Module 3 : Superconductivity phenomenon

Lecture 2 : Solution of London equations and free energy calculations

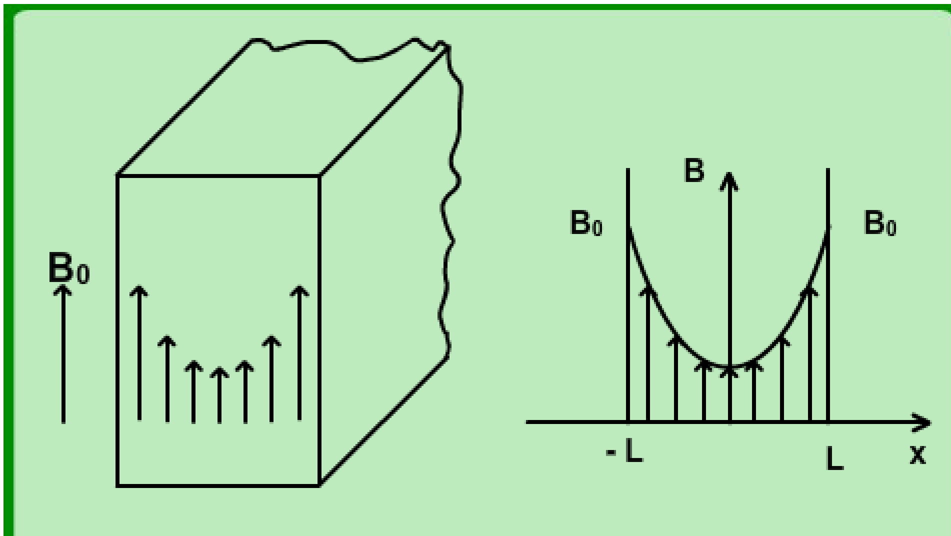
Solution of London Equations for sample cases

Flat slab in a magnetic field

Consider a flat superconducting slab of thickness d in a magnetic field H_{a} parallel to the slab. The boundary condition is that field match at $x = \pm d/2$. Subject to this condition, the solution to the London equation $\nabla^2 h = \frac{1}{\lambda^2} h$, where h is the microscopic value of the flux density, is easily determined to be a superposition of two exponentials. The result can be written as

$$h = H_a \frac{\cosh(x/\lambda)}{\cosh(d/2\lambda)} \qquad (13)$$

Clearly, the minimum value of the flux density is attained at the mid-plane of the slab where it has a value $H_a/\cosh(d/2\lambda)$.



Cross section of a superconducting slab in a magnetic field

The field variation inside a superconducting slab is shown.

Averaging this internal field over the sample thickness one gets

$$B \equiv \bar{h} = H_{\mathrm{a}} \frac{2\lambda}{d} \tanh(\frac{d}{2\lambda}) \qquad (14)$$

Let us consider the limit $d \gg \lambda$. This leads to $B = 0$ deep inside the superconductor. In the other limit, i.e., $d \ll \lambda$, we expand $\tanh(x) = x - x^3/3$. Therefore, B approaches $H_{\mathrm{a}}(1 - \frac{d^2}{12\lambda^2})$. Since $B = H_{\mathrm{a}} + 4\pi M$, we get

$$M = -\frac{H_{\mathrm{a}}}{4\pi} (\frac{d^2}{12\lambda^2}) \qquad (15)$$

As a consequence, magnetisation measurements can be made on thin films of known thicknesses and the penetration depth can be estimated from such measurements. Since the magnetisation is reduced below its Meissner value, the effective critical field for a thin sample is greater than that for bulk. The difference in the free energy between the normal state and the superconducting state is

$$(F_n - F_s)\,|_{H=0} = -\int_0^{H_c} M\,dH \qquad (16)$$

For the case of complete flux expulsion, the above difference in free energies is

$$\Delta F = -\int_0^{H_c} \frac{H}{4\pi} dH = \frac{H_c^2}{8\pi} \tag{17}$$

This energy, which stabilises the superconducting state is called the condensation energy and H_c is called the thermodynamic critical field.

For a thin film sample (with a field applied parallel to the plane) we get,

$$\Delta F = \frac{1}{4\pi}\frac{d^2}{12\lambda^2}\int_0^{H_{c\,\mathrm{par}}} H dH = \frac{H_{c\,\mathrm{par}}^2}{8\pi}\frac{d^2}{12\lambda^2} \tag{18}$$

In terms of the bulk thermodynamic critical field

$$\frac{H_{c\,\mathrm{par}}^2}{8\pi}\frac{d^2}{12\lambda^2} = \frac{H_c^2}{8\pi} \ \text{ or } \ H_{c\,par} = \frac{\sqrt{12}\lambda}{d}H_c \tag{19}$$

Critical current of a wire

Consider a long superconducting wire having a circular cross-section of radius a. Also, assume that $\lambda \gg a$. A current I is passed through the wire. This gives rise to a circumferential magnetic field at the surface of the wire $H = \frac{2I}{ca}$. In a simple minded picture, when this field reaches H_c, the wire will become normal. Therefore, the critical current $I_c = \frac{ca}{2}H_c$ depends linearly on the radius and not on the area. The current flows only in a surface layer of thickness λ. Hence, the current density $J_c \approx \frac{I_c}{2\pi a\lambda} = \frac{H_c}{2\pi a\lambda}\frac{ca}{2}$. Therefore, $J_c = \frac{c}{4\pi}\frac{H_c}{\lambda}$.

Free energy calculations

Now consider the case of a type I superconductor in a relatively large field. First we will carry out some calculations assuming zero demagnetisation factor. For a normal sample of volume V in a magnetic field H_a, the Helmholtz free energy is given by

$$F_n = Vf_{n0} = V\frac{H_a^2}{8\pi} + V_{\mathrm{ext}}\frac{H_a^2}{8\pi} \tag{20}$$

Here f_{n0} is the free energy density in zero applied field. V_{ext} is the volume external to the sample volume where the field H_a exists. On the other hand, for a superconductor, the field is excluded from its interior and hence its free energy is given by

$$F_s = Vf_{s0} + V_{\mathrm{ext}}\frac{H_a^2}{8\pi} \tag{21}$$

Here we have ignored the fact that the field actually penetrates in a layer of depth $\sim\lambda$ from the surface. The difference between the above two free energies is then

$$F_n - F_s = V(f_{n0} - f_{s0}) + V\frac{H_a^2}{8\pi} \tag{22}$$

Since the condensation energy density is the stabilisation energy

$$F_n - F_s = V\frac{H_c^2}{8\pi} + V\frac{H_a^2}{8\pi} \tag{23}$$

For $H_a = H_c$

$$(F_n - F_s)\ |_{H_c} = V\frac{H_c^2}{4\pi} \tag{24}$$

This is the energy increase (sample plus the surroundings) when a sample becomes normal at

$H_{\mathrm{a}} = H_{\mathrm{c}}$. The increase comes about because the energy source (generator) maintaining the constant field does work against the back emf. This emf is induced as the flux threading the sample changes (starts entering the bulk of the superconductor). Actually, discussion in terms of a Helmholtz free energy is appropriate for a situation where B is held constant (i.e., no induced emf). Here, we are holding H constant so the appropriate thermodynamic potential is the Gibbs free energy G. Recall that the Gibbs free energy density g is related to the Helmholtz free energy density f as follows

$$g = f - \frac{hH}{4\pi} \tag{25}$$

In the normal state, the local flux density h is equal to the average flux density B which is the same as the applied field H_{a}. Therefore we get

$$G_n = V f_{n0} - V\frac{H_{\mathrm{a}}^2}{8\pi} - V_{ext}\frac{H_{\mathrm{a}}^2}{8\pi} \tag{26}$$

In the superconducting state, flux is excluded from the superconductor, so $h = B = 0$. This gives

$$G_s = V f_{s0} - V_{\mathrm{ext}}\frac{H_{\mathrm{a}}^2}{8\pi} \tag{27}$$

The difference between the two free energies is then

$$G_n - G_s = V(f_{n0} - f_{s0}) - V\frac{H_{\mathrm{a}}^2}{8\pi} \tag{28}$$

For an applied field equal to the thermodynamic critical field, we get $G_n = G_s$ i.e., there will be a phase equilibrium between the normal and the superconducting phase at $H_{\mathrm{a}} = H_{\mathrm{c}}$.

Field variation for a non-zero demagnetisation factor

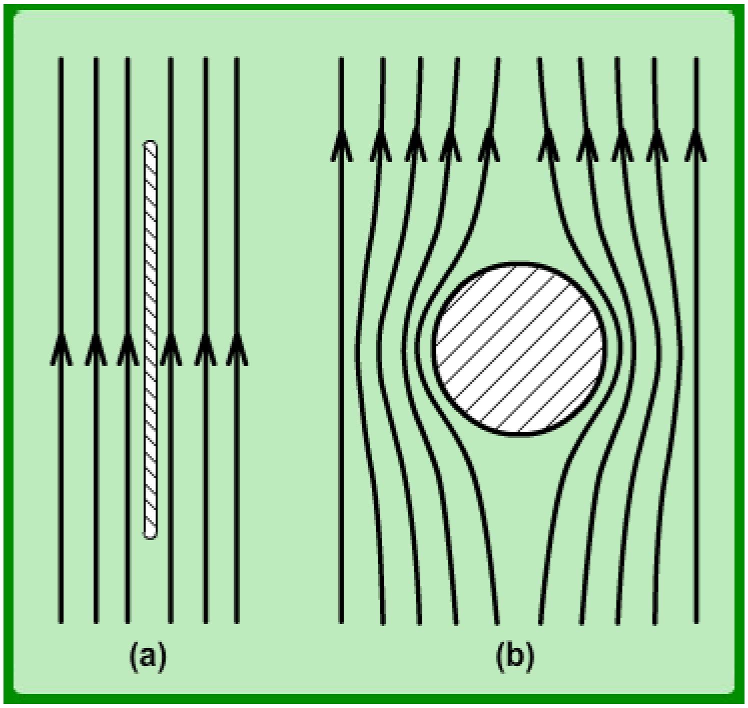
Consider a spherical sample of radius R. Outside the sphere, we have $\nabla \cdot B = 0$ and $\nabla \times B = 0$. Consequently $\nabla^2 B = 0$. Clearly, B approaches H_{a} as $r \to \infty$. Also, the perpendicular component of B is zero at $r = R$. The solution to to $\nabla^2 B = 0$ is then

$$B_{\mathrm{out}} = H_{\mathrm{a}} + \frac{H_{\mathrm{a}} R^3}{2}\vec{\nabla}(\frac{\cos\theta}{r^2}) \tag{29}$$

Therefore, the tangential component of B at the surface of the sphere can be calculated and is given below

$$B_\theta(r = R) = \frac{3}{2} H_{\mathrm{a}} \sin\theta \tag{30}$$

This exceeds the applied field at the equator. Even when the applied field is less than H_{c}, so long as it is greater than $\frac{2}{3}H_{\mathrm{c}}$, B can attain a value of H_{c} at the equator. Therefore, for $\frac{2}{3}H_{\mathrm{c}} < H_{\mathrm{a}} < H_{\mathrm{c}}$ there will be a coexistence of normal and superconducting regions. This has been called the ``intermediate'' state. Note that this is different from the ``mixed'' state which occurs at applied magnetic fields between H_{c1} and H_{c2}, even in the absence of demagnetisation effects.In general, for ellipsoidal samples (i.e., where a demagnetisation factor is well defined), when the applied field is in the range $1 - \eta < \frac{H_{\mathrm{a}}}{H_{\mathrm{c}}} < 1$(where η is the demagnetisation factor) an intermediate state will occur. The value of η for a sphere is $1/3$, for a flat plate with field perpendicular to it is 1, for a long cylinder with the field along the axis it is 0, and for a long cylinder with a field perpendicular to its axis it is $1/2$.



Contrast of exterior-field pattern
(a) when demagnetizing coefficient is nearly zero and
(b) when it is 1/3 for a sphere.
In (b) teh equatorial field is three halves the applied
field for the case of full Meissner effect, which is shown.

The field pattern inside a spherically shaped superconducting sample is shown. The larger concentration of field lines near the equator is a result of the demagnetisation factor.