Field Theory: A Modern Primer

Pierre Ramond

Institute for Fundamental Theory, Physics Department

University of Florida

To My Girls

Contents

2

How to Build an Action Functional

2.1 The Action Functional: Elementary Considerations

It is a most beautiful and awe-inspiring fact that all the fundamental laws of Classical Physics can be understood in terms of one mathematical construct called the Action. It yields the classical equations of motion, and analysis of its invariances leads to quantities conserved in the course of the classical motion. In addition, as Dirac and Feynman have shown, the Action acquires its full importance in Quantum Physics. As such, it provides a clear and elegant language to effect the transition between Classical and Quantum Physics through the use of the Feynman Path Integral (FPI).

Thus our task is clear: we first study the art of building acceptable Action functionals (AF) and later derive the quantum properties of the system a given AF describes by evaluating the associated Feynman Path Integral (FPI).

For a start, examine the AF for an elementary system: take a point particle, with position vector $x_i(t)$ $(i = 1, 2, 3)$, at a time t, moving in a time independent potential $V(x_i)$. The corresponding AF is given by

$$S\left([x_i]\,; t_1, t_2\right) \equiv \int_{t_1}^{t_2} dt \left(\frac{1}{2} m \frac{dx_i}{dt}\, \frac{dx_i}{dt} - V(x_i)\right) \quad . \tag{2.1}$$

It is a function of the initial and final times t_1 and t_2 and a functional of the path $x_i(t)$ for $t_1 < t < t_2$. Repeated Latin indices are summed over. This means that to a given path $x_i(t)$, we associate a number called the functional (in this case S). Functional relationship will be indicated by square brackets $[\cdots]$. For instance, the length of a path is a functional of the path.

Consider the response of S to a small deformation of the path

$$x_i(t) \to x_i(t) + \delta x_i(t) \quad . \tag{2.2}$$

Then

$$S\left[x_i + \delta x_i\right] = \int_{t_1}^{t_2} dt \left(\frac{1}{2} m \frac{d(x_i + \delta x_i)}{dt} \; \frac{d(x_i + \delta x_i)}{dt} - V(x_i + \delta x_i) \right) \quad . \tag{2.3}$$

Neglect terms of $O(\delta x)^2$ and use the chain rule to obtain

$$\frac{d(x_i + \delta x_i)}{dt} \; \frac{d(x_i + \delta x_i)}{dt} \simeq \frac{dx_i}{dt} \; \frac{dx_i}{dt} - 2 \frac{d^2 x_i}{dt^2} \delta x_i + 2 \frac{d}{dt} \left(\delta x_i \frac{dx_i}{dt} \right) \tag{2.4}$$

$$V(x_i + \delta x_i) \simeq V(x_i) + \delta x_i \partial_i V \quad . \tag{2.5}$$

[Here $\partial_i \equiv \frac{\partial}{\partial x_i}$]. Thus

$$S[x_i + \delta x_i] \simeq S[x_i] + \int_{t_1}^{t_2} dt \, \delta x_i \left(-\partial_i V - m \frac{d^2 x_i}{dt^2} \right) + m \int_{t_1}^{t_2} dt \frac{d}{dt} \left(\delta x_i \frac{dx_i}{dt} \right) \quad . \tag{2.6}$$

The last term is just a "surface" term. It can be eliminated by restricting the variations to paths which vanish at the end points: $\delta x_i(t_1) = \delta x_i(t_2) = 0$. With this *proviso* the requirement that S not change under an arbitrary δx_i leads to the classical equations of motion for the system. We symbolically write it as the vanishing of the functional derivative introduced by

$$S[x_i + \delta x_i] = S[x_i] + \int dt \, \delta x_i \frac{\delta S}{\delta x_i} + \cdots \quad . \tag{2.7}$$

That is

$$\frac{\delta S}{\delta x_i} = - \left(m \frac{d^2 x_i}{dt^2} + \partial_i V \right) = 0 \quad . \tag{2.8}$$

Thus we have the identification between equations of motion and extremization of S. Note, however, that extremization of S only leads to a class of possible paths. Which of those is followed depends on the boundary conditions, given as initial values of x_i and $\frac{dx_i}{dt}$.

A further, and most important point to be made is the correspondence between the symmetries of S and the existence of quantities conserved in the course of the motion of the system. An example will suffice. Take $V(x_i)$

to be a function of the length of x_i, *i.e.*, $r = (x_i x_i)^{1/2}$. Then S is manifestly invariant under a rotation of the three-vector x_i since it depends only on its length. Under an arbitrary infinitesimal rotation

$$\delta x_i = \epsilon_{ij} x_j \quad , \qquad \epsilon_{ij} = -\epsilon_{ji} \quad , \text{ with } \epsilon_{ij} \text{ time independent.} \tag{2.9}$$

Now, since S is invariant, we know that $\delta S = 0$, but as we have seen above, δS consists of two parts: the functional derivative which vanishes for the classical path, and the surface term. For this particular variation, however, we cannot impose boundary conditions on $\delta x_i(t)$, so the invariance of S together with the equations of motion yield

$$0 = \delta S = \int_{t_1}^{t_2} dt \frac{d}{dt} \left(m \frac{dx_i}{dt} \delta x_i \right) = \epsilon_{ij} m x_j \frac{dx_i}{dt} \bigg|_{t_1}^{t_2} \quad . \tag{2.10}$$

As this is true for any ϵ_{ij}, it follows that the

$$\ell_{ij}(t) \equiv m \left(x_i \frac{dx_j}{dt} - x_j \frac{dx_i}{dt} \right) \quad , \tag{2.11}$$

satisfy

$$\ell_{ij}(t_1) = \ell_{ij}(t_2) \quad , \tag{2.12}$$

and are therefore conserved during the motion. These are, as you know, the components of the angular momentum. An infinitesimal form of the conservation laws can be obtained by letting t_2 approach t_1. We have just proved in a simple case the celebrated theorem of Emmy Noether, relating an invariance (in this case, rotational) to a conservation law (of angular momentum). To summarize the lessons of this elementary example:

(i) 1) Classical equations of motion are obtained by extremizing S.

(ii) 2) Boundary conditions have to be supplied externally.

(iii) 3) The symmetries of S are in correspondence with conserved quantities and therefore reflect the basic symmetries of the physical system. This example dealt with particle mechanics: it can be generalized to

Classical Field Theory, as in Maxwell's Electrodynamics or Einstein's General Relativity.

The Action is just a mathematical construct, and therefore unlimited in its possibilities. Yet, it also affords a description of the physical world which we believe operates in a definite way. Hence, there should be *one* very special AF out of many which describes correctly what is going on. The problem is to find ways to characterize this unique Action. Noether's theorem gives us a hint since it allows us to connect the symmetries of the system with those of the functional. Certain symmetries, such as those implied by the Special Theory of Relativity, are well documented. Thus, any candidate action must reflect this fact. Other symmetries, such as electric charge conservation, further restrict the form of *the* AF. It is believed that Nature is partial to certain types of actions which are loaded with all kinds of invariances that vary from point to point. These give rise to the gauge theories which will occupy us later in this course. For the time being, let us learn how to build AF's for systems that satisfy the laws of the Theory of Special Relativity. Technically, these systems can be characterized by their invariance under transformations generated by the inhomogeneous Lorentz group, a.k.a., the Poincaré group, which is what will concern us next.

2.1.1 PROBLEMS

Notes: Problems are given in order of increasing complexity. Use the Action Functional as the main tool, although you may be familiar with more elementary methods of solution.

A. i) Prove that linear momentum is conserved during the motion described by $S = \int dt \frac{1}{2} m\dot{x}^2$, $\dot{x} = \frac{dx}{dt}$.

ii) If $V(x_i) = v\left(1 - \cos\frac{r}{a}\right)$, find the rate of change of the linear momentum.

B. For a point particle moving in an arbitrary potential, derive the expression for the rate of change in angular momentum.

*C. For a point particle moving in a potential $V = -\frac{a}{r}$, find the invariances of the AF. Hint: recall that the Newtonian orbits do not precess, which leads to a non-trivial conserved quantity, the Runge-Lenz vector.

*D. Given an AF invariant under uniform time translations, derive the expression for the associated conserved quantity. Use as an example a point particle moving in a time-independent potential. What happens if the potential is time-dependent?

2.2 The Lorentz Group (A Cursory Look)

The postulates of Special Relativity tell us that the speed of light is the same in all inertial frames. This means that if x_i is the position of a light signal at time t in one frame, and the same light ray is found at x'_i at time t' in another frame, we must have

$$s^2 \equiv c^2 t^2 - x_i x_i = c^2 t'^2 - x'_i x'_i \ . \tag{2.13}$$

The set of linear transformations which relate (x'_i, t') to (x_i, t) while preserving the above expressions form a group called the Lorentz group (see problem). Choose units such that $c = 1$, and introduce the notation

$$x^\mu \quad \mu = 0,\, 1,\, 2,\, 3 \text{ with } x^0 = t, \quad \left(x^1,\, x^2,\, x^3\right) = \left(x^i\right) = \vec{x} \ \ ,$$

i.e.

$$\begin{aligned} x^\mu &= (x^0, x^i) \quad i = 1,\, 2,\, 3 \\ &= (t, \vec{x}). \end{aligned}$$

In this compact notation, s^2 can be written as

$$s^2 = x^0 x^0 - x^i x^i \equiv x^\mu x^\nu g_{\mu\nu} \ , \tag{2.14}$$

where the metric $g_{\mu\nu} = g_{\nu\mu}$ is zero except for $\mu = \nu$ when $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$. Repeated indices are summed except when otherwise indicated. Then Eq. (2.13) becomes

$$g_{\mu\nu} x^\mu x^\nu = g_{\mu\nu} x'^\mu x'^\nu \ . \tag{2.15}$$

Now look for a set of linear transformations

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu = \Lambda^\mu{}_0 x^0 + \Lambda^\mu{}_i x^i \ , \tag{2.16}$$

which preserves s^2. The $\Lambda^\mu{}_\nu$ must therefore satisfy

$$g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho x^\sigma = g_{\rho\sigma} x^\rho x^\sigma \ . \tag{2.17}$$

As (2.17) must hold for any x^μ, we conclude that

$$g_{\rho\sigma} = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \ \ . \tag{2.18}$$

It is more convenient for certain purposes to use a matrix notation: regard x^μ as a column vector $\mathbf{x}$ and $g_{\mu\nu}$ as a matrix $\mathbf{g}$. Then

$$s^2 = \mathbf{x}^T \mathbf{g} \mathbf{x} \quad , \tag{2.19}$$

and

$$\mathbf{x}' = \mathbf{L}\mathbf{x} \quad , \tag{2.20}$$

where $\mathbf{L}$ is the matrix equivalent of the $\Lambda^\mu{}_\nu$ coefficients, and T means the transpose. The $\mathbf{L}$'s must obey

$$\mathbf{g} = \mathbf{L}^T \mathbf{g} \mathbf{L} \ , \tag{2.21}$$

to be Lorentz transformations (LT). Examine the consequences of Eq. (2.21). First take its determinant

$$\det \mathbf{g} = \det \mathbf{L}^T \det \mathbf{g} \det \mathbf{L} \ , \tag{2.22}$$

from which we deduce that

$$\det \mathbf{L} = \pm 1 \ . \tag{2.23}$$

The case $\det \mathbf{L} = 1(-1)$ corresponds to proper (improper) LT's. As an example, the LT given numerically by $\mathbf{L} = \mathbf{g}$ is an improper one; physically it corresponds to $x^0 \to x^0$, $x^i \to -x^i$, *i.e.*, space inversion. Second, take the 00 entry of Eq. (2.19)

$$1 = \Lambda^\rho{}_0 g_{\rho\sigma} \Lambda^\sigma{}_0 = \left(\Lambda^0{}_0\right)^2 - \left(\Lambda^i{}_0\right)^2 \quad , \tag{2.24}$$

which shows that

$$\left|\Lambda^0{}_0\right| \geq 1 \ . \tag{2.25}$$

When $\Lambda^0{}_0 \geq 1$, the LT's are said to be orthochronous, while $\Lambda^0{}_0 \leq -1$ gives non-orthochronous LT's. It follows then that LT's can be put in four categories (see problem):

1) proper orthochronous, called restricted ($L_+^\uparrow$) with $\det \mathbf{L} = +1$, $\Lambda^0{}_0 \geq 1$
2) proper non-orthochronous ($L_+^\downarrow$) with $\det \mathbf{L} = +1$, $\Lambda^0{}_0 \leq -1$
3) improper orthochronous ($L_-^\uparrow$) with $\det \mathbf{L} = -1$, $\Lambda^0{}_0 \geq 1$
4) improper non-orthochronous ($L_-^\downarrow$) with $\det \mathbf{L} = -1$, $\Lambda^0{}_0 \leq -1$.

Let us give a few examples:

a) Rotations: $x'^0 = x^0$, $x'^i = a^{ij}x^j$ with a^{ij} an orthogonal matrix. Then we can write L in the block form

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{a} \end{pmatrix} \tag{2.26}$$

so that $\det \mathbf{L} = \det \mathbf{a}$. We can have $\det \mathbf{a} = \pm 1$, corresponding to proper and improper rotations, with $\mathbf{L}$ belonging to $L_+^\uparrow$ and $L_-^\uparrow$, respectively.

b) Boosts: the transformations

$$x'^0 = x^0 \cosh\eta - x^1 \sinh\eta \tag{2.27}$$
$$x'^1 = -x^0 \sin\eta + x^1 \cosh\eta \tag{2.28}$$
$$x'^{2,3} = x^{2,3} \tag{2.29}$$

describe a boost in the 1-direction. Then in (2×2) block form

$$\mathbf{L} = \begin{pmatrix} \begin{matrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{matrix} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} , \tag{2.30}$$

$$\det \mathbf{L} = \cosh^2\eta - \sinh^2\eta = 1 , \tag{2.31}$$
$$\Lambda^0{}_0 = \cosh\eta \geq 1 . \tag{2.32}$$

This transformation therefore belongs to $L_+^\uparrow$. Note that the identifications

$$\cosh\eta = \left(1 - v^2\right)^{-1/2} \quad , \qquad \sinh\eta = v\left(1 - v^2\right)^{-1/2} , \tag{2.33}$$

where v is the velocity of the boosted frame leads to the more familiar form.

c) Time inversion: defined by $x'^0 = -x^0$, $x'^i = x^i$. It has $\det \mathbf{L} = -1$, $\Lambda^0{}_0 = -1$, and therefore belongs to $L_-^\downarrow$.

d) Full inversion: defined by $x'^\mu = -x^\mu$. It has $\det \mathbf{L} = +1$, $\Lambda^0{}_0 = -1$, and belongs to $L_+^\downarrow$. Full inversion can be obtained as the product of a space and a time inversion.

Any Lorentz transformation can be decomposed as the product of transformation of these four types (see problem). Thus, it suffices to concentrate on rotations and boosts. Since there are three rotations and three boosts, one for each space direction, the Lorentz transformations are described in terms of six parameters. We now proceed to build the six corresponding generators. Consider an infinitesimal LT

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \epsilon^{\mu}{}_{\nu} \ , \tag{2.34}$$

where $\delta^{\mu}{}_{\nu}$ is the Kronecker delta which vanishes when $\mu \neq \nu$ and equals $+1$ otherwise. Evaluation of Eq. (2.18) yields to $O(\epsilon)$

$$0 = g_{\nu\rho}\epsilon^{\rho}{}_{\mu} + g_{\mu\rho}\epsilon^{\rho}{}_{\nu} \ . \tag{2.35}$$

We use the metric $g_{\mu\nu}$ to lower indices, for example

$$x_{\mu} \equiv g_{\mu\nu}x^{\nu} = \left(x^0, -\vec{x}\right) \ . \tag{2.36}$$

Eq. (2.34) becomes

$$0 = \epsilon_{\nu\mu} + \epsilon_{\mu\nu} \quad , \tag{2.37}$$

that is, $\epsilon_{\mu\nu}$ is an antisymmetric tensor with (as advertised) $\frac{4\cdot 3}{2} = 6$ independent entries. Introduce the Hermitian generators

$$L_{\mu\nu} \equiv i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right) \quad , \tag{2.38}$$

where

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \quad . \tag{2.39}$$

In terms of these we can write

$$\delta x^{\mu} = \frac{i}{2}\epsilon^{\rho\sigma}L_{\rho\sigma}x^{\mu} = \epsilon^{\mu\rho}x_{\rho} \ . \tag{2.40}$$

It is easy to see that the $L_{\mu\nu}$'s satisfy a Lie algebra

$$[L_{\mu\nu}, L_{\rho\sigma}] = ig_{\nu\rho}L_{\mu\sigma} - ig_{\mu\rho}L_{\nu\sigma} - ig_{\nu\sigma}L_{\mu\rho} + ig_{\mu\sigma}L_{\nu\rho} \quad , \tag{2.41}$$

to be identified with the Lie algebra of $SO(3,1)$. The most general representation of the generators of $SO(3,1)$ that obeys the commutation relations (2.41) is given by

$$M_{\mu\nu} \equiv i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) + S_{\mu\nu} \ , \tag{2.42}$$

where the Hermitian $S_{\mu\nu}$ satisfy the same Lie algebra as the $L_{\mu\nu}$ and commute with them. The Hermitian generators M_{ij} for i, $j = 1, 2, 3$ form an algebra among themselves

$$[M_{ij}, M_{k\ell}] = -i\delta_{jk}M_{i\ell} + i\delta_{ik}M_{j\ell} + i\delta_{j\ell}M_{ik} - i\delta_{i\ell}M_{jk} \ , \tag{2.43}$$

which is that of the rotation group $SU(2)$. A more familiar expression can be obtained by introducing the new operators

$$J_i \equiv \frac{1}{2}\epsilon_{ijk}M_{jk} \ , \tag{2.44}$$

where ϵ_{ijk} is the Levi-Civita symbol, totally antisymmetric in all of its three indices, and with $\epsilon_{123} = +1$. (Repeated Latin indices are summed over.) Then, we find

$$[J_i, J_j] = i\epsilon_{ijk}J_k \ . \tag{2.45}$$

Define the boost generators

$$K_i \equiv M_{0i} \ . \tag{2.46}$$

It follows from the Lie algebra that

$$[K_i, K_j] = -i\epsilon_{ijk}J_k \ , \tag{2.47}$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k \ . \tag{2.48}$$

Here, both K_i and J_i are Hermitian generators; the K_i are non-compact generators. We can disentangle these commutation relations by introducing the new linear combinations

$$N_i \equiv \frac{1}{2}\left(J_i + iK_i\right) \ . \tag{2.49}$$

Although not Hermitian, $N_i \neq N_i^\dagger$, they have the virtue of yielding simple commutation relations

$$\left[N_i, N_j^\dagger\right] = 0 \ , \tag{2.50}$$

$$[N_i, N_j] = i\epsilon_{ijk}N_k \ , \tag{2.51}$$

$$\left[N_i^\dagger, N_j^\dagger\right] = i\epsilon_{ijk}N_k^\dagger \ , \tag{2.52}$$

This means that the N_i and the $N_i^\dagger$ independently obey the Lie algebra of $SU(2)$. We can therefore appeal to its well-known representation theory. In particular, we have two Casimir operators (operators that commute with all the generators)

$$N_iN_i \qquad \text{with eigenvalues } n(n+1)$$

and

$$N_i^\dagger N_i^\dagger \qquad \text{with eigenvalues } m(m+1) \ .$$

where $m, n = 0,\ 1/2,\ 1,\ 3/2,\ \cdots$, using well-known results from the representation theory of the $SU(2)$ (spin) group. These representations are labelled by the pair $(\mathbf{n}, \mathbf{m})$ while the states within a representation are further distinguished by the eigenvalues of N_3 and $N_3^\dagger$. Observe that the two $SU(2)$'s are not independent as they can be interchanged by the operation of parity, P:

$$J_i \to J_i \ , \qquad K_i \to -K_i \ ,$$

and the operation of Hermitian conjugation which changes the sign of i and therefore switches N_i to $N_i^\dagger$. In general, representations of the Lorentz group are neither parity nor (Hermitian) conjugation eigenstates. Since $J_i = N_i + N_i^\dagger$, we can identify the spin of the representation with $m + n$. As an example, consider the following representations:

a) $(\mathbf{0}, \mathbf{0})$ with spin zero is the scalar representations, and it has a well-defined parity (can appear as scalar or pseudoscalar):

b) $(\mathbf{\frac{1}{2}}, \mathbf{0})$ has spin $\frac{1}{2}$ and represents a left-handed spinor (the handedness is a convention):

c) $(\mathbf{0}, \mathbf{\frac{1}{2}})$ describes a right-handed spinor.

These spinors have two components ("spin up" and "spin down"); they are called Weyl spinors. When parity is relevant, one considers the linear combination $(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}) \oplus (\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0})$, which yields a Dirac spinor.

The fun thing is that given these spinor representations, we can generate any other representation by multiplying them together. This procedure is equivalent to forming higher spin states by taking the (Kronecker) product of many spin $\frac{1}{2}$ states in the rotation group. Let us give a few examples:

a) $(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}) \otimes (\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}) = (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}})$ gives a spin 1 representation with four components. In tensor notation it will be written as a 4-vector.

b) $(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}) \otimes (\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}) = (\mathbf{0}, \mathbf{0}) \oplus (\mathbf{1}, \mathbf{0})$. Here the scalar representation is given by the antisymmetric product. The new representation $(\mathbf{1}, \mathbf{0})$ is represented by an antisymmetric, self-dual second rank tensor, *i.e.*, a tensor $B_{\mu\nu}$ which obeys

$$B_{\mu\nu} = -B_{\nu\mu} \tag{2.53}$$

$$B_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma} B_{\rho\sigma} \ , \tag{2.54}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol in four dimensions with $\epsilon^{0123} = +1$, and total antisymmetry in its indices. The $(\mathbf{0}, \mathbf{1})$ representation would correspond to a tensor that is antiself-dual

$$B_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma} B_{\rho\sigma} \ . \tag{2.55}$$

For example, Maxwell's field strength tensor $F_{\mu\nu}$ transforms under the Lorentz group as $(\mathbf{0}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{0})$.

Finally, let us emphasize an important point. Suppose that we had considered LT's in the so-called "Euclidean space", where t is replaced by $\sqrt{-1}\,t$. Then the commutation relations would have gone through except that $g_{\mu\nu}$ would have been replaced by $-\delta_{\mu\nu}$, the Kronecker delta, giving the Lie algebra of $SO(4)$, the rotation group in four dimensions. The split-up of two commuting $SU(2)$ groups is now achieved with the Hermitian combinations $J_i \pm K_i$. These two $SU(2)$'s are completely independent since they cannot be switched by conjugation. Parity can still relate the two, but it loses much of its interest in Euclidean space where all directions are equivalent.

2.2.1 PROBLEMS

A. Show that the Lorentz transformations satisfy the group axioms, *i.e.*, if $\mathbf{L}_1$ and $\mathbf{L}_2$ are two LT's so is $\mathbf{L}_1\mathbf{L}_2$; the identity transformation exists, and if $\mathbf{L}$ is an LT, so is its inverse $\mathbf{L^{-1}}$.

B. Show that $\det \mathbf{L}$ and the sign of $\Lambda^0{}_0$ are Lorentz-invariant, and can therefore be used to catalog the Lorentz transformations.

C. Show that if $\mathbf{L}$ is restricted LT ($\det \mathbf{L} = +1$, $\Lambda^0{}_0 \geq 0$), all Lorentz transformations can be written in the forms

$$\begin{aligned}
&\mathbf{L} \quad \times \quad \text{space inversion for } L_-^{\uparrow} \ , \\
&\mathbf{L} \quad \times \quad \text{time inversion for } L_-^{\downarrow} \ , \\
&\mathbf{L} \quad \times \quad \text{space inversion} \times \text{time inversion for } L_+^{\downarrow} \ .
\end{aligned}$$

D. Show that a restricted Lorentz transformation can be uniquely written as the product of a boost and a rotation.

*E. Index shuffling problem: Show that the components of a self-dual antisymmetric second rank tensor transform among themselves, *i.e.*, irreducibly under the Lorentz group.

2.3 The Poincaré Group

Another fundamental principle is the invariance of the behavior of an isolated physical system under uniform translations in space and time. (This principle has to be extended to include arbitrary translations to generate gravitational interactions) Such a transformation is given by

$$x^\mu \to x'^\mu = x^\mu + a^\mu \ , \tag{2.56}$$

where a^μ is an arbitrary constant four-vector. Hence the general invariance group is a ten-parameter group called the Poincaré group, under which

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \ . \tag{2.57}$$

The translations (2.56) do not commute with the LT's. Indeed consider two successive PG transformations with parameters (Λ_1, a_1) and (Λ_2, a_2)

$$x^\mu \to \Lambda^\mu_{1\nu} x^\nu + a^\mu_1 \to \Lambda^\mu_{2\rho}\Lambda^\rho_{1\nu} x^\nu + \Lambda^\mu_{2\rho} a^\rho_1 + a^\mu_2 \ , \tag{2.58}$$

and we see that the translation parameters get rotated. Nothing surprising here since this is what four-vectors do for a living! Such a coupling of the translation and Lorentz groups is called a semi-direct product. Still, as indicated by their name, the PG transformations form a group (see problem). In order to obtain the algebra of the generators, observe that we can write the change in x under a small translation as

$$\delta x^\mu = \epsilon^\mu = i\epsilon^\rho P_\rho x^\mu \ , \tag{2.59}$$

where ϵ^μ are the parameters, and

$$P_\rho = -i\partial_\rho \ , \tag{2.60}$$

are the Hermitian generators of the transformation. They clearly commute with one another

$$[P_\mu, P_\nu] = 0 \ , \tag{2.61}$$

but not with the Lorentz generators (how can they? they are four-vectors!),

$$[M_{\mu\nu}, P_\rho] = -ig_{\mu\rho} P_\nu + ig_{\nu\rho} P_\mu \ . \tag{2.62}$$

The commutation relations (2.61), (2.62) and those among the $M_{\mu\nu}$ define the Lie algebra of the Poincaré group. The "length" $P_\mu P^\mu$ of the vector P_ρ is obviously invariant under Lorentz transformations and in view of (2.61) is therefore a Casimir operator. The other Casimir operator is not so obvious to construct, but as we just remarked, the length of any four-vector which commutes with the P_ν's will do. The Pauli-Lubanskí four-vector, W^μ, is such a thing; it is defined by

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \ . \tag{2.63}$$

Use of (2.61), (2.62) and of the antisymmetry of the Levi-Civita symbol gives

$$[W^\mu, P^\rho] = 0 \ , \tag{2.64}$$

while W_ρ transforms as a four-vector,

$$[M_{\mu\nu}, W_\rho] = -ig_{\mu\rho}W_\nu + ig_{\nu\rho}W_\mu \ . \tag{2.65}$$

Its length $W^\mu W_\mu$ is therefore a Casimir invariant. The most general representation of the ten Poincaré group generators is

$$\begin{aligned} P_\rho &= -i\partial_\rho \ , \\ M_{\mu\nu} &= i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) + S_{\mu\nu} \ , \end{aligned}$$

so that

$$W^\mu = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}S_{\rho\sigma}\partial_\nu \ . \tag{2.66}$$

The representation theory of the Poincaré group was worked out long ago by E. Wigner. Its unitary representations fall into four classes.

1) The eigenvalue of $P_\rho P^\rho \equiv m^2$ is a real positive number. Then the eigenvalue of $W_\rho W^\rho$ is $-m^2 s(s+1)$, where s is the spin, which assumes discrete values $s = 0,\ \frac{1}{2},\ 1,\ \cdots$. This representation is labelled by the mass m and the spin s. States within the representation are distinguished by the third component of the spin $s_3 = -s,\ -s+1, \cdots, s-1,\ s$, and the continuous eigenvalues of P_i. Physically a state corresponds to a particle of mass m, spin s, three-momentum P_i and spin projection s_3. Massive particles of spin s have $(2s+1)$ degrees of freedom. The massive neutrinos belong to this category

2) The eigenvalues of $P_\rho P^\rho$ is equal to zero, corresponding to a particle of zero rest mass. $W_\rho W^\rho$ is also zero and, since $P^\rho W_\rho = 0$, it follows that W_μ and P_μ are proportional. The constant of proportionality is called the helicity λ is equal to $\pm s$, where $s = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ \cdots$ is the spin of the representation. They describe massless particles with spin have two degrees of freedom. They are further distinguished by the three values of their momenta along the x, y and z directions. Examples of particles in this category are the photon with spin 1, its two states of polarization labelled by helicities $\lambda = \pm 1$, the graviton with two states of polarization $\lambda = \pm 2$.

3) $P_\rho P^\rho = 0$, $W^\mu W_\mu < 0$ and continuous spin. The are only two representations of this type, describing a "particle" of zero rest mass, one with infinite number of integer helicities $\lambda = ... - 2, -1, 0, +1, +2, ...$, and the other with half-odd integer helicities $\lambda = ... - 3/2, -1/2, +1/2, +3/2, ...$ These do not seem to be realized in nature.

4) $P_\rho P^\rho < 0$, tachyon representations which describe "particles" moving faster than light. There is no evidence for tachyons in nature.

For more details on these, see V. Bargman and E. P. Wigner, *Proceedings of the National Academy of Sciences*, Vol. 34, No. 5, 211 (1946).

There are other representations of the Poincaré group; however they are not unitary. Quantum Mechanics allows for the identification of only the unitary representations with particle states. The Wigner representations are infinite dimensional, corresponding to particles with unbounded momenta. The situation is to be compared with that of the Lorentz group where we discussed finite dimensional but non-unitary representations. The introduction of fields will enable us to make use of these representations.

2.3.1 PROBLEMS

A. Show that the transformations (2.56) form a group.

B. Show that when $P_\rho P^\rho = m^2 > 0$, the eigenvalue of $W_\rho W^\rho$ is indeed given by $-m^2 s(s+1)$.

*C. Find the representation of the Poincaré group generators on the space like surface $x_0 = 0$ in the case $m^2 = 0$, $s = 0$. Hint: by setting $x_0 = 0$, one has to re-express its conjugate variable P_0 in terms of the remaining variables. Use a Casimir operator to do this. Then re-express all of the Poincaré group generators in terms of x_i, P_i and m^2. See P. A. M. Dirac, *Rev. Mod. Phys.* **21**, 392 (1949).

*D.Repeat the previous problem on the spacelike surface $x^0 = x^3$.

**E. Repeat problem D when $m^2 > 0$ and $s \neq 0$.

2.4 Behavior of Local Fields under the Poincaré Group

Consider an arbitrary function of the spacetime point P. In a given inertial frame, where P is located at x^μ, this function will be denoted by $f(x^\mu)$; in another where P is at x'^μ it will be written as $f'(x'^\mu)$ because the functional relationship will in general be frame-dependent. Write for an infinitesimal transformation the change in the function as

$$\begin{aligned} \delta f &= f'(x') - f(x) \ , \\ &= f'(x + \delta x) - f(x) \ , \\ &= f'(x) - f(x) + \partial x^\mu \partial_\mu f' + O(\delta x^2) \ . \end{aligned} \tag{2.67}$$

To $\mathcal{O}(\delta x^\mu)$, we can replace $\partial_\mu f'$ by $\partial_\mu f$. Then

$$\delta f = \delta_0 f + \delta x^\mu \partial_\mu f \quad , \tag{2.68}$$

where we have introduced the functional change at the same x

$$\delta_0 f \equiv f'(x) - f(x) \ . \tag{2.69}$$

The second term in Eq. (2.68) is called the transport term. We can formally write (2.68) as an operator equation

$$\delta = \delta_0 + \delta^\mu \partial_\mu \ . \tag{2.70}$$

Under spacetime translations, there is no change in a local field, that is

$$\delta f = 0 = \delta_0 f + \epsilon^\mu \partial_\mu f \tag{2.71}$$

from which

$$\delta_0 f = -\epsilon^\mu \partial_\mu f = -i\epsilon^\mu P_\mu f \ , \tag{2.72}$$

with P_μ defined by (2.60). However, under Lorentz transformations, the situation is more complicated and requires several examples for clarification.

a) The Scalar Field

We build (or imagine) a function of x^μ, $\varphi(x)$, which has the same value when measured in different inertial frames related by a Lorentz transformation

$$\varphi'(x') = \varphi(x) \ . \tag{2.73}$$

This condition defines a scalar field (under LT's). Specializing to an infinitesimal transformation, we have, using (2.74) and (2.68)

$$0 = \delta\varphi = \delta_0\varphi + \delta x^\mu \partial_\mu \varphi \ , \tag{2.74}$$

with δx^μ given by (2.40). Setting

$$\delta_0 \varphi = -\frac{i}{2} \epsilon^{\rho\sigma} M_{\rho\sigma} \varphi \ , \tag{2.75}$$

and comparing with (2.74) tells us that for a scalar field the representation of the Lorentz group generators $M_{\mu\nu}$ is just $i(x_\mu \partial_\nu - x_\nu \partial_\mu)$. That is, the operator $S_{\mu\nu}$ we had introduced earlier vanishes when acting on a scalar field. We can see how a non-trivial $S_{\mu\nu}$ can arise by considering the construct $\partial_\mu \varphi(x)$. Note that it is a scalar under translations just as φ was, because the derivative operator is not affected by translations, (true for uniform translations only!). We have

$$\delta \partial_\mu \varphi = [\delta, \partial_\mu] \, \varphi + \partial_\mu \delta \varphi \ . \tag{2.76}$$

Now $\delta\varphi$ vanishes since φ is a Lorentz-scalar. However, from (2.70) we see that

$$[\delta, \partial_\mu] = [\delta_0, \partial_\mu] + [\delta x^\nu \partial_\nu, \partial_\mu] \ . \tag{2.77}$$

Since δ_0 does not change x^μ, it commutes with ∂_μ, but δx^ν does not. Evaluation of the last commutator yields

$$[\delta, \partial_\mu] = \epsilon_\mu{}^\nu \partial_\nu \ . \tag{2.78}$$

Putting it all together we find

$$\delta_0 \partial_\mu \varphi = -\frac{1}{2} \epsilon^{\rho\sigma} L_{\rho\sigma} \partial_\mu \varphi - \frac{i}{2} \left(\epsilon^{\rho\sigma} S_{\rho\sigma} \right)_\mu{}^\nu \partial_\nu \varphi \ , \tag{2.79}$$

where

$$\left(S_{\rho\sigma} \right)_\mu{}^\nu = i \left(g_{\rho\mu} g^\nu{}_\sigma - g_{\sigma\mu} g^\nu{}_\rho \right) \ . \tag{2.80}$$

One can check that they obey the same commutation relations as the $L_{\mu\nu}$'s. Comparison with the canonical form

$$\delta_0(\text{anything}) = -\frac{i}{2} \epsilon^{\rho\sigma} M_{\rho\sigma}(\text{anything}) \ , \tag{2.81}$$

yields the representation of the Lorentz generator on the field $\partial_\mu \varphi$. A field transforming like $\partial_\mu \varphi(x)$ is called a vector field. Note that the role of the "spin part" of $M_{\mu\nu}$ is to shuffle indices.

A tensor field with many Lorentz indices will transform like (2.79). The action of $S_{\rho\sigma}$ on it will be the sum of expressions like (2.80), one for each index. For instance, the action of $S_{\rho\sigma}$ on a second rank tensor $B_{\mu\nu}$ is given by

$$(S_{\rho\sigma}B)_{\mu\nu} = -i\left(g_{\sigma\mu}B_{\rho\nu} + g_{\rho\nu}B_{\sigma\mu} + g_{\sigma\nu}B_{\mu\rho} - g_{\rho\mu}B_{\nu\sigma}\right) \ . \tag{2.82}$$

It is now easy to make Poincaré invariants out of scalar fields. Candidates are any scalar function of $\varphi(x)$ such as φ^n, $\cos\varphi(x)$, etc. $\cdots$, $\partial_\mu\partial^\mu\varphi(x)$, $(\partial_\mu\varphi)(\partial^\mu\varphi)$ (see problem) etc ... . Note that while the expression $x^\mu\partial_\mu\varphi$ is Lorentz invariant, it is not Poincaré invariant.

b) The Spinor Fields

The spinor representations of the Lorentzgroup $\left(\frac{\mathbf{1}}{\mathbf{2}},\mathbf{0}\right)$ and $\left(\mathbf{0},\frac{\mathbf{1}}{\mathbf{2}}\right)$ are realized by two-component complex spinors. Call these spinors $\psi_L(x)$ and $\psi_R(x)$, respectively. The two-valued spinor indices are not written explicitly. [In the literature L-like (R-like) spinor indices appear dotted (undotted).] We write

$$\begin{aligned} \psi_L(x) \to \psi_L'(x') &= \mathbf{\Lambda}_L\psi_L(x) \ \text{ for } \left(\frac{\mathbf{1}}{\mathbf{2}},\mathbf{0}\right) \\ \psi_R(x) \to \psi_R'(x') &= \mathbf{\Lambda}_R\psi_R(x) \ \text{ for } \left(\mathbf{0},\frac{\mathbf{1}}{\mathbf{2}}\right) \ , \end{aligned}$$

where $\mathbf{\Lambda}_{R,L}$ are (2×2) matrices with complex entries. When the transformation is a rotation we know the form of $\mathbf{\Lambda}_{L,R}$ from the spinor representation of $SU(2)$

$$\mathbf{\Lambda}_{L(R)} = e^{i\frac{\vec\sigma}{2}\cdot\vec\omega} \ . \ (rotation) \tag{2.83}$$

The ω^i are the rotation parameters and the σ^i are the Hermitian 2×2 Pauli spin matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{2.84}$$

They obey

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k \ . \tag{2.85}$$

After thus identifying the rotation generators J_i with $\frac{1}{2}\sigma^i$, we have to write the non-compact boost generators in this (2×2) notation. The K_i cannot be represented unitarily because unitary representations of non-compact groups such as the Lorentz group are infinite-dimensional. The representation

$$\vec{K} = -\frac{i}{2}\vec{\sigma} \ , \tag{2.86}$$

satisfies all the required commutation relations. So we write

$$\mathbf{\Lambda}_L = e^{\frac{i}{2}\vec{\sigma}\cdot(\vec{\omega}-i\vec{\nu})} \ , \tag{2.87}$$

where $\vec{\nu}$ are the boost parameters associated with $\vec{K}$. Since the $\left(\mathbf{\frac{1}{2}},\mathbf{0}\right)$ and $\left(\mathbf{0},\mathbf{\frac{1}{2}}\right)$ representations are related by parity, we construct $\mathbf{\Lambda}_R$ from $\mathbf{\Lambda_L}$ by changing the sign of the boost parameters:

$$\mathbf{\Lambda}_R = e^{\frac{i}{2}\vec{\sigma}\cdot(\vec{\omega}+i\vec{\nu})} \ . \tag{2.88}$$

These explicit forms for $\mathbf{\Lambda}_L$ and $\mathbf{\Lambda}_R$ enable us to describe important properties. First, we see that $\mathbf{\Lambda}_L$ and $\mathbf{\Lambda}_R$ are related by

$$\mathbf{\Lambda}_L^{-1} = \mathbf{\Lambda}_R^{\dagger} \ . \tag{2.89}$$

Secondly, the magic of the Pauli matrices

$$\sigma^2\sigma^i\sigma^2 = -\sigma^{i\ *} \ , \tag{2.90}$$

where the star denotes ordinary complex conjugation, enables us us to write

$$\sigma^2\mathbf{\Lambda}_L\sigma^2 = e^{-\frac{i}{2}\vec{\sigma}^*\cdot(\vec{\omega}-i\vec{\nu})} = \mathbf{\Lambda}_R{}^* \ . \tag{2.91}$$

Thirdly, the Hermitian conjugate equation of (1.4.25) with the Hermiticity of the Pauli matrices yields

$$\mathbf{\Lambda}_L^T = \sigma^2\mathbf{\Lambda}_L^{-1}\sigma^2 \ , \tag{2.92}$$

whence

$$\sigma^2\mathbf{\Lambda}_L^T\sigma^2\mathbf{\Lambda}_L = 1 \qquad \text{or} \qquad \mathbf{\Lambda}_L^T\sigma^2\mathbf{\Lambda}_L = \sigma^2 \ . \tag{2.93}$$

The same equation holds for $\mathbf{\Lambda}_R$. These relations will prove useful in the construction of Lorentz-invariant expressions involving spinor fields. As a first application, under a Lorentz transformation,

$$\begin{aligned}\sigma^2\psi_L^* &\to \sigma^2\mathbf{\Lambda}_L^*\psi_L^* \\ &= \sigma^2\mathbf{\Lambda}_L^*\sigma^2\sigma^2\psi_L^* \\ &= \mathbf{\Lambda}_R\sigma^2\psi_L^* \ ,\end{aligned} \tag{2.94}$$

using the complex conjugate of (2.91). Equation (2.93) indicates that given a spinor ψ_L which transforms as $\left(\frac{\mathbf{1}}{\mathbf{2}},\mathbf{0}\right)$, we can construct a related spinor $\sigma^2\psi_L^*$ which transforms as $\left(\mathbf{0},\frac{\mathbf{1}}{\mathbf{2}}\right)$. In a similar way we can see that $\sigma^2\psi_R^*$ transforms as $\left(\frac{\mathbf{1}}{\mathbf{2}},\mathbf{0}\right)$ if ψ_R transforms as $\left(\mathbf{0},\frac{\mathbf{1}}{\mathbf{2}}\right)$.

We noted earlier that by taking the antisymmetric product of two $\left(\frac{\mathbf{1}}{\mathbf{2}},\mathbf{0}\right)$ representations we can construct the scalar representation. Now we can now show it explicitly. Let ψ_L and χ_L be two spinors that transform as $\left(\frac{\mathbf{1}}{\mathbf{2}},\mathbf{0}\right)$. As a consequence of (2.93), under a Lorentz transformation

$$\chi_L^T\sigma^2\psi_L \to \chi_L^T\mathbf{\Lambda}_L^T\sigma^2\mathbf{\Lambda}_L\psi_L = \chi_L^T\sigma^2\psi_L \ . \tag{2.95}$$

This is our scalar. Group theory tells us that the scalar representation is in the antisymmetric product, so by taking $\chi_L = \psi_L$, the scalar invariant should not exist. Explicitly,

$$\psi_L^T\sigma^2\psi_L = \left(\psi_{L_1}\psi_{L_2}\right)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\psi_{L_1}\\ \psi_{L_2}\end{pmatrix} = -i\psi_{L_1}\psi_{L_2} + i\psi_{L_2}\psi_{L_1} \ . \tag{2.96}$$

which vanishes if ψ_{L_1} and ψ_{L_2} are normal numbers.

However, suppose that ψ_{L_1} and ψ_{L_2} anticommute with one another, the scalar invariant (2.96) will not vanish. This is important because, as we shall see later, classical spinor fields will always be "Grassmann numbers", which (as their name indicates) are to be treated just like regular numbers, except that they all anticommute with one another.

If ψ and χ are two Grassmann numbers, there are two ways to define complex conjugation.

The most popular is to simply reverse the order under a new complex conjugation,

$$(\psi\chi)^* \equiv \chi^*\psi^* = -\psi^*\chi^* \ , \tag{2.97}$$

in which case the product of two real Grassman numbers is purely imaginary! The world of Grassmann numbers can be counter-intuitive.

This book follows the less popular definition,

$$(\psi\chi)^* \equiv \psi^* \chi^* \ , \tag{2.98}$$

which has the advantage that the product of two real Grassmann numbers is also real (this convention was adopted in the earlier editions of this book).

One can always switch fom one convention to the other by sprinkling a bunch of i here and there, as long as the physical quantities are real.

We can evaluate the scalar invariant for $\chi_L = \sigma^2 \psi_R^*$

$$i \left(\sigma^2 {\psi_R}^*\right)^T \sigma^2 \psi_L = -i\psi_R^\dagger \psi_L \ . \tag{2.99}$$

In our convention, none of these scalar invariants are real. One obtains their complex conjugates by switching L to R.

The next exercise is to construct a four-vector representation out of two spinors. The simplest way is to start from a single $\psi_L \sim \left(\mathbf{\frac{1}{2}}, \mathbf{0}\right)$ since out of it we can build a $\left(\mathbf{0}, \mathbf{\frac{1}{2}}\right)$ spinor and then multiply them together. One knows that the quantity $\psi_L^\dagger \psi_L$ is invariant under rotations which are represented by unitary operators on the spinors. Not so however under the boosts for which

$$\psi_L^\dagger \psi_L \to \psi_L^\dagger e^{\vec{\sigma}\cdot\vec{\nu}} \psi_L = \psi_L^\dagger \psi_L + \vec{\nu} \cdot \psi_L^\dagger \vec{\sigma} \psi_L + \mathcal{O}(\nu^2) \ . \tag{2.100}$$

The new quantity, fortunately, behaves nicely

$$\begin{aligned} \psi_L^\dagger \sigma^i \psi_L \to \psi_L^\dagger e^{\frac{\vec{\sigma}}{2}\cdot\vec{\nu}} \sigma^i e^{\frac{\vec{\sigma}}{2}\cdot\vec{\nu}} \psi_L &= \psi_L^\dagger \sigma^i \psi_L + \frac{1}{2} \nu^j \psi_L^\dagger \{\sigma^i, \sigma^j\} \psi_L + O(\nu^2) \\ &= \psi_L^\dagger \sigma^i \psi_L + \nu^i \psi_L^\dagger \psi_L + \mathcal{O}(\nu^2) \ . \end{aligned} \tag{2.101}$$

where $\{\ ,\ \}$ denotes the anticommutator. Thus under a boost these quantities transform into one another

$$\delta\ \psi_L^\dagger \psi_L = \nu^i \psi_L^\dagger \sigma^i \psi_L \ , \tag{2.102}$$

$$\delta\ \psi_L^\dagger \sigma^i \psi_L = \nu^i \psi_L^\dagger \psi_L \ , \tag{2.103}$$

and $\psi_L^\dagger \sigma^i \psi_L$ behaves as a three-vector under rotations. Equations (2.102) and (2.103) compare with the transformation laws of a four-vector,

$$\delta V^\mu = \epsilon^\mu{}_\nu V^\nu \ , \tag{2.104}$$

where $\epsilon^{0i} = -\nu^i$ are the boost parameters. Thus the quantity

$$i\psi_L^\dagger \sigma^\mu \psi_L = i\left(\psi_L^\dagger \psi_L,\, \psi_L^\dagger \sigma^i \psi_L\right) \ , \tag{2.105}$$

is a four-vector. We have identified σ^0 with the (2×2) unit matrix. Another four-vector can be obtained starting from ψ_R and changing the sign of the space components

$$i\psi_R^\dagger \bar{\sigma}^\mu \psi_R \equiv i\left(\psi_R^\dagger \psi_R, -\psi_R^\dagger \vec{\sigma} \psi_R\right) \ . \tag{2.106}$$

These two vectors are real since ψ_L and ψ_R are Grassmann variables. In our convention, $(\psi_L^\dagger \psi_R)^* = \psi_L^T \psi_R^* = -\psi_R^\dagger \psi_L$, and their sum (difference) is even (odd) under parity.

Either of these combined with another four-vector can yield Lorentz invariants. The simplest four-vector, as we saw earlier, is the derivative operator ∂_μ which has the added virtue of preserving translational invariance. Since ∂_μ can act on any of the fields, we have the following bilinear invariants in the spinor fields

$$\partial_\mu \psi_R^\dagger \bar{\sigma}^\mu \psi_R \ , \quad \psi_R^\dagger \bar{\sigma}^\mu \partial_\mu \psi_R \ , \quad \partial_\mu \psi_L^\dagger \sigma^\mu \psi_L \ , \quad \psi_L^\dagger \sigma^\mu \partial_\mu \psi_L \ . \tag{2.107}$$

The derivative operator is understood to act to the right and on its nearest neighbor only. These Lorentz invariants are no longer real; however, real linear combinations can be formed, such as

$$\frac{1}{2}\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L - \frac{1}{2}\partial_\mu \psi_L^\dagger \sigma^\mu \psi_L \equiv \frac{1}{2}\psi_L^\dagger \sigma^\mu \overleftrightarrow{\partial}_\mu \psi_L \ , \tag{2.108}$$

and a similar expression with L replaced by R and σ^μ by $\bar{\sigma}^\mu$.

If parity is a concern, one has to assemble $\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$ and $\left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ representations. Since we cannot equate ψ_L with $\sigma_2 \psi_L^*$ without leading to a contradiction, we have to build a four-component spinor called a Dirac spinor

$$\Psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \ , \tag{2.109}$$

on which the operation of parity is well-defined.

$$P: \quad \Psi \to \Psi^P = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi \equiv \gamma_0 \Psi \ , \tag{2.110}$$

where we have defined the (4×4) matrix γ_0. We can project only the left and right spinors by means of the projection operators

$$\frac{1}{2}\left(1 \pm \gamma_5\right) \ , \tag{2.111}$$

where in (2×2) block form

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{2.112}$$

We can rewrite all the invariants we have built in terms of Dirac spinors. For instance,

$$\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R = \Psi^\dagger \gamma^0 \Psi \equiv \bar{\Psi}\Psi \ , \tag{2.113}$$

where $\bar{\Psi} = \Psi^\dagger \gamma_0$ is the Pauli adjoint. Since (2.113) is Lorentz invariant it transforms contragrediently to Ψ. Similarly

$$\frac{1}{2}\left(\psi_L^\dagger \sigma^\mu \overleftrightarrow{\partial}_\mu \psi_L + \psi_R^\dagger \bar{\sigma}^\mu \overleftrightarrow{\partial}_\mu \psi_R\right) = \frac{1}{2}\bar{\Psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \Psi \ , \tag{2.114}$$

where we have introduced the (4×4) matrices

$$\gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \ . \tag{2.115}$$

Since (2.114) is Lorentz invariant the μ index on the γ-matrices is a true four-vector index. They are, of course, the Dirac matrices in the Weyl representation. They obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ . \tag{2.116}$$

The γ_5 matrix is related to the others by

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ . \tag{2.117}$$

By means of the equivalence under Lorentz transformations of ψ_L and $\sigma^2\psi_R^*$, we can build an associated Dirac spinor

$$\Psi^c \equiv \begin{pmatrix} \sigma^2\psi_R^* \\ -\sigma^2\psi_L^*\} \end{pmatrix} \ . \tag{2.118}$$

Note that

$$(\Psi^c)^c = \Psi \ . \tag{2.119}$$

Ψ^c is called the (charge) conjugate spinor. Since $\sigma^2\psi_L^*$ transforms like ψ_R, we can construct a special type of four-component spinor called a Majorana spinor

$$\Psi^M = \begin{pmatrix} \psi_L \\ -\sigma^2\psi_L^* \end{pmatrix} . \tag{2.120}$$

It is self-conjugate under (charge) conjugation. The Majorana spinor is a Weyl spinor in four-component form. Its physical interpretation will be discussed when we build Actions out of spinor fields. Suffice it to say that Majorana and/or Weyl spinors describe objects with half as many degrees of freedom as Dirac spinors.

As we remarked at the end of Section 2, one cannot relate in Euclidean space the two $SU(2)$ groups that form the (Euclidean) Lorentz group. We can now see explicitly why. Since both are unitarily realized we have the new expressions Behavior

$$\mathbf{\Lambda}_L \to \mathbf{\Lambda}_L^E = e^{\frac{i}{2}\vec{\sigma}\cdot(\vec{\omega}+\vec{\nu})} \tag{2.121}$$

$$\mathbf{\Lambda}_R \to \mathbf{\Lambda}_R^E = e^{\frac{i}{2}\vec{\sigma}\cdot(\vec{\omega}-\vec{\nu})} \ , \tag{2.122}$$

and there is no possible relation between $\mathbf{\Lambda}_L^E$ and $\mathbf{\Lambda}_R^E$. Thus, Majorana spinors do not exist in Euclidean space because one cannot relate ψ_L^E to ψ_R^E. However, one is free to deal with ψ_L^E and ψ_R^E separately and even form Dirac spinors Ψ^E, with the understanding that the conjugation operation introduced earlier ceases to exist.

c) The Vector Field

The vector field transforms according to the $\left(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}\right)$ representation. We have already seen the effect of $S_{\rho\sigma}$ on an arbitrary vector field, $A_\mu(x)$. We might add that there is another representation of the vector field as a Hermitean (2×2) matrix

$$A^\mu \to \mathbf{A} = \begin{pmatrix} A^0 + A^3 & A^1 + iA^2 \\ A^1 - iA^2 & A^0 - A^3 \end{pmatrix} \ . \tag{2.123}$$

Lorentz transformations are defined to be those that preserve the condition

$\mathbf{A} = \mathbf{A}^\dagger$ and leave $\det \mathbf{A}$ invariant. One can consider many invariants such as

$$A_\mu(x)A^\mu(x) \ , \quad \partial_\mu A_\nu(x)\partial^\nu A^\mu(x) \ , \quad \partial_\mu A_\nu(x)\partial^\mu A^\nu(x) \ , \quad \partial^\mu A_\mu(x) \ , \text{ etc.} \tag{2.124}$$

Since parity is defined for the vector representation we can define both vector and axial vector fields.

d) The Spin-3/2 Field

There are several ways to define a spin-3/2 field depending on the role we want parity to play. One procedure is to take the product of three $\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$ representations

$$\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) \otimes \left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) \otimes \left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) = \left(\frac{\mathbf{3}}{\mathbf{2}}, \mathbf{0}\right) \oplus \left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) \oplus \left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) \ . \tag{2.125}$$

The spin-3/2 corresponds to the completely symmetric product (the two $\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$ have mixed symmetry). Thus, we can represent a spin-3/2 field by a field totally symmetric in the interchange of its three L-like spinor indices. Its transformation properties are obtained by a suitable generalization of the action of $S_{\rho\sigma}$ on an L-like index (see problem). The parity eigenstate is then a combination of $\left(\frac{\mathbf{3}}{\mathbf{2}}, \mathbf{0}\right) \oplus \left(\mathbf{0}, \frac{\mathbf{3}}{\mathbf{2}}\right)$. However, this representation is rather awkward because of the many indices on the field symbol.

A more convenient representation of a spin-3/2 field is obtained through the product of a vector and a spinor

$$\left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right) \otimes \left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) = \left(\mathbf{1}, \frac{\mathbf{1}}{\mathbf{2}}\right) \oplus \left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right) \ . \tag{2.126}$$

The corresponding field quantity has a four-vector and a spinor index. The parity eigenstate is the four-component "Rarita–Schwinger" field

$$\Psi_\mu = \begin{pmatrix} \psi_{\mu L} \\ \psi_{\mu R} \end{pmatrix} \ . \tag{2.127}$$

where the spinor indices have been suppressed. As written, Ψ_μ describes all the states in the product (2.126) together with their parity partners. Hence we must project out the extra $\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right) \oplus \left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\right)$ components in a Lorentz invariant way. We impose on it the subsidiary conditions

$$\sigma^\mu \psi_{\mu L} = \bar{\sigma}^\mu \psi_{\mu R} = 0 \ , \tag{2.128}$$

or in Dirac language

$$\gamma^\mu \Psi_\mu = 0 \ . \tag{2.129}$$

The same kind of covariants and invariants can be assembled as in the spinor case except we have the extra vector index to play with. A sample of quadratic invariants is

$$\psi^T_{\mu L}\sigma_2\psi^\mu_L \ , \quad \psi^T_{\mu R}\sigma_2\psi^\mu_R \ , \quad \psi^\dagger_{\mu R}\psi^\mu_L \ , \quad \cdots \ . \tag{2.130}$$

We can use the set of four-vectors

$$\psi^\dagger_{\mu L}\sigma_\rho\psi_{\nu L}\epsilon^{\mu\rho\nu\sigma} \quad , \qquad \psi^\dagger_{\mu R}\sigma_\rho\psi_{\nu R}\epsilon^{\mu\rho\nu\sigma} \ , \tag{2.131}$$

in combination with ∂_ρ to make invariants of the form

$$\partial_\mu\psi^\dagger_{\sigma L}\sigma_\rho\psi_{\nu L}\epsilon^{\mu\sigma\rho\nu} \quad , \quad \text{etc.} \ . \tag{2.132}$$

The real scalar invariant is then given by

$$\frac{1}{2}\left(\psi^\dagger_{\mu L}\sigma_\rho\overleftrightarrow{\partial_\sigma}\psi_{\nu L} - \psi^\dagger_{\mu R}\bar{\sigma}_\rho\overleftrightarrow{\partial_\sigma}\psi_{\nu R}\right)\epsilon^{\mu\rho\sigma\nu} = \frac{1}{2}\bar{\Psi}_\mu\gamma_5\gamma_\rho\overleftrightarrow{\partial_\sigma}\Psi_\nu\epsilon^{\mu\rho\sigma\nu} \ . \tag{2.133}$$

The presence of the relative minus sign *i.e.*, of the γ_5, is dictated by the parity properties of the ϵ-operation. Finally, note that we can, as in the spin-$\frac{1}{2}$ case, impose a Majorana condition on the R–S fields.

e) The Spin 2 Field

Again there are many possible ways to describe a spin-2 field: ($\mathbf{2}$,$\mathbf{0}$, ($\mathbf{0}$,$\mathbf{2}$),($\mathbf{1}$,$\mathbf{1}$). We choose the latter which appears in the product

$$\left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}\right)\otimes\left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}\right) = [(\mathbf{0},\mathbf{0})\oplus(\mathbf{1},\mathbf{1})]_S \oplus [(\mathbf{0},\mathbf{1})\oplus(\mathbf{1},\mathbf{0})]_A \ , \tag{2.134}$$

where $S(A)$ denotes the symmetric (antisymmetric) part. Thus a spin-2 field can be described by a second-rank symmetric tensor $h_{\mu\nu}$. The scalar component corresponds to its trace which can be subtracted by the tracelessness condition

$$g^{\mu\nu} h_{\mu\nu}(x) = 0 \ . \tag{2.135}$$

Invariants are easily constructed by saturating the vector indices, and by the use of ∂_ρ. Low level examples are

$$h_{\mu\nu} h^{\mu\nu} \quad , \quad \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} \quad , \quad \partial_\rho h_{\mu\nu} \partial^\mu h^{\rho\nu} \quad , \quad \text{etc.} \quad . \tag{2.136}$$

This tensor field appears in General Relativity where it is used to describe the graviton.

To conclude this section we note that many other fields with definite Lorentz transformation properties can be constructed. However, we have chosen to discuss in some detail only those which prove useful in the description of physical phenomena. They are the ones to which we can associate fundamental particles. To Dirac spinors we associate charged fermions such as electron, muon, tau, quarks; to Weyl spinors the neutrinos ν_e, ν_μ, ν_τ. To vector fields we find the gluons which mediate Strong Interactions, the W-bosons, Z-boson and the photon that mediate the Electro-Weak Interactions; and finally, to a tensor field the graviton that mediates Gravitation.

2.4.1 PROBLEMS

A. Build explicitly the action of $S_{\rho\sigma}$ on ψ_L and ψ_R.

B. Write the action of $S_{\rho\sigma}$ on a Dirac spinor in terms of the Dirac matrices, *i.e.*, in a representation independent way.

C. Build explicitly a field bilinear in the spinor χ_L and ψ_L that transforms as the $(\mathbf{1},\mathbf{0})$ representation. Can you build the same field out of one ψ_L field?

D. Find the form Lorentz transformations take acting on the matrix (2.123) in terms of $\mathbf{\Lambda}_L$ and $\mathbf{\Lambda}_R$.

E. Given $\psi_L(x)$, $A_\mu(x)$, build at least two invariants where the two fields appear.

F. Find a representation for the Dirac matrices where the components of a Majorana spinor are real. Such a representation is called the Majorana representation.

2.5 General Properties of the Action

In the previous sections we have learned how to build Poincaré invariant expressions out of fields which have well-defined transformation properties under the Poincaré group. Now comes the time to assemble these invariants into Actions that describe reasonable physical theories. The requirement of Poincaré invariance insures that these theories will obey the axioms of Special Relativity. Yet as we become more and more adept at this game we will learn that there are too many candidate theories and that the single prescription of Poincaré invariance is not sufficient to pinpoint the true Action of the world. In an attempt to narrow our search we try to enumerate certain *ad hoc* prescriptions which have been found sufficient to yield acceptable theories.

First, we deal with Action Functionals of the form

$$S \equiv \int_{\tau_1}^{\tau_2} d^4x \, \mathcal{L} \ , \tag{2.137}$$

where τ_1 and τ_2 denote the limits of integrations and

$$d^4x = dt \, dx^1 dx^2 dx^3 \ , \tag{2.138}$$

is the integration measure in four-dimensional Minkowski space. Sometimes we might alter for mathematical purposes, the number of space-time dimensions or even consider the measure in Euclidean space with d^4x replaced by the Euclidean measure

$$d^4\bar{x} = d\bar{x}^0 d\bar{x}^1 d\bar{x}^2 d\bar{x}^3 \ , \tag{2.139}$$

where $\bar{x}^0 = ix^0$, $\bar{x}^i = x^i$. The integrand, $\mathcal{L}$, is called the Lagrange density, Lagrangian for short. It is a *function* of the fields and their derivatives limited in form by the requirement of translation invariance. Also, it depends on the fields taken at one space-time point x^μ only, leading to a *local* field theory. This is clearly the simplest choice to make: one can easily imagine non-local field theories but they are necessarily more complicated in nature. In fact, our faith in local field theory is such that we believe it to be sufficient even in the description of non-local phenomena!

Secondly, we demand that S be real. It is found (in retrospect) that this is a crucial requirement in obtaining satisfactory quantum field theories where total probability is conserved. In Classical Physics a complex potential leads to absorption, *i.e.*, disappearance of matter into nothing; it is not a satisfactory situation.

Third, we demand that S leads to classical equations of motions that involve no higher than second-order derivatives. Classical systems described by higher order differential equations will typically develop non-causal solutions. A well-known example is the Lorentz–Dirac equation of Electrodynamics, a third order differential equation that incorporates the effects of radiation reaction and shows non-causal effects such as the preacceleration of particles yet to be hit by radiation. To satisfy this requirement we take $\mathcal{L}$ to contain at most two ∂_μ operations. As a consequence the classical equations (or their square for spinor fields) will display the operator $\partial_\mu\partial^\mu$ acting on a field. When the equations of motion turn into an eigenvalue condition on this operator, we will say we are dealing with a free field theory because we can identify $\partial_\mu\partial^\mu$ with a Casimir operator of the Poincaré group, with the equations of motion restricting us to a (free) particle representation.

Fourth, we take S to be invariant under the Poincaré group, as we have already discussed.

Fifth, there may be further invariance requirements on S. In fact the phenomenological success of gauge theories suggests that the relevant Action Functional is invariant under peculiar new transformations which involve new degrees of freedom such as electric charge, weak charge, color charge and other charges yet to be discovered. Gauge theories are described by actions which are invariant under local (*i.e.*, x-dependent) transformations among these internal degrees of freedom. We will be much more specific on this subject later on.

In classical theory the Action has the definite units of angular momentum ML^2T^{-1} or equivalently units of $\hbar$. In a natural unit system where $\hbar = 1$, S is taken to be "dimensionless." Then in four dimensions the Lagrange density has natural dimensions of L^{-4}.

Consider the action

$$S\left(\tau_1,\, \tau_2,\, [\varphi]\right) \equiv \int_{\tau_1}^{\tau_2} d^4x\, \mathcal{L}\left(\Phi, \partial_\mu\Phi\right) \ , \tag{2.140}$$

where $\varphi(x)$ in any local field or any collection of local fields (it could be scalars, spinors, $\cdots$; we suppress all indices); τ_1 and τ_2 are the boundaries of integrations. Under an arbitrary change in Φ, $\delta\Phi$, the resulting change in S is

$$\delta S = \int_{\tau_1}^{\tau_2} d^4x\,\delta\mathcal{L}\ , \tag{2.141}$$

$$= \int_{\tau_1}^{\tau_2} d^4x \left[\frac{\partial\mathcal{L}}{\partial\Phi}\delta\Phi + \frac{\partial\mathcal{L}}{\partial[\partial_\mu\Phi]}\delta(\partial_\mu\Phi)\right]\ . \tag{2.142}$$

Since x does not change in this variation

$$\delta(\partial_\mu\Phi) = \partial_\mu\delta\Phi\ . \tag{2.143}$$

Use of the chain rule yields

$$\delta S = \int_{\tau_1}^{\tau_2} d^4x \left[\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial[\partial_\mu\Phi]}\right]\delta\varphi + \int_{\tau_1}^{\tau_2} d^4x\,\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial[\partial_\mu\Phi]}\delta\Phi\right]\ . \tag{2.144}$$

The last term is just a surface term which can be rewritten as a surface integral

$$\oint_\sigma d\sigma_\mu \frac{\partial\mathcal{L}}{\partial[\partial_\mu\Phi]}\delta\Phi\ , \tag{2.145}$$

where σ is the boundary surface and $d\sigma_\mu$ the surface element. Finally, we demand that $\delta\Phi$ vanishes on σ. By requiring that S be stationary under an arbitrary change $\delta\Phi$ vanishing on the boundaries we obtain the Euler–Lagrange equations

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial[\partial_\mu\Phi]} - \frac{\partial\mathcal{L}}{\partial\Phi} = 0, \tag{2.146}$$

which are the classical field equations for the system described by S. We can identify (2.146) with the functional derivative of S with respect to Φ. Again note that it is well-defined only for variations that vanish on the boundaries of integration. As an important consequence of dropping the surface term, observe that the same equations of motion would have been obtained if we had started from the new Lagrangian density

$$\mathcal{L}' = \mathcal{L} + \partial_\mu\Lambda^\mu, \tag{2.147}$$

with Λ^μ arbitrary. Such a change in $\mathcal{L}$ produces a change in S that entirely depends on the choice of boundary conditions on the fields that enter in $\mathcal{L}'$

[this freedom is no longer tolerated in the presence of gravity]. In Classical Mechanics, the transformation between $\mathcal{L}$ and $\mathcal{L}'$ is called a canonical transformation. Also, note that the addition of a constant to $\mathcal{L}$ does not change the nature of the classical system although it affects the coupling of the system to gravity as it generates an infinite energy.

Next, we consider the response of the Action to yet unspecified (but not arbitrary) changes in the coordinates and in the fields, δx^μ and $\delta\Phi$, respectively. To the coordinate change corresponds the change in the integration measure given by the Jacobi formula

$$\delta(d^4x) = d^4x\,\partial_\mu \delta x^\mu. \tag{2.148}$$

Thus it follows that

$$\delta S = \int_{\tau_1}^{\tau_2} d^4x\left[\partial_\mu \delta x^\mu \mathcal{L} + \delta\mathcal{L}\right] \ . \tag{2.149}$$

Use of (2.70) yields

$$\begin{aligned} \delta\mathcal{L} &= \delta x^\mu \partial_\mu \mathcal{L} + \delta_0 \mathcal{L} && (2.150)\\ &= \delta x^\mu \partial_\mu \mathcal{L} + \frac{\partial\mathcal{L}}{\partial\Phi}\delta_0\Phi + \frac{\partial\mathcal{L}}{\partial[\partial_\mu\Phi]}\delta_0\partial_\mu\Phi \ . && (2.151) \end{aligned}$$

Now δ_0 is just a functional change, therefore

$$\begin{aligned} \delta_0\partial_\mu\Phi &= [\delta_0, \partial_\mu]\,\Phi + \partial_\mu\delta_0\Phi \ , && (2.152)\\ &= \partial_\mu\delta_0\Phi \ . && (2.153) \end{aligned}$$

Use of the chain rule yields

$$\delta\mathcal{L} = \delta x^\rho \partial_\rho \mathcal{L} + \left[\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial[\partial_\mu\Phi]}\right]\delta_0\Phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\Phi]}\delta_0\Phi\right) \ . \tag{2.154}$$

By invoking the classical equations of motion the change in the Action is

$$\begin{aligned} \delta S &= \int_{\tau_1}^{\tau_2} d^4x\left[\mathcal{L}\partial_\mu\delta x^\mu + \delta x^\mu\partial_\mu\mathcal{L} + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\Phi]}\delta_0\Phi\right)\right] \ , && (2.155)\\ &= \int_{\tau_1}^{\tau_2} d^4x\,\partial_\mu\left[\mathcal{L}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial[\partial_\mu\Phi]}\delta_0\Phi\right] && (2.156)\\ & && (2.157) \end{aligned}$$

Alternatively, by re-expressing δ_0 in terms of δ, we can obtain

$$\delta S = \int_{\tau_1}^{\tau_2} d^4x\,\partial_\mu \left[\left(\mathcal{L} g^\mu_\rho - \frac{\partial \mathcal{L}}{\partial[\partial_\mu \Phi]} \partial_\rho \Phi \right) \delta x^\rho + \frac{\partial \mathcal{L}}{\partial[\partial_\mu \Phi]} \delta \Phi \right] . \qquad (2.158)$$

Next we write the variations in the coordinates and the fields in terms of the *global*, *i.e.*, x-independent) parameters of the transformation

$$\delta x^\rho = \frac{\delta x^\rho}{\delta \omega^a} \delta \omega^a \ , \qquad (2.159)$$

$$\delta \Phi = \frac{\delta \Phi}{\delta \omega^a} \delta \omega^a \ . \qquad (2.160)$$

Here a is an index which enumerates the parameters of the transformation. Consequently

$$\delta S = \int_{\tau_1}^{\tau_2} d^4x\,\partial_\mu \left[\left(\mathcal{L} g^\mu_\rho - \frac{\partial \mathcal{L}}{\partial[\partial_\mu \Phi]} \partial_\rho \Phi \right) \frac{\delta x^\rho}{\delta \omega^a} + \frac{\partial \mathcal{L}}{\partial[\partial_\mu \Phi]} \, \frac{\delta \Phi}{\delta \omega^a} \right] \delta \omega^a \ . \qquad (2.161)$$

If the Action is invariant under the transformations (2.159) and (2.160), it follows that the current density

$$j^\mu_a \equiv - \left[\mathcal{L} g^\mu_\rho - \frac{\partial \mathcal{L}}{\partial[\partial_\mu \Phi]} \partial_\rho \Phi \right] \frac{\delta x^\rho}{\delta \omega^a} - \frac{\partial \mathcal{L}}{\partial[\partial_\mu \Phi]} \, \frac{\delta \Phi}{\delta \omega^a} \ , \qquad (2.162)$$

is conserved, *i.e.*,

$$\partial_\mu j^\mu_a = 0 \ . \qquad (2.163)$$

This conservation equation is a consequence of the validity of (2.161) *for all* $\delta\omega^a$. We have just proved *E.* Noether's theorem for classical field theory which relates a conservation equation to an invariance of the Action.

On the other hand, if the Action is not conserved the conservation equation is no longer valid. For example it has a particularly simple form when $\delta x^\rho = 0$

$$\partial_\mu j^\mu_a = - \frac{\delta \mathcal{L}}{\delta \omega^a} \ . \qquad (2.164)$$

Now suppose we have found a set of transformations (2.159) and (2.160) which leave the Action invariant. Integrate (2.163) over an infinite range of the space directions and a finite interval over the time direction. We get

$$0 = \int_{T_1}^{T_2} dx^0 \int_{-\infty}^{+\infty} d^3x\, \partial_\mu j^\mu_a = \int_{T_1}^{T_2} dx^0 \frac{\partial}{\partial x^0} \int_{-\infty}^{+\infty} d^3x\, j^0_a + \int_{T_1}^{T_2} dx^0 \int d^3x \partial_i j^i_a \, . \tag{2.165}$$

The last term vanishes if the space boundaries are suitably chosen. We are left with

$$0 = \int_{-\infty}^{+\infty} d^3x\, j^0_a(T_1, \vec{x}) - \int_{-\infty}^{+\infty} d^3x\, j^0_a(T_2, \vec{x}) \; . \tag{2.166}$$

Therefore, the charges defined by

$$Q_a(T) \equiv \int_{-\infty}^{+\infty} d^3x\, j^0_a(T, \vec{x}) \; , \tag{2.167}$$

are time independent since the above argument does not depend on the choice of the time integration limits:

$$\frac{dQ_a}{dt} = 0 \; . \tag{2.168}$$

So, from $\delta S = 0$, we have been able to deduce the existence of conserved charges.

When the parameters of the transformations are dimensionless as in the case of Lorentz transformations and internal transformations (but not translations) the resulting currents always have the dimensions of L^{-D+1} in D dimensions, so that the charges are dimensionless.

Further, we remark that a conserved current does not have a unique definition since we can always add to it the four-divergence of an antisymmetric tensor $\partial_\rho t^{\rho\mu}_a$. This is most clearly seen in the light of (2.147). Also since j^μ_a is conserved only after use of the equations of motion we have the freedom to add to it any quantity which vanishes by virtue of the equations of motion. This is particularly relevant when a is a Lorentz index, as in the case of a translation

$$\delta x^\mu = \epsilon^\mu \quad : \quad \frac{\delta x^\mu}{\delta \omega^a} \to g^\mu_\rho \quad , \qquad (a = \rho) \; . \tag{2.169}$$

or a Lorentz transformation

$$\delta x^\mu = \epsilon^{\mu\nu} x_\nu \quad ; \quad \frac{\delta x^\mu}{\delta \omega^a} \to \frac{1}{2} \left(g^\mu_\rho x_\nu - g^\mu_\nu x_\rho \right) \; . \tag{2.170}$$

In the latter case the parameter a is replaced by the antisymmetric pair $[\rho\nu]$.

Finally we note that a transformation that leaves S invariant may change $\mathcal{L}$ by a total divergence, which means that the symmetry operation is accompanied by a canonical transformation. In quantum theory where one cannot rely on the equations of motion the statement of current conservation will lose its significance but will be replaced by relations between Green's functions known as Ward identities.

2.5.1 PROBLEMS

A. Consider the conformal transformations

$$\delta x^\mu = \left(2x^\mu x^\rho - g^{\mu\rho} x_\tau x^\tau\right) c_\rho \quad ,$$

where c_ρ is an infinitesimal four-vector. Show that these transformations together with the dilatations

$$\delta x^\mu = \alpha x^\mu \quad , \quad \alpha \text{ infinitesimal} \quad ,$$

and the Poincaré group transformations form a fifteen-parameter group, called the Conformal group.

B. The dilatations and the Poincaré group form a group called the Weyl group. Under dilatations a field Φ of dimension d transforms as

$$\delta\Phi = \alpha d \Phi \quad .$$

Assuming that S is invariant under the Weyl group and contains Φ, find the conserved current corresponding to dilatations.

2.6 The Action for Scalar Fields

The most general form of the Lagrange density containing only one scalar field $\varphi(x)$ is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi(x) \partial^\mu \varphi\varphi(x) - V\left[\varphi(x)\right] \ , \tag{2.171}$$

where the $\frac{1}{2}$ is purely conventional and V is a scalar function. The first term is called the kinetic term, the second the potential. The kinetic term has a larger invariance group than the potential: it is invariant under a shift of the field $\varphi \to \varphi + a$, where a is a global constant. In four dimensions, $\varphi(x)$ therefore has natural dimensions of L^{-1} (or of mass). The form of $V\left[\varphi(x)\right]$ is unrestricted in the classical theory. Some special examples are

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 \ , \tag{2.172}$$

where m has the dimension of mass. This Action describes a free particle of mass m (as we shall deduce later from its path integral treatment). Note that $\mathcal{L}_0$ is also invariant under the discrete symmetry

$$\varphi(x) \to -\varphi(x). \tag{2.173}$$

A more complicated example is given by

$$\mathcal{L} = \mathcal{L}_0 - \frac{\lambda}{4!}\varphi^4 \ , \tag{2.174}$$

which describes a self-interacting theory. Observe that in four dimensions λ is a dimensionless parameter. The minus sign ensures the positivity of V (for positive λ). This Action leads to an acceptable quantum field theory. Another popular example is the Sine-Gordon Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\partial^\mu\varphi + \frac{m^4}{\lambda}\left[\cos\frac{\sqrt{\lambda}\,\varphi}{m} - 1\right], \tag{2.175}$$

where λ is dimensionless. For $\frac{\sqrt{\lambda}\varphi}{m} << 1$, it reproduces the previous examples, except for the sign of the φ^4 term. Alas, it is not known to lead to an acceptable quantum field theory in four dimensions; however, it yields a healthy quantum theory in two dimensions!

Whatever the form of V, the equations of motion are easily obtained

$$\partial_\mu\partial^\mu\varphi = -V'(\varphi) \ , \tag{2.176}$$

where the prime denotes the derivative with respect to φ. We can construct the conserved quantity using the procedures of the previous section.

a) Under an infinitesimal translation characterized by $\delta x^\mu = \epsilon^\mu$ and $\delta\varphi = 0$, Eqs. (2.162) and (2.163) become

$$\begin{aligned} j_{\mu\nu} &= -g_{\mu\nu}\mathcal{L} + \partial_\mu\varphi\partial_\nu\varphi \ , &(2.177)\\ \partial^\mu j_{\mu\nu} &= 0 \ . &(2.178)\end{aligned}$$

Observe that in this case $j_{\mu\nu}$ is a symmetric tensor; it is called the energy momentum tensor. The corresponding conserved charge is

$$P_\mu = \int d^3x\, j_{\mu 0} = \int d^3x\, (-g_{\mu 0}\mathcal{L} + \partial_0\varphi\partial_\mu\varphi) \ . \tag{2.179}$$

Then since P_0 is the energy of the system the energy density is

$$\begin{aligned} j_{00} &= -\mathcal{L} + \partial_0\varphi\partial_0\varphi \ , &(2.180)\\ &= \frac{1}{2}\partial_0\varphi\partial_0\varphi + \frac{1}{2}\vec{\nabla}\varphi\vec{\nabla}\varphi + V(\varphi) \ , &(2.181)\end{aligned}$$

and is seen to be positive definite when $V > 0$. The ground state field configuration is that which gives the lowest value for j_{00}. Since the derivative terms give a positive contribution it always occurs for a static field φ_c $(\partial_0\varphi_c = \partial_i\varphi_c = 0)$, in which case the energy density is the value of the potential $V(\varphi_c)$ for this particular field.

b) Under a Lorentz transformation the conserved Noether current is a three-indexed quantity given by

$$\begin{aligned} j_{\mu\nu\rho} &= (-g_{\mu\lambda}\mathcal{L} + \partial_\mu\varphi\partial_\lambda\varphi)\left(g^\lambda_\nu x_\rho - g^\lambda_\rho x_\nu\right) \ , &(2.182)\\ &= j_{\mu\nu}x_\rho - j_{\mu\rho}x_\nu \ . &(2.183)\end{aligned}$$

The corresponding conserved charges are the generators of the Lorentz transformations

$$M_{\nu\rho} = \int d^3x\, j_{0\nu\rho} = \int d^3x\, (j_{0\nu}x_\rho - j_{0\rho}x_\nu) \ . \tag{2.184}$$

The conservation of these charges is a consequence of the invariance of the Action under Poincaré transformations.

As an example of the application of Noether's theorem to transformations which are not necessarily invariances of S, consider an infinitesimal dilatation

$$\delta x^\mu = \alpha x^\mu \qquad \delta\varphi = -\alpha\varphi \ . \tag{2.185}$$

The Noether current is

$$\begin{aligned} j^\mu_D &= \left(-g^\mu_\rho\mathcal{L} + \partial^\mu\varphi\partial_\rho\varphi\right)x^\rho + \varphi\partial^\mu\varphi \ , &(2.186)\\ &= j^{\mu\rho}x_\rho + \frac{1}{2}\partial^\mu\varphi^2 \ . &(2.187)\end{aligned}$$

We see that using (2.178)

$$\partial_\mu j^\mu_D = j^\mu_\nu + \frac{1}{2}\partial^\mu \partial_\mu \varphi^2 \ . \tag{2.188}$$

When $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{\lambda}{4!}\varphi^4$, it is easy to show that j^μ_D is divergenceless (in four dimensions: see problem). However, had we added to $\mathcal{L}$ the "mass term" $-\frac{1}{2}m^2\varphi^2$, its contribution would have been

$$\partial_\mu j^\mu_D = m^2\varphi^2 \neq 0 \ . \tag{2.189}$$

The reason for the failure of conservation of j^μ_D in this case is that a dimensional parameter appears in $\mathcal{L}$.

Recall that there are ambiguities in the form of $j_{\mu\nu}$. As an example consider the new definition

$$\Theta_{\mu\nu} \equiv j_{\mu\nu} + a\left(\partial_\mu\partial_\nu - g_{\mu\nu}\partial_\rho\partial^\rho\right)\varphi^2 \ , \tag{2.190}$$

where a is a dimensionless number. We still have

$$\partial^\mu\Theta_{\mu\nu} = 0 \ . \tag{2.191}$$

We fix a by demanding that, for a dilatation invariant theory, $\Theta_{\mu\nu}$ be traceless. Taking as an example the Lagrangian given by (2.174) with $m^2 = 0$, we find

$$\Theta^\mu_\mu = (1+6a)\left[-\partial_\rho\varphi\partial^\rho\varphi - \varphi\partial_\rho\partial^\rho\varphi\right] \ , \tag{2.192}$$

which sets $a = -\frac{1}{6}$. Furthermore, the difference between $\Theta_{\mu\nu}$ and $j_{\mu\nu}$ is a surface term which does not alter the conserved charges. Now we can define a new dilatation current as

$$j^{\mu\prime}{}_D \equiv x_\rho \Theta^{\mu\rho} \ . \tag{2.193}$$

Then using (2.191)

$$\partial_\mu j'^\mu_D = \Theta^\mu{}_\mu \ , \tag{2.194}$$

which shows that dilatation invariance is equivalent to tracelessness of $\Theta_{\mu\nu}$. This new dilatation current is related to the old one by

$$\begin{aligned} j'^{\mu}_{D} &= x_\rho j^{\mu\rho} - \frac{1}{6} x_\rho \left(\partial^\mu \partial^\rho - g^{\mu\rho}\partial_\tau \partial^\tau\right)\varphi^2 & (2.195)\\ &= j^\mu_D - \frac{1}{2}\partial^\mu \varphi^2 - \frac{1}{6} x_\rho \left(\partial^\mu \partial^\rho - g^{\mu\rho}\partial_\tau \partial^\tau\right)\varphi^2 & (2.196)\\ &= j^\mu_D + \frac{1}{6}\partial_\rho \left[x^\mu \partial^\rho - x^\rho \partial^\mu\right]\varphi^2 \ , & (2.197)\end{aligned}$$

using the form (2.187). They are seen to differ from one another by a total divergence and thus the dilatation charge is not affected. The tensor $\Theta_{\mu\nu}$ is called the "new improved energy momentum tensor" [see F. Gürsey, *Annals of Physics* **24**, 211 (1963) and S. Coleman and R. Jackiw, *Annals of Physics* **67**, 552 (1971)]. The differences between $\Theta_{\mu\nu}$ and $j_{\mu\nu}$ and between $j_{\mu D}$ and $j'_{\mu D}$ are all surface terms.

These new forms for the energy momentum tensor and the dilatation current can be obtained canonically if we add to the scalar field Lagrangian a surface term of the form $\partial_\mu \Lambda^\mu$ where

$$\Lambda^\mu = \frac{1}{6}\left(x^\mu \partial^\rho - x^\rho \partial^\mu\right)\partial_\rho \varphi^2 \ ; \qquad (2.198)$$

it therefore corresponds to a canonical transformation.

In field theories of higher spin fields the dilatation invariance is always linked to a traceless energy momentum tensor. As we shall see later, dilatation invariance even when present in the original Lagrangian is broken by quantum effects.

The field theory of many scalar fields goes in much the same way except that interesting new symmetries arise. As an example consider N real scalar fields $\varphi_a, a = 1, \cdots, N$ and the Lagrangian

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^{N} \partial_\mu \varphi_a \partial^\mu \varphi_a \ . \qquad (2.199)$$

Besides the usual invariances, $\mathcal{L}$ is obviously invariant under a global (*i.e.*, x-independent) rotation of the N real scalar fields into one another

$$\delta\varphi_a = \epsilon_{ab}\varphi_b \ , \qquad \epsilon_{ab} = -\epsilon_{ba} \ . \qquad (2.200)$$

As a result there are $\frac{1}{2}N(N-1)$ conserved Noether currents

$$j^{\mu}_{ab} = \varphi_a \partial^{\mu} \varphi_b - \varphi_b \partial^{\mu} \varphi_a \ . \tag{2.201}$$

This constitutes an example of an internal symmetry stemming from the presence of many fields of the same type. If the theory is supplemented by a potential that depends only on the rotation invariant "length" $\varphi_a \varphi_a$, the internal rotation invariance is preserved.

2.6.1 PROBLEMS

A. In four-dimensions show that the canonical dilatation current is divergenceless when $\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{\lambda}{4!}\varphi^4$.

B. In D dimensions, derive the expression for the divergence of the dilatation current when $\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - V(\varphi)$.

*C. In general the canonical energy momentum tensor need not be symmetric. Show that one can always find a term $B^{\rho\mu\nu}$ antisymmetric under $\rho \to \mu$ such that the Belinfante tensor

$$j^{\mu\nu}_B = j^{\mu\nu} + \partial_\rho B^{\rho\mu\nu} \quad ,$$

is symmetric and the conserved Noether current for LT's is written in the form

$$j^{\mu\nu\rho} = \left(j^{\mu\nu}_B x^\rho - j^{\mu\rho}_B x^\nu \right) \quad .$$

Hint: $B^{\rho\mu\nu} = 0$ for scalar fields so it has to do with $S^{\mu\nu}$.

*D. Find $\delta\varphi$ for a conformal transformation. Show that $S = \int d^4x \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi$ is invariant under a conformal transformation. Construct the conserved Noether current.
vskip .3cmE. Derive the form of the conserved currents corresponding to the transformations (2.200) when $\mathcal{L}$ is given by (2.199).

2.7 The Action for Spinor Fields

In this section we concern ourselves primarily with the construction of Actions that involve the spinor Grassmann fields ψ_L and ψ_R. Using the results of Section 4 the simplest candidates for a spinor kinetic term are

$$\mathcal{L}_L \quad = \quad \frac{1}{2}\psi^\dagger_L \sigma^\mu \overleftrightarrow{\partial}_\mu \psi_L \quad , \quad \mathcal{L}_L = \mathcal{L}^*_L \ , \tag{2.202}$$

$$\mathcal{L}_R = \frac{1}{2}\psi_R^\dagger\bar{\sigma}^\mu \overleftrightarrow{\partial}_\mu\psi_R \quad , \quad \mathcal{L}_R = \mathcal{L}_R^* \ . \tag{2.203}$$

of if parity is of interest

$$\mathcal{L}_{\text{Dirac}} = \frac{1}{2}\bar{\Psi}\gamma^\mu\overleftrightarrow{\partial}_\mu\Psi \ , \tag{2.204}$$

$$= \mathcal{L}_L + \mathcal{L}_R \ . \tag{2.205}$$

In the special case $\psi_R = -\sigma_2\psi_L^*$ it is easy to see that $\mathcal{L}_R$ is equivalent to $\mathcal{L}_L$ up to a total divergence (see problem). Thus, if Ψ_M is a four-component Majorana spinor the Lagrangian is written as

$$\mathcal{L}_{\text{Maj}} = \frac{1}{4}\bar{\Psi}_M\gamma^\mu\overleftrightarrow{\partial}_\mu\Psi_M, \tag{2.206}$$

and is equal to $\mathcal{L}_L$ as can be seen by using the Grassmann properties of ψ_L. In the literature, one often sees the Dirac kinetic term (2.205) written with the derivative operator acting only to the right and without the factor of $\frac{1}{2}$. Although superficially different from (2.205) this form differs from it by a total divergence. The distinction is not important as long as the system is not coupled to gravity.

These possible kinetic terms are invariant under conformal transformations (see problem), just as the scalar field kinetic term was, but in addition have phase invariances of their own. For instance, consider $\mathcal{L}_L$ (the same can be said for $\mathcal{L}_R$). Since ψ_L is a complex spinor, one can perform on it a phase transformation

$$\psi_L \to e^{i\delta}\psi_L \ , \tag{2.207}$$

which leaves $\mathcal{L}_L$ invariant as long as δ does not depend on x. The Dirac Lagrangian (2.205 has two of these invariances. They may be reshuffled in four-component language as an overall phase transformation

$$\Psi \to e^{i\alpha}\Psi \ , \tag{2.208}$$

and a chiral transformation

$$\Psi \to e^{i\beta\gamma_5}\Psi \ . \tag{2.209}$$

Lastly, as in the scalar case, the Action with $\mathcal{L} = \mathcal{L}_L$ (or $\mathcal{L}_R$) is invariant under a constant shift in the fields since

$$\mathcal{L}_L\left(\psi_L+\alpha_L\right)=\mathcal{L}_L+\frac{1}{2}\partial_\mu\left(\alpha_L^\dagger\sigma^\mu\psi_L-\psi_L^\dagger\sigma^\mu\alpha_L\right)\ . \tag{2.210}$$

By means of Noether's theorem we can build the conserved currents corresponding to the transformations (2.208) and (2.209). They are

$$j^\mu=i\bar{\Psi}\gamma^\mu\Psi=i\psi_L^\dagger\sigma^\mu\psi_L+i\psi_R^\dagger\bar{\sigma}^\mu\psi_R\ , \tag{2.211}$$

and

$$j_5^\mu=i\bar{\Psi}\gamma^\mu\gamma^5\Psi=i\psi_L^\dagger\sigma^\mu\psi_L-i\psi_R^\dagger\bar{\sigma}^\mu\psi_R\ , \tag{2.212}$$

respectively, while the conserved charges are

$$Q=i\int d^3x\,\bar{\Psi}\gamma^0\Psi=i\int d^3x\left(\psi_L^\dagger\psi_L+\psi_R^\dagger\psi_R\right)\ , \tag{2.213}$$

and

$$Q_5=i\int d^3x\,\bar{\Psi}\gamma^0\gamma^5\Psi=i\int d^3x\left(\psi_L^\dagger\psi_L-\psi_R^\dagger\psi_R\right)\ . \tag{2.214}$$

For a Majorana field only the chiral transformation exists since ψ_R is the conjugate of ψ_L; therefore ψ_L and ψ_R have opposite phase transformation.

Other non-kinetic quadratic invariants can be constructed out of spinor fields (see Section 4). Using ψ_L only, we have the real combinations

$$\begin{aligned}\mathcal{L}_L^m &= \frac{im}{2}\left(\psi_L^T\sigma^2\psi_L+\psi_L^\dagger\sigma^2\psi_L^*\right)\ , &(2.215)\\ \mathcal{L}_{L5}^m &= \frac{m}{2}\left(\psi_L^T\sigma^2\psi_L-\psi_L^\dagger\sigma^2\psi_L^*\right)\ . &(2.216)\end{aligned}$$

where m is a parameter with dimensions of mass (in any number of dimensions). These are known as mass terms. Since ψ_L can be used to describe a Majorana spinor Ψ_M, it follows that (2.215) can serve as the mass term for a Majorana spinor. In four-component notation (2.215) is known as the Majorana mass. Thus, having only ψ_L does not guarantee masslessness as is so often wrongly stated (*e.g.*, the Glashow–Weinberg–Salam model of weak and electromagnetic interactions where the neutrino is represented by a two-component left-handed spinor without a right-handed partner. There the masslessness of the neutrino results from the absence of certain Higgs bosons, in which case the fermion number conservation keeps the neutrino massless even after radiative corrections). This remark assumes special relevance in the conventional description of the neutrino in terms of a left-handed field. Note that $\mathcal{L}_L^m$ breaks the continuous phase symmetry (2.208) leaving only the discrete remnant $\psi_L \to -\psi_L$. In Majorana notation

$$\mathcal{L}_L^m = -\frac{im}{2}\bar{\Psi}_M\Psi_M \ , \tag{2.217}$$

$$\mathcal{L}_{L5}^m = -\frac{m}{2}\bar{\Psi}_M\gamma_5\Psi_M \tag{2.218}$$

$$. \tag{2.219}$$

When both ψ_L and ψ_R are present, two more real quadratic invariants are available, namely

$$\mathcal{L}_D^m = im\bar{\Psi}\Psi = im\left(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L\right) \ , \tag{2.220}$$

$$\mathcal{L}_{D5}^m = m\bar{\Psi}\gamma_5\Psi = -m\left(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L\right) \ . \tag{2.221}$$

Both are left invariant by the overall phase transformation (2.208), but not by the chiral transformation (2.209) under which

$$\Psi \to e^{i\beta\gamma_5}\Psi \quad , \qquad \bar{\Psi} = \Psi^\dagger\gamma^0 \to \bar{\Psi}e^{i\beta\gamma_5}, \tag{2.222}$$

and thus

$$\mathcal{L}_D^m \to im\,\bar{\Psi}e^{2i\beta\gamma_5}\Psi \ . \tag{2.223}$$

Application of (2.164) yields

$$\partial_\mu j_5^\mu = -2m\bar{\Psi}\gamma_5\Psi \ , \tag{2.224}$$

while j^μ of (2.211) is still divergenceless. This is not to say that we cannot have terms quadratic in Dirac fields, free of derivatives that respect chiral invariance as the following example will show. Consider the term

$$\sigma(x)\bar{\Psi}(x)\Psi(x) + i\pi(x)\bar{\Psi}(x)\gamma_5\Psi(x) \ , \tag{2.225}$$

which is the sum of $\mathcal{L}_D^m$ and $\mathcal{L}_{D5}^m$, with the coefficients depending this time on x. To preserve chiral invariance, σ and π must transform under chiral transformations as

$$[\sigma(x) + i\gamma_5\pi(x)] \to \left[\sigma'(x) + i\gamma_5\pi'(x)\right] = e^{-i\beta\gamma_5}\left[\sigma(x) + i\gamma_5\pi(x)\right]e^{-i\beta\gamma_5}; \tag{2.226}$$

for infinitesimal β, the σ and π fields are rotated into one another

$$\delta\sigma = +2\beta\pi \qquad \delta\pi = -2\beta\sigma \ . \tag{2.227}$$

This transformation leaves $\sigma^2 + \pi^2$ invariant. Hence the Lagrangian

$$\mathcal{L}_f = \frac{1}{2}\bar{\Psi}\gamma^\mu\overleftrightarrow{\partial_\mu}\Psi + ih\bar{\Psi}\left[\sigma + i\gamma_5\pi\right]\Psi \ , \tag{2.228}$$

is chirally invariant. If σ and π are canonical fields, h is a dimensionless constant (usually called the Yukawa coupling constant). One can give σ and π a life of their own by adding to $\mathcal{L}$ their kinetic terms as well as self-interactions that preserve (2.227), leading to the Lagrangian

$$\mathcal{L} = \frac{1}{2}\bar{\Psi}\gamma^\mu\overleftrightarrow{\partial_\mu}\Psi + ih\bar{\Psi}\left[\sigma + i\gamma_5\pi\right]\Psi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi - V\left(\sigma^2 + \pi^2\right) \ . \tag{2.229}$$

This Lagrangian has the following symmetries (all global)

a) an overall Dirac phase symmetry

$$\Psi \to e^{i\alpha}\Psi \ ; \quad \sigma \to \sigma \ ; \quad \pi \to \pi \tag{2.230}$$

b) a chiral symmetry

$$\delta\Psi = i\beta\gamma_5\Psi \ , \quad \delta\sigma = 2\beta\pi \ , \quad \delta\pi = -2\beta\sigma \ . \tag{2.231}$$

c) a discrete parity transformation $\Psi \to \gamma_0\Psi$, $\sigma \to \sigma$, $\pi \to -\pi$; thus $\sigma(x)$ is a scalar field while $\pi(x)$ is a pseudoscalar field.

We see that the demand that the symmetry of the kinetic term be preserved in interaction leads to the introduction of extra fields. This is a general feature: linear enlargement of symmetries $\Rightarrow$ additional fields.

Note that in four dimensions invariant terms involving more than two spinor fields have dimensions of at least -6 so that dimensionful constants are needed to recover the dimension of $\mathcal{L}$. In two dimensions, however, terms like $(\bar{\Psi}\Psi)^2$ or $\bar{\Psi}\gamma^\mu\Psi\bar{\Psi}\gamma_\mu\Psi$ have the same dimension as $\mathcal{L}$.

Since the two-component spinor fields are always complex, the equations of motion are obtained by varying independently with respect to ψ_L and $\psi_L^\dagger$. Extra care must be exercised because we treat ψ_L and $\psi_L^\dagger$ as Grassmann fields and we cannot push a $\delta\psi$ past a ψ without changing sign. For instance, we write

$$\delta\mathcal{L}_L = \frac{1}{2}\left(\delta\psi_L^\dagger\sigma^\mu\partial_\mu\psi_L - \partial_\mu\delta\psi_L^\dagger\sigma^\mu\psi_L + \psi_L^\dagger\sigma^\mu\partial_\mu\delta\psi_L - \partial_\mu\psi_L^\dagger\sigma^\mu\delta\psi_L\right) \quad (2.232)$$

$$= \delta\psi_L^\dagger\sigma^\mu\partial_\mu\psi_L - \left(\partial_\mu\psi_L^\dagger\sigma^\mu\right)\delta\psi_L + \text{surface terms} \; . \quad (2.233)$$

which leads to the conjugate equations

$$\sigma^\mu\partial_\mu\psi_L = 0 \quad \text{or} \quad \partial_\mu\psi_L^\dagger\sigma^\mu = 0 \; . \quad (2.234)$$

In the case of the Dirac spinor, independent variations for Ψ and $\bar{\Psi}$ lead to the equations of motion.

Finally, let us note that one can build more complicated invariants involving spinor fields such as $\partial_\mu\bar{\Psi}\partial^\mu\Psi$. While there is nothing wrong with this type of term as far as invariance requirements, it does not lead to satisfactory theories in the sense that it violates the connection between spin and statistics. We will come back to this subject later, when we consider gauge theories.

2.7.1 PROBLEMS

A. Show that $\mathcal{L}_R$ with $\psi_R = \sigma^2\psi_L^*$ is equal to $\mathcal{L}_L$ plus a total divergence.

B. Find the Belinfante energy momentum tensor for $\mathcal{L}_{\text{Dirac}}$.

C. Show that for $\mathcal{L} = \mathcal{L}_{\text{Dirac}}$ the dilatation current can be written as

$$j_D^\mu = x_\rho j_B^{\mu\rho} \; , \quad (2.235)$$

where $j_B^{\mu\rho}$ is the Belinfante form of the energy momentum tensor, thus showing that the Belinfante tensor coincides with the new improved energy momentum tensor for the Dirac field.

D. Given

$$\mathcal{L} = \frac{1}{2}\bar{\Psi}\gamma^\mu\overleftrightarrow{\partial}_\mu\Psi + im\bar{\Psi}\Psi + m'\bar{\Psi}\gamma_5\Psi \; , \quad (2.236)$$

use a chiral transformation to transform the pseudoscalar term away. What is the mass of the resultant Dirac field?

*E. Given a quadratic Lagrangian with both ψ_L and ψ_R

$$\mathcal{L} = \mathcal{L}_L + \mathcal{L}_R + \mathcal{L}_L^m + \mathcal{L}_R^{m'} + iM\left(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R\right) , \qquad (2.237)$$

involving Dirac and Majorana masses. Rediagonalize the fields to obtain unmixed masses. What are the masses of the fields? What is the physical interpretation of the various degrees of freedom?

*F. For the σ-model Lagrangian, a) use Noether's theorem to derive the expression for the conserved chiral current; b) suppose we add to $\mathcal{L}$ a term linear in σ; find the divergence of the chiral current. The last equation embodies the PCAC (partially conserved axial current) hypothesis of pion physics.

*G How does ψ_L transform under a conformal transformation? Show that $\mathcal{L}_L$ is conformally invariant.

2.8 An Action with Scalar and Spinor Fields and Supersymmetry

There are several differences between the simplest kinetic term for spinor fields, $\mathcal{L}_L$ and its counterpart for a scalar field $S(x)$. While $\mathcal{L}_L(x)$ involves one derivative, the scalar kinetic term involves two; while ψ_L is a Grassmann field, $S(x)$ is a normal field, and finally $\mathcal{L}_L$ has the phase invariance (2.208) whereas the kinetic term for one scalar field has none. Yet there are similarities since they both are conformally invariant. In this section we address ourselves to the possibility that there might exist a symmetry on the fields that relate the fermion and scalar kinetic terms. Such a symmetry is called a *supersymmetry* — it has the virtue of allowing non-trivial interactions between the scalar and spinor fields. To increase the odds we made the scalar field kinetic term resemble $\mathcal{L}_L$ as much as possible. This is achieved by taking the kinetic term for two scalar fields, which we call $S(x)$ and $P(x)$, and by comparing it with the kinetic term for a four component Majorana spinor field we call χ. In this way both kinetic terms have a phase invariance of their own. Indeed, the Lagrangian

$$\mathcal{L}_0^{\text{Susy}} = \frac{1}{2}\partial_\mu S \partial^\mu S + \frac{1}{2}\partial_\mu P \partial^\mu P + \frac{1}{4}\bar{\chi}\gamma^\mu \overleftrightarrow{\partial}_\mu \chi \ , \qquad (2.238)$$

besides being conformally invariant has two independent global phase invariances

$$\chi \to e^{i\alpha\gamma_5}\chi \ , \qquad (S+iP) \to e^{i\beta}(S+iP) \ , \tag{2.239}$$

Any further invariance will involve transformations that change the spinless fields S and P into the spinor field χ. The general characteristics of this type of transformation are: 1) Its parameter must itself be a Grassmann spinor field, call it α, a global "infinitesimal" Majorana spinor parameter; 2) In its simplest form, the transformation of S and P must involve no derivative operator and that of χ must involve one since the fermion kinetic term has one less derivative than the scalar kinetic term. Thus we are led to

$$\delta(S \text{ or } P) = \bar{\alpha}M\chi \ , \tag{2.240}$$

where M is some (4×4) matrix. Since no four-vector indices are involved, it must only contain the unit matrix or γ_5. Hence we fix it to be

$$\delta S = ia\bar{\alpha}\chi \ , \qquad \delta P = b\bar{\alpha}\gamma_5\chi \ , \tag{2.241}$$

where a and b are unknown real coefficients. Here we have used the phase invariance (2.239) to define the variation of S to be along the unit matrix and that of P along $i\gamma_5$. The right-hand side of the variations is arranged to be real. [In a Majorana representation for the Dirac matrices, all four components of the Majorana spinors are real and all the matrix elements of the γ-matrices are pure imaginary, so as to have real matrix elements for $i\gamma_5$.] Then we have, (assuming that ∂_μ does not change: see Problem F)

$$\delta\left[\frac{1}{2}\partial_\mu S\partial^\mu S + \frac{1}{2}\partial_\mu P\partial^\mu P\right] = \left(ia\partial^\mu S\bar{\alpha} + b\partial^\mu P\bar{\alpha}\gamma_5\right)\partial_\mu\chi \ . \tag{2.242}$$

What can the variation of χ be? First note that

$$\delta\left[\frac{1}{4}\bar{\chi}\gamma^\mu\overleftrightarrow{\partial}_\mu\chi\right] = \frac{1}{2}\delta\bar{\chi}\gamma^\mu\partial_\mu\chi - \frac{1}{2}\partial_\mu\bar{\chi}\gamma^\mu\delta\chi \ , \tag{2.243}$$

up to a total divergence. Now we use the vector part of "Majorana-flip" properties: $\bar{\xi}\eta$, $\bar{\xi}\gamma_5\eta$, and $\bar{\xi}\gamma_\mu\gamma_5\eta$ are even as $\xi \to \eta$ while $\bar{\xi}\gamma_\mu\eta$ and $\bar{\xi}\sigma_{\mu\nu}\eta$ are odd. These hold for any two Majorana spinors ξ and η (see problem). Their application to (2.243) yields

$$\delta\left[\frac{1}{4}\bar{\chi}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\chi\right] = \delta\bar{\chi}\gamma^{\mu}\partial_{\mu}\chi \ , \tag{2.244}$$

up to surface terms. Putting it all together we see that, up to total derivative "surface" terms,

$$\begin{aligned} \delta\mathcal{L}_0^{\mathrm{Susy}} &= \left(\delta\bar{\chi}\gamma_{\mu} + ia\partial_{\mu}S\bar{\alpha} + b\partial_{\mu}P\bar{\alpha}\gamma_5\right)\partial^{\mu}\chi \ , & (2.245)\\ &= -\left(\partial_{\mu}\delta\bar{\chi}\gamma^{\mu} + ia\partial_{\mu}\partial^{\mu}S\bar{\alpha} + b\partial_{\mu}\partial^{\mu}P\bar{\alpha}\gamma_5\right)\chi \ , & (2.246)\end{aligned}$$

where a partial integration has been performed to obtain (2.246) from (2.245). Thus $\mathcal{L}_0^{\mathrm{Susy}}$ changes only by a total divergence if $\delta\bar{\chi}$ obeys the following equation ($\Box \equiv \partial_{\mu}\partial^{\mu}$)

$$\partial_{\mu}\delta\bar{\chi}\gamma^{\mu} + ia\,\Box\,S\bar{\alpha} + b\,\Box\,P\bar{\alpha}\gamma_5 = 0 \ . \tag{2.247}$$

A solution is easily found,

$$\delta\chi = ia\gamma_{\rho}\alpha\partial^{\rho}S + b\gamma_{\rho}\gamma_5\alpha\partial^{\rho}P \ . \tag{2.248}$$

Here, use of $\gamma_{\rho}\gamma_{\sigma}\partial^{\rho}\partial^{\sigma} = \partial_{\rho}\partial^{\rho}$ has been made. We have therefore achieved our goal by finding a set of transformations between spinless and spin 1/2 fields which leaves the sum of their kinetic terms invariant (up to a canonical transformation). To further convince ourselves of the veracity of our find, we have to see if these transformations close among themselves and form a group.

As a starter, examine the effect of two supersymmetry transformations with Grassmann parameters α_1 and α_2 on the fields. Explicitly

$$\begin{aligned} [\delta_1,\delta_2]\,S &= ia\bar{\alpha}_2\delta_1\chi - (1\leftrightarrow 2) \ , \\ &= ia\bar{\alpha}_2\left[ia\gamma_{\rho}\alpha_1\partial^{\rho}S + b\gamma_{\rho}\gamma_5\alpha\partial^{\rho}P\right] - (1\leftrightarrow 2) \ , & (2.249)\\ &= -2a^2\bar{\alpha}_2\gamma_{\rho}\alpha_1\partial^{\rho}S \ . \end{aligned}$$

To get the last equation we have used the Majorana flip property of the axial vector part. Thus, the effect of two supersymmetry transformations on S is to translate S by an amount $-2a^2\bar{\alpha}_2\gamma_{\rho}\alpha_1$. Let us see what happens to P:

$$\begin{aligned} [\delta_1,\delta_2]\,P &= b\bar{\alpha}_2\gamma_5\delta_1\chi - (1\leftrightarrow 2) \ , \\ &= -2b^2\bar{\alpha}_2\gamma_{\rho}\alpha_1\partial^{\rho}P \ , & (2.250)\end{aligned}$$

where again the Majorana flip identity for axial vector has been used. Since transformations must be the same for S, P and χ, we must have

$$b = \pm a \ . \tag{2.251}$$

Let us finally verify that the action of two supersymmetry transformations on χ is itself a translation:

$$\begin{aligned} [\delta_1, \delta_2]\,\chi &= ia\gamma_\rho \alpha_2 \partial^\rho \delta_1 S + b\gamma_\rho\gamma_5\alpha_2\partial^\rho\delta_1 P - (1 \leftrightarrow 2)\ , && (2.252)\\ &= -a^2\gamma_\rho\alpha_2\bar{\alpha}_1\partial^\rho\chi + b^2\gamma_\rho\gamma_5\alpha_2\bar{\alpha}_1\gamma_5\partial^\rho\chi - (1 \leftrightarrow 2)\ . && (2.253) \end{aligned}$$

We would like to rewrite the right-hand side of this equation in a form similar to the others, that is involving $\bar{\alpha}_2\gamma_\rho\alpha_1$ and not the matrix $\alpha_2\bar{\alpha}_1$ that appears in (1.8.16). We do this using a trick due to Fierz: Take any two Dirac spinors (not necessarily Majorana), Ψ and Λ. The 4×4 matrix $\Lambda\bar{\Psi}$ can be expanded as a linear combination of the 16 Dirac covariants, 1, γ_5, $\gamma_5\gamma_\mu$, γ_μ, $\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$. The coefficients are evaluated by taking the relevant traces. The result is

$$\Lambda\bar{\Psi} = -\frac{1}{4}\bar{\Psi}\Lambda - \frac{1}{4}\gamma_5\bar{\Psi}\gamma_5\Lambda + \frac{1}{4}\gamma_5\gamma_\rho\bar{\Psi}\gamma_5\gamma^\rho\Lambda - \frac{1}{4}\gamma_\rho\bar{\Psi}\gamma^\rho\Lambda + \frac{1}{2}\sigma_{\rho\sigma}\bar{\Psi}\sigma^{\rho\sigma}\Lambda\ . \tag{2.254}$$

The numbers in front of the various terms constitute the first row of the celebrated Fierz matrix. They contain all the necessary information to generate the whole matrix. Application to our case yields

$$\alpha_2\bar{\alpha}_1 - \alpha_1\bar{\alpha}_2 = -\frac{1}{2}\bar{\alpha}_1\gamma^\rho\alpha_2\gamma_\rho + \bar{\alpha}_1\sigma^{\rho\sigma}\alpha_2\sigma_{\rho\sigma}, \tag{2.255}$$

where we have used the Majorana flip properties. Use of (2.251) and (2.255) leads to

$$[\delta_1, \delta_2]\,\chi = -a^2\bar{\alpha}_2\gamma^\mu\alpha_1\gamma_\rho\gamma_\mu\partial^\rho\chi\ . \tag{2.256}$$

By using the anticommutator of the γ-matrices, we rewrite it as

$$[\delta_1, \delta_2]\,\chi = -2a^2\bar{\alpha}_2\gamma^\mu\alpha_1\partial_\mu\chi + a^2\bar{\alpha}_2\gamma^\mu\alpha_1\gamma_\mu\gamma^\rho\partial_\rho\chi\ . \tag{2.257}$$

The first term on the right-hand side is the expected result, but unfortunately we have an extra term proportional to $\gamma^\rho\partial_\rho\chi$. This extra term vanishes only when the classical equations of motion are valid. In order to eliminate this term, we have to enlarge the definition of $\delta\chi$ and see where it leads us. Note that if we add to $\delta\chi$ of (2.248) an extra variation of the form

$$\delta_{\text{extra}}\chi = (F + i\gamma_5 G)\,\alpha\ , \tag{2.258}$$

where F and G are also functions of x, but not canonical fields since they have dimensions of L^{-2}, the relations (1.8.13) and (1.8.14) are not affected because of the Majorana flip conditions. For example

$$\begin{aligned}
[\delta_1, \delta_2]_{\text{extra}}\, S &= ia\bar{\alpha}_2\delta_{1\ \text{extra}}\chi - (1 \leftrightarrow 2)\ , \\
&= ia\bar{\alpha}_2\,(F + i\gamma_5 G)\,\alpha_1 - (1 \leftrightarrow 2)\ , \\
&= 0\ .
\end{aligned} \tag{2.259}$$

However, this extra variation gives a contribution on χ, namely

$$[\delta_1, \delta_2]_{\text{extra}}\,\chi = (\delta_1 F + i\gamma_5\delta_1 G)\,\alpha_2 - (1 \leftrightarrow 2)\ . \tag{2.260}$$

The extra term in (2.257) can be rewritten in a suggestive way by means of the Fierz rearrangement

$$\bar{\alpha}_2\gamma^\mu\alpha_1\gamma_\mu = -\alpha_1\bar{\alpha}_2 + \gamma_5\alpha_1\bar{\alpha}_2\gamma_5 - (1 \leftrightarrow 2)\ . \tag{2.261}$$

Comparison with (2.260) now shows that by choosing

$$\delta_1 F = -a^2\bar{\alpha}_1\gamma^\rho\partial_\rho\chi\ , \tag{2.262}$$

$$\delta_1 G = -ia^2\bar{\alpha}_1\gamma_5\gamma^\rho\partial_\rho\chi\ , \tag{2.263}$$

we cancel the extra term and obtain the desired result. We leave it as an exercise (see problem) to show that the full operator relation

$$[\delta_1, \delta_2] = -2a^2\bar{\alpha}_2\gamma^\mu\alpha_1\partial^\mu\ , \tag{2.264}$$

is satisfied when acting on F and G.

Unfortunately, the new $\delta\chi$ does not leave the original Action invariant because of δ_{extra}. But we observe that (up to surface terms)

$$\begin{aligned}
\delta_{\text{extra}}\mathcal{L}_0^{\text{Susy}} &= \delta_{\text{extra}}\bar{\chi}\gamma^\mu\partial_\mu\chi\ , \\
&= F\bar{\alpha}\gamma^\rho\partial_\rho\chi + iG\bar{\alpha}\gamma_5\gamma^\rho\partial_\rho\chi\ ,
\end{aligned} \tag{2.265}$$

$$= -\frac{1}{2a^2}\delta\left(F^2+G^2\right)$$

Therefore the Action

$$S_0^{\rm Susy} = \int d^4x \left[\frac{1}{2}\partial_\mu S\partial^\mu S + \frac{1}{2}\partial_\mu P\partial^\mu P + \frac{1}{4}\bar{\chi}\gamma^\rho\overleftrightarrow{\partial}_\rho\chi + \frac{1}{2a^2}\left(F^2+G^2\right)\right] , \tag{2.266}$$

is invariant under the supersymmetry transformations

$$\begin{aligned} \delta S &= ia\bar{\alpha}\chi \ ; \quad \delta P = a\bar{\alpha}\gamma_5\chi \ ; \quad \delta F = -a^2\bar{\alpha}\gamma^\rho\partial_\rho\chi \ ; \\ \delta G &= -ia^2\bar{\alpha}\gamma_5\gamma^\rho\partial_\rho\chi \ ; \quad \delta\chi = ia\gamma_\rho\alpha\partial^\rho S + a\gamma_\rho\gamma_5\alpha\partial^\rho P + (F+i\gamma_5 G)\,\alpha \ . \end{aligned} \tag{2.267}$$

These transformations now all satisfy the operator equation (2.264). This Action was first written down Gol'fand and Likthman reference? and by Wess and Zumino, *Nucl. Phys.* **B78** (1974) 1.

With the introduction of the auxiliary fields F and G, we now have the same number of spinless (S, P, F and G) and spinor (the four real components of χ) fields irrespective of the equations of motion. The reader can convince himself that "on mass-shell" (*i.e.*, on the classical path) where F and G are not necessary, the balance between spinless and spinor degrees of freedom is still true. This balance between the number of boson (even spin) and fermion (odd spin) degrees of freedom is a general feature of relativistic supersymmetric theories.

From (2.264) we see that the effect of two supersymmetry transformations is a translation. In addition, since the supersymmetry parameters are spinors, it follows that the generators of the supersymmetry transform as spinors. Therefore we have an enlargement of the Poincaré group to include the supersymmetry generators (see problem). The F and G fields have no kinetic terms; they serve as auxiliary fields which are totally uncoupled for the free theory.

The beauty of the supersymmetry transformations (2.267) is their generalizability to interacting theories. For instance, one can introduce a supersymmetric Yukawa coupling term which leaves one global chiral invariance

$$\mathcal{L}_y^{\rm Susy} = ih\left(\bar{\chi}\chi S + i\bar{\chi}\gamma_5\chi P - \frac{i}{a}F\left(P^2-S^2\right) - \frac{2i}{a}GSP\right) , \tag{2.268}$$

or

$$\mathcal{L}_y^{\mathrm{WZ}'} = ih' \left(\bar{\chi}\chi P - i\bar{\chi}\gamma_5\chi S + \frac{i}{a} G \left(S^2 - P^2\right) + \frac{2i}{a} FSP \right) . \tag{2.269}$$

Even mass terms can be written down

$$\mathcal{L}_m^{\mathrm{WZ}} = -i\frac{m}{2} \left(\bar{\chi}\chi + \frac{2i}{a} SF - \frac{2i}{a} PG \right) . \tag{2.270}$$

These combinations are invariant under supersymmetric transformations only after being integrated over space-time.

We can use this term to find an important (and fatal for phenomenology) property of theories with *exact* supersymmetry. Consider the equations of motion for the GLWZ Lagrangian with a mass term. They are

$$\gamma^\mu \partial_\mu \chi \;=\; im\chi \; ; \quad \Box S = \frac{m}{a} F \; ; \quad \Box P = -\frac{m}{a} G \; ; \tag{2.271}$$

$$0 \;=\; \frac{1}{a^2} F + \frac{m}{a} S \; ; \quad 0 = \frac{1}{a^2} G - \frac{m}{a} P \; label A231 \tag{2.272}$$

The last two equations can be solved for F and G in terms of S and P without great difficulty and their result substituted in the equations for S and P, yielding

$$\Box S = -m^2 S \; ; \qquad \Box P = -m^2 P \; . \tag{2.273}$$

Hence, the three fields χ, S and P all have the same mass. This is a general feature of relativistic supersymmetry : all fields entering a supermultiplet have the same mass. This is because the mass operator $P_\mu P^\mu$ commutes with all supersymmetry generators. As an immediate consequence, we see that *exact* supersymmetry cannot exist in nature because particles of different spins show no mass degeneracy.

[Finally, this little calculation hints at the role of the auxiliary fields when equations of motion can be solved for. The following embryonic model of how auxiliary fields work will illustrate the point independently of the equations of motion: let $\varphi(x)$ be a scalar field and $A(x)$ be an auxiliary field. Take

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2} A^2 + A\varphi^2 \; , \tag{2.274}$$

and complete the square to obtain

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\left(A + \varphi^2\right)^2 - \frac{1}{2}\varphi^4 \ . \tag{2.275}$$

Redefine the new auxiliary field $A' = A + \varphi^2$; it decouples from φ, and we are left with the interaction Lagrangian $\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}\varphi^4$].

This is the simplest example of a supersymmetric theory in four dimensions. Supersymmetry remains as of this writing a purely "theoretical symmetry" with no experimental support. However, we felt it instructive to alert the reader to the existence of non-trivial symmetries among field of different spins. After all, there must be a reason why Nature entertains particles with both integer and half-integer spin!

2.8.1 PROBLEMS

A. Prove the Majorana flip properties.

B. Verify the Fierz decomposition (2.254) by using γ-matrix identities.

C. Identify the chiral invariance of $S_0^{\rm Susy}$ and express its action on the fields.

D. Show that $\int (a\bar{\chi}\gamma_5\chi - 2SG - 2PF)\, d^4x$ is a supersymmetric invariant.

*E. Introduce the Majorana spinor generators of supersymmetry Q by writing a finite supersymmetry transformation as $e^{i\bar{\alpha}Q}$. Derive the expression for the anticommutator of two Q's and the commutator of Q with the Poincaré generators. The ensuing algebra involving both commutators and anticommutators form a graded Lie algebra (superalgebra). As a consequence show that Q commutes with the mass.

*F. Find the change of the coordinate x^μ under a supersymmetry, and verify that ∂_μ is invariant.

**G. Use Noether's theorem to derive the expression for the conserved supersymmetric current. Use caution because $\mathcal{L}_0^{\rm Susy}$ picks up a total divergence under supersymmetric variations.