearest (2), second (2') and third (2'') neighbour coupling and four 4-spin interactions. In table I the 4-spin interactions are listed as 4, 4' and 4''. 4' has two variants $4'_b$ and $4'_c$ distinguishable by the fact whether they allow a $3'_b$ or $3'_c$ as subfigure.

For the fixed point we find the values:

$$K^* = K_2^* = 0.331, \quad L^* = K_2^* = -0.0275, \quad M^* = K_4^* = 0.00858,$$
$$N^* = K_{2'}^* = -0.0267, \quad K_{4_b}^* = 0.0063, \quad K_{4_c}^* = 0.0096, \quad K_{4''}^* = -0.0037. \tag{6.16}$$

The even–even part of $T_{\alpha\beta}$ has eigenvalues:

$$\lambda_T = \lambda_1 = 1.784, \quad \lambda_2 = 0.585, \quad \lambda_3 = 0.195, \quad \lambda_4 = 0.112,$$
$$\lambda_5 = -0.00003, \quad \lambda_6 = -0.040, \quad \lambda_7 = -0.150. \tag{6.17}$$

The left eigenvector ϕ_α^T reads ($\phi_{n.n}^T = 1$)

$$\phi_\alpha^T = (1, 1.708, 1.237, 1.917, 2.9888, 2.9894, 2.742),$$
$$\alpha = (2, 2', 4, 2'', 4_b', 4_c', 4''). \tag{6.18}$$

The magnetic perturbations take place in the 1, 3, $3_b'$, $3_c'$, $3''$, $3_b''$, $3_c'''$ and 5. The eigenvalues are

$$\lambda_H = \lambda_1 = 3.186, \quad \lambda_2 = 0.716, \quad \lambda_3 = 0.271, \quad \lambda_4 = 0.199,$$
$$\lambda_5 = -0.078, \quad \lambda_6 = 0.002, \quad \lambda_7 = -0.0007, \quad \lambda_8 = -0.027. \tag{6.19}$$

7. *Discussion.* A transformation in the space of spin hamiltonians for a triangular lattice, formally defined in (2.6), is studied in various approximations. The most remarkable point is that in all these approximations the transformation exhibits a nontrivial fixed point, with properties close to those to be expected from known results for an Ising model. We consider this as evidence that the true transformation (2.6) also exhibits a fixed point in the neighbourhood of our approximate findings. This fixed point then determines the critical properties (exponents) for a wide class of spin hamiltonians having basically short-range interactions.

The appearance of a fixed point in a transformation, like the approximations to (2.6) presented here, is a delicate matter. We have, for example, treated the five-cell approximation with omission of the last three (4-spin) interactions, assuming that their strengths (all < 0.01) are so small that they could safely be omitted. Then, however, there no longer is a nontrivial fixed point! The reason is that different interactions of the same range have opposite signs and therefore tend to compensate each other. The approximations all have in common, that in the fixed-point hamiltonian, the nearest-neighbour coupling dominates by a factor 10 over the second and third-neighbour coupling, which in turn are 3 to 4 times larger than the 4-spin interactions.

TABLE II

Survey of approximations. ϕ_2^T, gives the value of the variation of T_c with second-neighbour interaction strength. The "exact" value is taken from the numerical analysis of Dalton and Wood.

Approximations	λ_T	λ_H	K_c	ϕ_2^T
1st order perturbations	1.634	3.036	0.3356	—
2nd order perturbations	1.784	—	0.251	0.754
 2 cells	1.544	3.036	0.365	—
 3 cells	1.501	2.501	0.252	—
 4 cells	1.567	2.497	0.253	1.205
 5 cells	1.782	3.186	0.281	1.66
exact	1.737	2.801	0.2744	1.35

The results for the eigenvalues λ_T and λ_H, the locations of the Ising critical point K_c and the variations of T_c with next-nearest neighbour coupling strength [according to (3.1)] are summarized in table II. For the thermal eigenvalues λ_T and for K_c one observes a steady approach to the exact values such that the five-cell approximation is within 3%. The magnetic eigenvalue λ_H shows a more erratic behaviour. In particular the five-cell approximation is disappointingly high. One should note, however, that in each approximation the fixed point shifts and the eigenvalues, in particular the magnetic one, are sensitive to the location of the fixed point. Also the five-cell approximation is not an optimal choice as far as symmetry is concerned and this too may be the cause of the unexpected behaviour of λ_H. A more preferable choice would in our opinion be a seven-cell figure, arranged in a ring of six around a center cell. This, however, requires a number of changes in the computer program generating the spin configurations in order to take full profit of the symmetries of the basic figure.

It is quite remarkable that in all approximations the "thermal" as well as the "magnetic" matrices have only one eigenvalue larger than 1 and that all eigenvalues are real. One would expect the eigenvalues that are smaller than 1 to approach definite values for larger and larger clusters; the fact that they show a

rather irregular behaviour probably means that the convergence as a whole is not yet very good.

Besides the problem of the best choice of the basic figure one has the problem of longer-ranged interactions. It is hoped that they are not too important, but whether it is safe to set them zero beyond the third neighbour, as is done here, remains an open question. To treat them by making the basic figure larger is only possible in principle since the calculational problems soon become insurmountable.

The method is not restricted to a triangular lattice or to the way we have drawn the cells on the lattice. The case treated had the advantage of a small cell (3 sites) and a complete occupation of the lattice by the cells. But one could allow for intermediate sites not belonging to any cell and consider these sites as part of the internal configuration. Various arrangements on a square lattice have been investigated partly and will be explored further. The case treated here has a shortcoming in the sense that the fixed-point hamiltonian has not the triangular symmetry, due to the arrangement of the cells chosen. One observes from the cell system that only rotations over $\pi/3$ are allowed and no reflections. This minor drawback shows up in the treatment of the five-cell approximation, which cluster is large enough to allow for the asymmetric interactions $4'_b$ and $4'_c$.

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Note added in proof. Subsequent calculations on the seven cluster approximation (hinted at in the discussion), *i.e.*, one central cell symmetrically surrounded by the six other cells, yields as main result: a thermal eigenvalue $\lambda_T = 1.7590$, a magnetic eigenvalue $\lambda_H = 1.8024$ and $K_c = 0.27416$, which purports a remarkable improvement over the smaller clusters treated here and confirms the conclusions of our paper. Numerical results, such as the strengths of the fixed-point hamiltonian and comparison with the present paper have been given in a short note[10]).

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