3

The Feynman Path Integral in Field Theory

We now want to generalize the path integral to Field Theory. Reasoning by analogy with Quantum Mechanics, and taking for convenience a real scalar field as an example, we can describe the state of the system at a given time by the ket $|\,\varphi(\vec{x})\,\rangle_t$. We call it a shape state. We can compute the transition amplitude between a shape at t_0 and a new shape at a later time t, but we are immediately faced with the need to resolve the shape ket in terms of the possible physical states of the system. The way one identifies the states of the system usually relies very much on the success of the perturbation theory: one starts with a zeroth order approximation to the theory in which the states are easily recognized. The full theory is recovered by adding a small perturbation to the idealized zeroth-order theory. Then the effects of this small perturbation on the zeroth-order states are computed. These steps are possible only when one can recognize the full theory in terms of a small perturbation on a simple system and, having done that, calculate corrections to the idealized zeroth-order states. As an example of this procedure,consider quantum electrodynamics, QED, where a small parameter $\alpha \approx (137)^{-1}$ is easily recognized. Then the zeroth order theory is that for which $\alpha = 0$. It is easily identified in terms of *idealized* photon and electron (or any other charged particle such as the muon, tau, quarks, $\cdots$) states. The effect of the interaction is then computed on these states and their interactions order by order in α. After some trickery (*i.e.*, renormalization theory), these corrections are found to yield *physical* electron and photon states and their interactions. The point of this discussion is to emphasize the reliance of the success of QED on our ability to recognize the emergence of idealized electron- and photon-like states in a zeroth order theory. This procedure is successful only because α happens to be a *small* number, otherwise such an identification cannot be made. An example of a yet unresolved theory is QCD, quantum-chromodynamics, which is thought to

describe the interactions of quarks and gluons (the QCD equivalent of photons). It is believed that quarks are not physical particles but that bound states of quarks such as protons, π-mesons, etc., are physical. Yet there is nothing to tell us *a priori* that quarks are *not* physical states. Operationally then we have to decide the size of the quark couplings among themselves. If they are small, then quarks could serve as physical states (here physical applies to states that survive in isolation); if they are big, it is not good form to talk of quarks because quarks would tend to bind strongly among themselves, and not appear as asymptotic states. Again Nature smiles upon us because *in some regime* quarks bind very weakly, which enables us to apply perturbation theory to QCD.

Thus the knowledge of the physical states of a Field Theory depends very much on the solution. But this is what we are trying to find! We have to obtain a path integral formulation that does not rely on the knowledge of its physical states (we'll derive that). The trick that gets us off the hook is very simple. Whatever the states, everyone agrees there must be a state of least energy, call it the vacuum state. It may be a very complicated structure (*e.g.*, superconductivity), and may be inhabited by all kinds of strange objects, but nevertheless it is thought to exist. Suppose now we ask for the transition amplitude of the system from the vacuum state at $t = -\infty$ to the vacuum state at $+\infty$ *in the presence of an arbitrary driving force*. This means that at any time we reserve the right to drive the system in any way we please and watch it respond. This ought to tell us all we want to know provided we are sufficiently clever to apply probes that will give recognizable responses. This will then be the strategy: a) work out the amplitude $\langle \Omega \mid \Omega \rangle_J$ for an arbitrary source $J(x)$, b) interpret (more exactly recognize) the results in terms of scattering amplitudes, c) use these amplitudes to calculate the physical consequences of the theory.

Traditionally, the source will be attached to a local field, the rationale being that this provides a generic driving term since all possible sources can be built in terms of it. When perturbation theory is applicable, the local fields will naturally be interpreted in terms of particles.

We start with the simplest Field Theory: a self-interacting scalar field, described by the action

$$S = \int d^4x \left[\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - V(\varphi) \right] , \tag{3.1}$$

$$= \int d^4x \, \mathcal{L}(\varphi, \partial_\mu \varphi) \ . \tag{3.2}$$

To construct the Hamiltonian density $\mathcal{H}$, define the canonical momentum

$$\pi(x) - \frac{\partial \mathcal{L}}{\partial\left[\partial_0 \varphi\right]} = \partial_0 \varphi \equiv \dot{\varphi} \ , \tag{3.3}$$

and then perform a Legendre transformation

$$\begin{aligned} \mathcal{H}(\pi, \varphi, \vec{\nabla}\varphi) &= \pi\dot{\varphi} - \mathcal{L} \ , & (3.4) \\ &= \frac{1}{2}\left(\pi^2 + \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi + m^2\varphi^2\right) + V(\varphi) \ . & (3.5) \end{aligned}$$

The vacuum to vacuum amplitude is **defined** to be

$$\langle \Omega \mid \Omega \rangle_J \equiv W[J] = N \int \mathcal{D}\varphi \mathcal{D}\pi \, e^{i\langle \pi\dot{\varphi} - \mathcal{H} + J\varphi \rangle} \ , \tag{3.6}$$

where N is a constant (usually ill-defined), the $\langle \cdots \rangle$ now means integration over spacetime, and $J(x)$ is an arbitrary source. We use the techniques of the previous chapter to integrate over π to obtain

$$W[J] = N' \int \mathcal{D}\varphi \, \exp\left\{ i\langle \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - V(\varphi) + J\varphi \rangle \right\} \ . \tag{3.7}$$

In this case $\mathcal{D}\varphi$ (or $\mathcal{D}\pi$) stands for the product of all the $d\varphi_k$ where φ_k is the value of φ at $x = x_k$.

The integrand in (3.1.7) is oscillatory and even this path integral is not well-defined. There are two ways to remedy this problem:

-a) put in a convergence factor $e^{-1/2\,\epsilon\langle\varphi^2\rangle}$ with $\epsilon > 0$, or

-b) define W in Euclidean space by setting

$$x_0 = -i\bar{x}_0 \qquad d^4x = -id^4\bar{x} \qquad \partial_\mu\varphi\partial^\mu\varphi = -\bar{\partial}_\mu\varphi\bar{\partial}_\mu\varphi \ , \tag{3.8}$$

where the bar denotes Euclidean space variables, $\bar{\partial}_\mu = \partial/\partial\bar{x}^\mu$. Then Eq. (3.1.7) becomes

$$W_E[J] = N_E \int \mathcal{D}\varphi \, \exp\left\{ -\langle \frac{1}{2}\bar{\partial}_\mu\varphi\bar{\partial}_\mu\varphi + \frac{1}{2}m^2\varphi^2 + V(\varphi) - J\varphi \rangle \right\} \ . \tag{3.9}$$

The exponent of the integrand is now negative definite for positive m^2 and V when $J = 0$.

In either case, the generating functional is used to manufacture the Green's functions which are the coefficient of the functional expansion

$$W[J] = \sum_{N=0}^{\infty} \frac{(i)^N}{N!} \left\langle J_1 J_2 \cdots J_N G^{(N)}(1,\, 2,\, \cdots, N) \right\rangle_{1\cdots N} \ , \tag{3.10}$$

or

$$G^{(N)}(1,\, 2, \cdots, N) = \frac{1}{(i)^N} \frac{\delta}{\delta J_1} \frac{\delta}{\delta J_2} \cdots \frac{\delta}{\delta J_N} W[J] \bigg|_{J=0} \tag{3.11}$$

where $J_i = J(\bar{x}_i)$, etc. , $\cdots$, $\langle \cdots \rangle_{1\cdots N}$ means integration over $d^4\bar{x}_1 \cdots d^4\bar{x}_N$. The task at hand is to compute the functions $G^{(N)}(\bar{x}_1, \cdots, \bar{x}_N)$, perturbatively or otherwise. In p-space they will be identified as transition amplitudes. This is not trivial since transition amplitudes must satisfy unitarity and completeness criteria. The Euclidean space functional $W_E[J]$ is used to construct these functions, $G_E^{(N)}(\bar{x}_1, \cdots, \bar{x}_N)$; they are related to the $G^{(N)}$ by analytic continuation (Wick rotation), which presupposes that no singularities are encountered in the process of contour rotation. This is sufficient to determine the singularity structure of $G^{(N)}$, but to show that it is consistent with unitarity is not a trivial matter. These rather obscure remarks will hopefully become transparent in the light of explicit calculations.

3.1 The Feynman Propagator

In this section we evaluate $W[J]$ in Minkowski space using the $-i\epsilon$ procedure when $V = 0$. Let

$$W_0[J] \equiv N \int \mathcal{D}\varphi \, \exp\left\{ i \left\langle \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} \left(m^2 - i\epsilon \right) \varphi^2 + J\varphi \right\rangle \right\} \ . \tag{3.12}$$

It is most easily evaluated in Fourier transform (momentum) space, following the same techniques used for the driven harmonic oscillator. We introduce the four-dimensional Fourier transform

$$\tilde{F}(p) = \int_{-\infty}^{+\infty} \frac{d^4x}{(2\pi)^2} \, e^{-ip\cdot x} F(x) \ , \tag{3.13}$$

$$F(x) = \int_{-\infty}^{+\infty} \frac{d^4p}{(2\pi)^2} \, e^{ip\cdot x} \tilde{F}(p) \ , \tag{3.14}$$

and

$$\delta^{(4)}(x - x') = \delta(x^0 - x^{0\prime}) \delta(\vec{x} - \vec{x}') \ , \tag{3.15}$$

$$= \int_{-\infty}^{+\infty} \frac{d^4p}{(2\pi)^4}\, e^{i(x-x')\cdot p} \ , \tag{3.16}$$

where $x \cdot p = x^0 p^0 - \vec{x} \cdot \vec{p}$, and F is any sufficiently well-behaved function. The exponent of the integrand is easily expressed in terms of the Fourier transforms of φ and J; it reads

$$\frac{i}{2} \int d^4p \left[\tilde{\varphi}'(p) \left[p^2 - m^2 + i\epsilon \right] \tilde{\varphi}'(-p) - \tilde{J}(p) \left[p^2 - m^2 + i\epsilon \right]^{-1} \tilde{J}(-p) \right] \ , \tag{3.17}$$

where

$$\tilde{\varphi}'(p) = \tilde{\varphi}(p) + \left[p^2 - m^2 + i\epsilon \right]^{-1} \tilde{J}(p) \ . \tag{3.18}$$

The new variable φ' differs from φ in function space by a constant, so that

$$\mathcal{D}\varphi = \mathcal{D}\varphi' \ . \tag{3.19}$$

Putting it all together, we find

$$W_0[J] = N \exp\left\{ -\frac{i}{2} \int d^4p \frac{|\tilde{J}(p)|^2}{p^2 - m^2 + i\epsilon} \right\} \int \mathcal{D}\varphi'\, e^{i\left\langle \frac{1}{2}\partial_\mu \varphi' \partial^\mu \varphi' - \frac{1}{2}(m^2 - i\epsilon)\varphi'^2 \right\rangle} \ . \tag{3.20}$$

We observe that the φ'-dependent term is just the same as the φ term in (3.2.1) with $J = 0$, allowing us to write

$$W_0[J] = W_0[0] \exp\left\{ -\frac{i}{2} \int d^4p \frac{\tilde{J}(p)\tilde{J}(-p)}{p^2 - m^2 + i\epsilon} \right\} \ . \tag{3.21}$$

By adjusting N, we can take $W_0[0] = 1$. Note that $W_0[0]$ can be formally calculated using formulae of Appendix A. The important thing is that we have succeeded in finding the explicit dependence of $W_0[J]$ on J. The use of the inverse Fourier transform yields

$$W_0[J] = W_0[0]\, e^{-\frac{i}{2}\langle J_1 \Delta_{F12} J_2 \rangle_{12}} \ , \tag{3.22}$$

where Δ_{F12} stands for $\Delta_F(x_1 - x_2)$:

$$\Delta_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \ . \tag{3.23}$$

It is the Feynman propagator. We now interpret the Green's functions obtained from W_0. From (3.1.10) we find

$$G_0^{(2)}(x_1, x_2) = i\Delta_F(x_1 - x_2) \tag{3.24}$$

$$G_0^{(4)}(x_1, x_2, x_3, x_4) = -[\Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) \tag{3.25}$$

$$+\Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3)] \ , \tag{3.26}$$

etc. $\cdots$ together with the vanishing of the G's with odd number of variables. This fact is easy to understand since $W_0[J]$ depends only on J^2. In passing, note that all G's are functions of only the difference of coordinates, reflecting the translation invariance of the theory. Another lesson is that the higher Green's functions can all be understood in terms of $G_0^{(2)}$. Hence it would appear more convenient to set

$$W[J] = e^{iZ[J]} \ , \tag{3.27}$$

and define new Green's functions in terms of $Z[J]$

$$iZ[J] = \sum_N \frac{(-1)^N}{N!} \left\langle G_c^{(N)}(1, \cdots, N) J_1 \cdots J_N \right\rangle_{1\cdots N} \ . \tag{3.28}$$

Then, at least in the case of W_0, we see that G_c is much simpler than G.

We now turn to the physical meaning of the Green's functions generated by W_0. By direct computation, we find

$$\left(\partial_\mu \partial^\mu + m^2\right) \Delta_F(x) = -\delta^{(4)}(x) \ , \tag{3.29}$$

thus identifying Δ_F with the Green's function of the operator $\Box + m^2$. Its boundary conditions are determined from the $-i\epsilon$ procedure by the path integral. We can therefore identify $\Delta_F(x-y)$ with the propagator of a signal from x to y. The signals it propagates are single particle and antiparticle states, since those are solutions of the Klein-Gordon equation

$$\left(\Box + m^2\right) \varphi = 0 \ . \tag{3.30}$$

The $-i\epsilon$ procedure tells us which of the solutions are propagated. One finds that positive energy solutions of the Klein-Gordon equations are propagated forward in time while negative energy solutions are propagated backwards in time (see problem).

Since these solutions are to be identified with particle (antiparticle) states

of energy $E = p^0 = \sqrt{\vec{p}^2 + m^2}\left(-\sqrt{\vec{p}^2 + m^2}\right)$, we arrive at the very nice physical picture that information propagates forward in time with particle and backward in time with antiparticle. As an example, ask how many ways a quantum number can be carried from x to y given that we have a particle which carries one unit and an antiparticle with minus one unit. The quantum number could be the electric charge and the particle a π^+ meson. There are two ways: by propagating a π^+ meson from x to y thus destroying a charge $+1$ at x and depositing it at y or by using a π^-, the antiparticle of π^+, to carry a negative charge from y to x.

The lesson is: 1) we have recognized the Green's function as the propagator of certain signals, and 2) we know which signals it propagates. Then it naturally follows that the states are, in our example, going to be particles of mass m^2, and we interpret $G_0^{(2)}(x-y)$ as the amplitude for this particle to go from x to y. We can invent a diagrammatic representation in x-space, associating with $\Delta_F(x-y)$ a line connecting the two space-like points x and y:

$$G_0^{(2)}(x,y): \qquad \dot{x} \qquad \dot{y}\ . \tag{3.31}$$

For higher Green's functions we just diagrammatically add the contribution of, say (3.2.13),

$$G_0^{(4)}(x_1,x_2,x_3,x_4): \quad \begin{pmatrix} \dot{x}_1 & \dot{x}_2 \\ \dot{x}_3 & \dot{x}_4 \end{pmatrix} + \begin{pmatrix} \dot{x}_1 & \dot{x}_3 \\ \dot{x}_2 & \dot{x}_4 \end{pmatrix} + \begin{pmatrix} \dot{x}_1 & \dot{x}_4 \\ \dot{x}_2 & \dot{x}_3 \end{pmatrix}\ . \tag{3.32}$$

It is clear that $G_0^{(4)}$ is a rather disconnected object. It can be interpreted as the amplitude for, say, a transition from x_1, x_2 to x_3, x_4. In this approximation there are only so many ways for signal propagation, all expressed diagrammatically in the above picture.

A much more transparent interpretation is obtained in the Fourier transformed space. We have seen that the nature of Δ_F impels us to interpret p_μ as the four-momentum of a particle state. This is consistent with translation invariance, leading to p-conservation. Indeed, since G depends only on differences of x's, the naive Fourier transform

$$\int d^4x_1 \cdots d^4x_N\, e^{-i(p_1x_1+\cdots+p_Nx_N)} G^{(N)}(x_1 \cdots x_N) \tag{3.33}$$

necessarily contains a δ-function of $(p_1 + \cdots + p_N)$. So, instead we set

$$\tilde{G}^{(N)} \quad (p_1, \cdots, p_N)(2\pi)^4 \delta^{(4)}(p_1 + \cdots + p_N) \tag{3.34}$$

$$= \int d^4x_1 \cdots d^4x_N \, e^{-i(p_1x_1 + \cdots + p_Nx_N)} G^{(N)}(x_1, \cdots, x_N) \ , \tag{3.35}$$

with $\tilde{G}^{(N)}(p_1, \cdots, p_N)$ defined only when $p_1 + \cdots + p_N = 0$. For example,

$$\tilde{G}_0^{(2)}(p, -p) = \frac{1}{p^2 - m^2 + i\epsilon} \tag{3.36}$$

gives the amplitude that a particle of momentum P and mass m^2 propagates. We can represent this diagrammatically as well

$$\tilde{G}_0^{(2)}(p) = .5.2 \ . \tag{3.37}$$

In general, however, we will represent the Green's function $\tilde{G}^{(N)}(p_1, \cdots, p_N)$ as a blob with N lines, labeled by $p_1, p_2, \cdots, p_N$, entering it and with $p_1 + p_2 + \cdots + p_N = 0$, which reflects the conservation of momentum:

$$\tilde{G}^{(N)}(p_1, \cdots, p_N) = 11 \ . \tag{3.38}$$

It will be interpreted, say, as the scattering amplitude of states of momenta $p_1, \cdots, p_j$ into states of momenta $p_{j+1}, \cdots, p_N$, if we take the lines $j+1, \cdots, N$ to be outgoing. Again, note that the form of $\tilde{G}_0^{(2)}$ is what suggests the nature of the external states. We will later discuss the unitarity constraints imposed on $\tilde{G}$.

3.1.1 PROBLEMS

A. Starting from the Euclidean formulation for $W_E[J]$, work out the corresponding expression of the Feynman propagator and show that by analytic continuation in Minkowski space it reduces to the usual one.

B. Given $\Delta_F(x)$, show that is propagates positive energy signals forward in time, and negative energy signals backward in time.

C. Find the real and imaginary parts of $\Delta_F(x)$; interpret physically. Can you express Δ_F in terms of $\Im\Delta_F$?

3.2 The Effective Action

Out of the generating functional we can construct local quantities which lend themselves to familiar interpretations. For instance,

$$\frac{\delta W_0}{\delta J(x)} = -i \left\langle \Delta_F(x-1) J_1 \right\rangle_1 W_0[J] \ , \tag{3.39}$$

so that

$$\varphi_{\rm cl}^{[0]}(x) \equiv -i \frac{\delta \ln W_0}{\delta J(x)} = \frac{\delta Z_0}{\delta J(x)} \ , \tag{3.40}$$

satisfies the classical equation of motion [using (3.2.16)]

$$\left(\Box + m^2\right) \varphi_{\rm cl}^{(0)}(x) = J(x) \ . \tag{3.41}$$

In fact, we can use (3.3.3) to replace $J(x)$ in terms of $\varphi_{\rm cl}^{(0)}(x)$. Formally, it comes down to performing a functional Legendre transformation; introducing

$$\Gamma_0[\varphi_{\rm cl}^{(0)}] = Z_0[J] - \langle J \varphi_{\rm cl}^{(0)} \rangle \ , \tag{3.42}$$

we see by using (3.3.2) that Γ_0 is independent of J. In this case it is easy to find the explicit form of Γ_0 by replacing J in terms of $\varphi_{\rm cl}^{(0)}$. We find (integrating by parts as we go along)

$$\begin{aligned} \Gamma_0 \left[\varphi_{\rm cl}^{(0)}\right] &= -\frac{1}{2} \left\langle \left[\left(\Box + m^2\right) \varphi_{\rm cl}^{(0)}\right]_1 \Delta_{F_{12}} \left[\left(\Box + m^2\right) \varphi_{\rm cl}^{(0)}\right]_2 \right\rangle & (3.43) \\ &\quad - \left\langle \varphi_{\rm cl}^{(0)} \left(\Box + m^2\right) \varphi_{\rm cl}^{(0)} \right\rangle & (3.44) \\ &= -\frac{1}{2} \left\langle \varphi_{\rm cl}^{(0)} \left(\Box + m^2\right) \varphi_{\rm cl}^{(0)} \right\rangle \ , & (3.45) \end{aligned}$$

using (3.2.16). Integration by parts now yields the final form

$$\Gamma_0 \left[\varphi_{\rm cl}^{(0)}\right] = \frac{1}{2} \int d^4x \left[\partial_\mu \varphi_{\rm cl}^{(0)} \partial^\mu \varphi_{\rm cl}^{(0)} - m^2 {\varphi_{\rm cl}^{(0)}}^2\right] \ , \tag{3.46}$$

which is the free action we had started from!

A similar procedure can be carried out in the general case $V \neq 0$. We form

$$\varphi_{\rm cl}(x) \equiv -i\frac{\delta \ln W}{\delta J} = \frac{\delta Z[J]}{\delta J}, \tag{3.47}$$

and try to compute the *effective action*

$$\Gamma\left[\varphi_{\rm cl}\right] = Z[J] - \langle J\varphi_{\rm cl}\rangle \ , \tag{3.48}$$

with now

$$J(x) = -\frac{\delta\Gamma[\varphi_{\rm cl}]}{\delta\varphi_{\rm cl}(x)} \ . \tag{3.49}$$

as seen by differentiating (3.3.8) with respect to $\varphi_{\rm cl}$. (Of course, $\Gamma[\varphi_{\rm cl}]$ depends only on $\varphi_{\rm cl}$ and $Z[J]$ only on J.) By the way, we observe that since Γ is an effective action, (3.3.9) is proportional to its equation of motion coming from extremizing Γ. In the $V = 0$ case this is obvious from (3.3.3).

In order to derive an equation of motion for $\varphi_{\rm cl}(x)$, we have to write $W[J]$ in a manageable form. We write

$$\begin{aligned} W[J] &\equiv N\int \mathcal{D}\varphi\, e^{i\ll\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\frac{1}{2}(m^2-i\epsilon)\varphi^2-V(\varphi)+\rangle} \ , & (3.50)\\ &= N\int \mathcal{D}\varphi\, e^{-i\ll V(\varphi)\rangle} e^{i\ll\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\frac{1}{2}(m^2-i\epsilon)\varphi^2+J\varphi\rangle} \ . & (3.51)\end{aligned}$$

Now comes the trick: observe that

$$\frac{1}{i}\ \frac{\delta}{\delta J(x)}\, e^{i\langle J\varphi\rangle} = \varphi(x)\, e^{i\langle J\varphi\rangle} \ , \tag{3.52}$$

and since J and φ are independent variables, the same will be true for any function of φ. In particular

$$e^{-i\ll V(\varphi)\rangle} e^{i\ll J\varphi\rangle} = e^{-i\ll V\left(\frac{1}{i}\ \frac{\delta}{\delta J}\right)\rangle} e^{i\ll J\varphi\rangle} \ . \tag{3.53}$$

This allows us to take the V dependent term out of the integral

$$\begin{aligned} W[J] &= e^{-i\ll V\left(\frac{1}{i}\ \frac{\delta}{\delta J}\right)\rangle} N\int \mathcal{D}\varphi e^{i\ll\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\frac{1}{2}(m^2-i\epsilon)\varphi^2+J\varphi\rangle} & (3.54)\\ &= e^{-i\ll V\left(\frac{1}{i}\ \frac{\delta}{\delta J}\right)\rangle} W_0[J] \ , & (3.55)\end{aligned}$$

or

$$e^{iZ[J]} = W[J] = N\, e^{-i\ll V\left(\frac{1}{i}\ \frac{\delta}{\delta J}\right)\rangle} e^{-\frac{i}{2}\ll J_1 \Delta_{F12} J_2\rangle} \ . \tag{3.56}$$

This equation will be the starting point of the perturbative evaluation of $W[J]$. For the moment, we use it to derive an equation for $\varphi_{\rm cl}$. From (3.3.16)

$$\begin{aligned} \frac{\delta W}{\delta J_x} &= -i\, e^{-i\ll V\left(-i\frac{\delta}{\delta J}\right)\rangle} \langle \Delta_{Fx1} J_1\rangle_1 W_0[J] && (3.57)\\ &= -i\, e^{-i\langle V\left(-i\frac{\delta}{\delta J}\right)\rangle} \langle \Delta_{Fx1} J_1\rangle_1\, e^{i\ll V\left(-i\frac{\delta}{\delta J}\right)\rangle} W[J] \ . && (3.58) \end{aligned}$$

it follows that

$$\left(\Box + m^2\right) \frac{\delta W}{\delta J_x} = i\mathcal{O}_x W[J] \ , \tag{3.59}$$

where

$$\mathcal{O}_x = e^{-i\ll V\left(-i\frac{\delta}{\delta J}\right)\rangle} J_x\, e^{i\ll V\left(-i\frac{\delta}{\delta J}\right)\rangle} \ . \tag{3.60}$$

We can evaluate $\mathcal{O}_x$ by means of yet another trick. Set

$$\mathcal{O}_x(\lambda) = e^{-i\lambda\ll V\left(-i\frac{\delta}{\delta J}\right)\rangle} J_x\, e^{i\lambda\ll V\left(-i\frac{\delta}{\delta J}\right)\rangle} \ , \tag{3.61}$$

where λ is a parameter. Clearly,

$$\frac{d\mathcal{O}_x(\lambda)}{d\lambda} = e^{-i\lambda\ll V\left(-i\frac{\delta}{\delta J}\right)\rangle} \left[-i \ll V\left(-i\frac{\delta}{\delta J}\right)\rangle, J_x\right] e^{i\lambda\ll V\left(-i\frac{\delta}{\delta J}\right)\rangle} \ . \tag{3.62}$$

But

$$\left[V\left(-i\frac{\delta}{\delta J_y}\right), J_x\right] = -iV'\left(-i\frac{\delta}{\delta J_y}\right)\delta^{(4)}(x-y) \ , \tag{3.63}$$

where V' is the derivative of V with respect to its argument. Integrating over y, we find

$$\frac{d\mathcal{O}_x(\lambda)}{d\lambda} = -V'\left(-i\frac{\delta}{\delta J_x}\right) \ . \tag{3.64}$$

This equation is now integrated over λ to yield

$$\mathcal{O}_x = \mathcal{O}_x(\lambda = 1) = J_x - V'\left(-i\frac{\delta}{\delta J_x}\right) \ . \tag{3.65}$$

Hence

$$\left(\Box_x + m^2\right)\frac{\delta W}{\delta J_x} = i\left(J_x - V'\left(-i\frac{\delta}{\delta J_x}\right)\right)W[J] \ , \tag{3.66}$$

or

$$\left(\Box_x + m^2\right)\varphi_{\rm cl}(x) = J(x) - \frac{1}{W[J]}V'\left(-i\frac{\delta}{\delta J(x)}\right)W[J] \ . \tag{3.67}$$

The last term clearly resembles a force. For example, take

$$V = \frac{\lambda}{4!}\varphi^4 \ , \quad \lambda \text{ dimensionless} \ . \tag{3.68}$$

Then

$$\frac{1}{W[J]}V'\left(-i\frac{\delta}{\delta J_x}\right)W[J] = \frac{\lambda}{3!}(-i)^3\frac{1}{W}\ \frac{\delta^3}{\delta J_x^3}W[J] \tag{3.69}$$

$$= \frac{\lambda}{3!}\left[\varphi_{\rm cl}^3(x) - \frac{\delta^2\varphi_{\rm cl}}{\delta J_x^2} - 3i\varphi_{\rm cl}\frac{\delta\varphi_{\rm cl}}{\delta J_x}\right] \ . \tag{3.70}$$

and finally

$$\left(\Box_x + m^2\right)\varphi_{\rm cl}(x) = J(x) - \frac{\lambda}{3!}\varphi_{\rm cl}^3(x) + \frac{\lambda}{3!}\ \frac{\delta^2\varphi_{\rm cl}(x)}{\delta J^2(x)} + \frac{i\lambda}{4}\ \frac{\delta\varphi_{\rm cl}^2(x)}{\delta J(x)} \ . \tag{3.71}$$

The first two terms on the right hand side give the classical equation of motion modified by the last two terms, which must amount to corrections from the quantum theory (see problem).

In the case $V \neq 0$, the explicit form of the effective action is, of course, not known. We can expand it functionally in terms of $\varphi_{\rm cl}$ as

$$\Gamma[\varphi_{\rm cl}] = \int d^4x\Big[-V^e(\varphi_{\rm cl}) + \frac{1}{2}F(\varphi_{\rm cl})\partial_\mu\varphi_{\rm cl}\partial^\mu\varphi_{\rm cl} \tag{3.72}$$

$$+\text{higher order derivatives}\Big] \ , \tag{3.73}$$

where we take into account now local effects by including arbitrarily high derivatives of $\varphi_{\rm cl}$. We have arbitrary functions $V^e(\varphi_{\rm cl})$, $F(\varphi_{\rm cl})$, etc., to be

determined. V^e is clearly an effective potential. By expressing J in terms of $\varphi_{\rm cl}$ using (3.3.29) and integrating (3.3.9), we see that

$$V^e(\varphi_{\rm cl}) = \frac{\lambda}{4!}\varphi_{\rm cl}^4 + \frac{M^2}{2}\varphi_{\rm cl}^2 + O(\hbar) \tag{3.74}$$

and

$$F(\varphi_{\rm cl}) = 1 + \text{corrections} \ . \tag{3.75}$$

Alternatively, we can expand the effective action in terms of $\varphi_{\rm cl}$ in a nonlocal way

$$\Gamma[\varphi_{\rm cl}] = \sum_N \frac{1}{N!} \ll \Gamma^{(N)}(1, \cdots, N)\varphi_{\rm cl}(1)\cdots\varphi_{\rm cl}(N)\rangle_{1,\cdots N} \ . \tag{3.76}$$

The coefficients $\Gamma^{(N)}(x_1, \cdots, x_N)$ are called the *proper vertices.* They depend on the differences $x_i - x_j$ because of translation invariance so that their Fourier transforms are introduced via

$$\tilde{\Gamma}^{(N)} \qquad (p_1, \cdots, p_N)(2\pi)^4\delta(p_1 + \cdots + p_N) = \tag{3.77}$$

$$\int d^4x_1 \cdots d^4x_N \, e^{-i(p_1x_1+\cdots+p_Nx_N)}\Gamma^{(N)}(x_1, \cdots, x_N) \ , \tag{3.78}$$

with $\tilde{\Gamma}^{(N)}$ being defined only when the sum of its arguments vanishes.

3.2.1 PROBLEMS

-A. By a judicious set of insertions of $\hbar$ when needed, show that the non-classical terms in the equation for $\varphi_{\rm cl}$ do indeed vanish as $\hbar \to 0$.

-B. Suppose that $V = \frac{1}{2}\delta m^2\varphi^2$ in the scalar field action. Find the equation obeyed by $\varphi_{\rm cl}$.

3.3 Saddle Point Evaluation of the Path Integral

Integrals of the form

$$I \equiv \int dx \, e^{-a(x)} \ , \tag{3.79}$$

where $a(x)$ is a function of x, can be approximated by expanding $a(x)$ around x_0 where $a(x)$ is stationary:

$$a(x) \simeq a(x_0) + \frac{1}{2}(x - x_0)^2 a''(x_0) + \cdots \ . \tag{3.80}$$

Then

$$I \simeq e^{-a(x_0)} \int dx e^{-\frac{1}{2}(x-x_0)^2 a''(x_0)} \ , \tag{3.81}$$

and the integral is easily performed if $a''(x_0) > 0$ (it's a Gaussian), neglecting the higher derivatives. The success of this approximation rests on the fact that the integrand is largest when $a(x)$ is smallest and that the points away from the minimum do not significantly contribute, as in the figure

$a(x)$ 11 $\qquad$ $a(x)$ 11

(good) $\qquad$ (bad)

In this section we apply this technique to the Euclidean space generating functional.

We start from the Euclidean space definition of the generating functional

$$W_E[J] = N_E \int \mathcal{D}\varphi \, e^{-S_E[\varphi,J]} f \ , \tag{3.82}$$

where

$$S_E[\varphi, J] = \int d^4\bar{x} \left[\frac{1}{2}\bar{\partial}_\mu\varphi\bar{\partial}_\mu\varphi + \frac{1}{2}m^2\varphi^2 + V(\varphi) - J\varphi \right] \ . \tag{3.83}$$

We then expand the action around a field configuration φ_0

$$\begin{aligned} \S_E[\varphi, J] &= S_E[\varphi_0, J] + \ll \frac{\delta S_E}{\delta\varphi}(\varphi - \varphi_0)\rangle & (3.84) \\ &+ \frac{1}{2} \ll \frac{\delta^2 S_E}{\delta\varphi_1\delta\varphi_2}(\varphi - \varphi_0)_1 (\varphi - \varphi_0)_2\rangle_{1,2} + \cdots \ . & (3.85) \end{aligned}$$

with the functional derivatives evaluated at φ_0. We take S_E to be stationary at φ_0, which means that φ_0 obeys the classical equations of motion with the source term

$$\left.\frac{\delta S_E}{\delta\varphi}\right|_0 = -\bar{\partial}^\mu\bar{\partial}_\mu\varphi_0 + m^2\varphi_0 + V'(\varphi_0) - J = 0 \ . \tag{3.86}$$

It follows that (after integration by parts)

$$S_E[\varphi_0, J] = \frac{1}{2}\int d^4\bar{x}\left\{2 - \varphi_0\frac{d}{d\varphi_0}\right\}\{-J\varphi_0 + V(\varphi_0)\} \ , \tag{3.87}$$

while

$$\frac{\delta^2 S}{\delta\varphi_1\delta\varphi_2} = \left[-\bar{\partial}_\mu\bar{\partial}^\mu + m^2 + V''(\varphi)\right]_1 \delta(\bar{x}_1 - \bar{x}_2) \tag{3.88}$$

is an operator. In the spirit of the saddle point evaluation, the generating functional now becomes

$$W_e[J] \simeq N_E\, e^{-S_E[\varphi_0,J]}\int \mathcal{D}\varphi\, \exp\left\{-\frac{1}{2} \ll \varphi_1 \frac{\delta^2 S_E}{\delta\varphi_1\delta\varphi_2}\varphi_2\rangle_{1,2}\right\} \ . \tag{3.89}$$

The Gaussian integral can be done (see Appendix A), with the formal result

$$W_E[J] \simeq N'_E\, e^{-S_E[\varphi_0,J]}\left\{\det\left(\left[-\bar{\partial}_\mu\bar{\partial}^\mu + m^2 + V''(\varphi_0)\right]\delta_{12}\right)\right\}^{-1/2} \ . \tag{3.90}$$

Clearly this expression needs some getting used to. We can rewrite it in a slightly more suggestive form by using the identity

$$\det \mathbf{M} = e^{\mathrm{Tr}\ln \mathbf{M}} \ , \tag{3.91}$$

as

$$W_E[J] = N'_E\, e^{-S_E[\varphi_o,J] - \frac{1}{2}\mathrm{Tr}\ln\left[\left\{-\bar{\partial}_\mu\bar{\partial}^\mu + m^2 + V''(\varphi_0)\right\}\delta_{12}\right]} \ , \tag{3.92}$$

which clearly indicates we are computing corrections to $Z[J]$. The physical meaning of this approximation can be understood by carefully putting back all the $\hbar$ factors. Then it is seen that it corresponds to an asymptotic series in $\hbar$ (see problem). The first term $S_E[\varphi, J]$ gives the classical contribution to the Green's functions (remember Dirac's identification). The next term, of $\mathcal{O}(\hbar)$ gives the first quantum corrections to the Green's functions. [The determinant of an operator is understood to mean the product of its eigenvalues.] We start by computing the classical contributions to $W[J]$. It must be remembered that φ_0, being the solution of (3.4.7), is a *functional* of J. The procedure is therefore very simple: a) calculate the functional dependence of φ_0 on J, b) insert it in (3.4.8) and, c) by comparing the resulting expression with the expansion (3.2.15), extract the Green's functions $G_c^{(N)}(1, \cdots, N)$. Alas, there are grave theoretical difficulties in carrying out step a). The equation obeyed at φ_0 is a nonlinear differential equation (for

$V' \neq 0$), and no one has succeeded in solving it in closed form. The best one can do it so solve it in perturbation theory. Specifically, take the φ^4 potential and expand around $\lambda = 0$.

We write

$$\varphi_0 = \varphi^{[0]} + \lambda\varphi^{[1]} + \lambda^2\varphi^{[2]} + \cdots \tag{3.93}$$

so that

$$\begin{aligned} S_E &= -\int \frac{d^4\bar{x}}{2}\left[J\left(\varphi^{(0)} + \lambda\varphi^{(1)} + \cdots\right) + \frac{\lambda}{12}\left(\varphi^{(0)} + \lambda\varphi^{(1)} + \cdots\right)^4\right] \\ &= -\frac{1}{2}\int d^4\bar{x} J\varphi^{(0)} - \frac{\lambda}{2}\int d^4\bar{x}\left[J\varphi^{(1)} + \frac{1}{12}\varphi^{(0)4}\right] + \mathcal{O}(\lambda^2) \quad . \end{aligned} \tag{3.94}$$

If we define the Euclidean Green's function (in an obvious notation)

$$\left(\bar{\partial}_\mu\bar{\partial}^\mu - m^2\right) G_{xy} = -\delta_{xy} \ , \tag{3.95}$$

if follows that

$$\varphi^{(0)}(x) = \ll G_{xa}J_a\rangle_a \tag{3.96}$$

$$\varphi^{(1)}(x) = -\frac{1}{6} \ll G_{xy}G_{ya}G_{yb}G_{yc}J_aJ_bJ_c\rangle_{abcy} \ , \text{ etc. .} \tag{3.97}$$

Thus,

$$\begin{aligned} S_E[J] &= -\frac{1}{2} \ll J_aG_{ab}J_b\rangle_{ab} + \frac{\lambda}{4!} \ll G_{xa}G_{xb}G_{xc}G_{xd}J_aJ_bJ_cJ_d\rangle_{abcdx} \\ &\quad -\frac{\lambda^2}{3\cdot 4!} \ll G_{xa}G_{xb}G_{xc}G_{xy}G_{yd}G_{ye}G_{yf}J_aJ_bJ_cJ_dJ_eJ_f\rangle\rangle_{abcdefxy} + \mathcal{O}(\lambda^3) \ . \end{aligned}$$

Correspondingly, the (connected) Euclidean Green's functions are given by

$$G_E^{(N)}(\bar{x}, \cdots, \bar{x}_N) = -\frac{\delta^N Z_E}{\delta J_1 \cdots \delta J_N} \ , \tag{3.98}$$

where

$$W_E[J] = N_E\, e^{-Z_E[J]} \ . \tag{3.99}$$

In this classical approximation we find the connected Green's functions to be

$$
\begin{aligned}
G_E^{(2)}\left(\bar{x}_1,\bar{x}_2\right) &= G\left(\bar{x}_1,\bar{x}_2\right) = \int \frac{d^4\bar{p}}{(2\pi)^4}\frac{e^{i\bar{p}\cdot\left(\bar{x}_1-\bar{x}_2\right)}}{\bar{p}^2+m^2} \\
G_E^{(4)}\left(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4\right) &= -\lambda\int d^4\bar{y}G\left(\bar{x}_1,y\right)G\left(\bar{x}_2,y\right)G\left(\bar{x}_3,y\right)G\left(\bar{x}_4,y\right) \qquad (3.100)
\end{aligned}
$$

$$
G_E^{(6)}\left(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4,\bar{x}_5,\bar{x}_6\right) = \lambda^2\int d^4\bar{x}d^4\bar{y}G\left(\bar{x},\bar{y}\right)P\left(\bar{x},\bar{y},\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4,\bar{x}_5,\bar{x}_6\right) \qquad (3.101)
$$

where

$$
P\left(\bar{x},\bar{y},\{\bar{x}_i\}\right) = \sum_{(ijk)} G\left(\bar{x},\bar{x}_i\right)G\left(\bar{x},\bar{x}_j\right)G\left(\bar{x},\bar{x}_k\right)G\left(\bar{y},\bar{x}_\ell\right)G\left(\bar{y},\bar{x}_m\right)G\left(\bar{y},\bar{x}_n\right) \ , \qquad (3.102)
$$

where the sum runs over all the following values of the triples, $(ijk) = (123)$, (124), (125), (126), (135), (136), (145), (146), (156), with (ℓmn) assuming the complementary value [*e.g.*, $(\ell mn) = (456)$ when $(ijk) = (123)$]. Note that ijk runs only over half of the possible values. This is because the expression for P is symmetric under the interchange $\bar{x} \to \bar{y}$.

In this classical approximation and to order λ^2 these are the only non-zero Green's functions.

The momentum space Green's functions, defined by

$$
\tilde{G}_E^{(N)} \qquad \left(\bar{p}_1,\cdots,\bar{p}_n\right)(2\pi)^4\delta\left(\bar{p}_1+\cdots+\bar{p}_N\right) \qquad (3.103)
$$

$$
\int d^4\bar{x}_1\cdots d^4\bar{x}_N\, e^{i\bar{p}_1\bar{x}_1+\cdots+i\bar{p}_N\bar{x}_N}G_E^{(N)}\left(\bar{x}_1,\cdots,\bar{x}_N\right) \qquad (3.104)
$$

are easily seen to be given at

$$
\begin{aligned}
\tilde{G}_E^{(2)}\left(\bar{p}_1,\bar{p}_2=-\bar{p}_1\right) &= \frac{1}{\bar{p}_1^2+m^2}+O(\hbar) \qquad (3.105) \\
\tilde{G}_E^{(4)}\left(\bar{p}_1,\bar{p}_2,\bar{p}_3,\bar{p}_4\right) &= \left(\frac{1}{\bar{p}_1^2+m^2}\ \frac{1}{\bar{p}_2^2+m^2}\ \frac{1}{\bar{p}_3^2+m^2}\ \frac{1}{\bar{p}_4^2+m^2}\right)\lambda+O(\hbar) \qquad (3.106) \\
\tilde{G}_E^{(6)}\left(\bar{p}_1,\cdots,\bar{p}_6\right) &= \prod_{i=1}^{6}\frac{1}{\left(\bar{p}_i^2+m^2\right)}\sum_{(ijk)}\frac{\lambda^2}{\left(\bar{p}_i+\bar{p}_j+\bar{p}_k\right)^2+m^2}+O(\hbar) \qquad (3.107)
\end{aligned}
$$

where again the sum is over the same triples as in (3.4.25). In the above

expressions the $\tilde{G}$'s are to be evaluated only when the sum of their arguments vanishes.

In the perturbative evaluation of φ_0, we note that $\varphi^{(k)}$ always depends on J^{1+2k}. Hence the λ^k order contributes solely to $G^{(2(k+1))}$. This is an artifact of the approximation, which neglects contributions of order $\hbar$.

Following Feynman, we develop a pictogram for these Green's functions. We represent $\tilde{G}_E^{(N)}(\bar{p}_1, \cdots, \bar{p}_N)$ by a blob with N external lines

$$11 \tag{3.108}$$

each line carrying a $\bar{p}$ label, and with all arrows pointing into the blob. This blob is them represented diagrammatically by the following rules:

$$.7.3 \text{ means } \frac{1}{\bar{p}^2 + m^2}, \text{ the propagator factor.7.7 means } -\lambda, \text{ the vertex .} \tag{3.109}$$

It is understood that the net amount of $\bar{p}$ flowing through the vertex is conserved. The vertex .25.25 has no arrows on its lines, thus indicating that the propagator for the lines is not included. Comparison with (3.4.27 – 3.4.29) leads to

$$1.5 \quad = 1.5 + \quad O(\hbar) \tag{3.110}$$

$$1.5 \quad = 1.5 + \quad O(\hbar) \tag{3.111}$$

$$1.5 \qquad = 1.5 \tag{3.112}$$

$$3.5 \tag{3.113}$$

$$3.5 \tag{3.114}$$

$$1.5 + \quad O(\hbar). \tag{3.115}$$

It appears then that the Feynman rules, to this order in $\hbar$, are to draw all possible arrangements using .4.05 and .25.25 as basic building blocks with no closed circuits (also called loops). Such diagrams are called tree diagrams. It is not hard to see that this approximation represents the Green's functions by their tree diagrams.

Consider a diagram with E external lines, I internal lines and V_n vertices, each with n lines (in our case $n = 4$). Each internal line hooks onto two vertex lines. Hence the number of external lines is just equal to the number of vertex lines minus twice the number of internal lines

$$E = nV_n - 2I \ . \tag{3.116}$$

To $\mathcal{O}(\hbar^\circ)$, we have $E = 2k + 2$, $V_4 = k$ hence

$$I = V_4 - 1 \qquad \mathcal{O}(\hbar^\circ) \ \text{ only} \ , \tag{3.117}$$

so that there cannot be any closed loop in these Feynman diagrams: they are called tree diagrams. The rule for an arbitrary (tree) diagram is to draw all topologically inequivalent diagrams with the external lines identified. For example, in $\tilde{G}^{(6)}$ when the same lines emanate from the vertex, the diagram enters only once because their rearrangements would give topological equivalent diagrams

1.51.5 , etc... .

These rules make it easy to write the expression for $\tilde{G}^{(8)}, \cdots$: we first draw all possible inequivalent tree arrangements with the external lines identified, and then use the rules in reverse to get the analytical expression. This use of the pictorial representation has proved to be an essential tool in the perturbative evaluation of the Green's functions.

It is interesting to rewrite these results in terms of the classical field $\varphi_{\rm cl}(x)$ and to see the form of the resulting effective action. In Euclidean space we define the classical field as

$$\varphi_{\rm cl}(\bar{x}) = -\frac{\delta Z_E}{\delta J(\bar{x})} \simeq -\frac{\delta S_E}{\delta J(\bar{x})} + \mathcal{O}(\hbar) \ , \tag{3.118}$$

which, using (3.4.19) gives us $\varphi_{\rm cl}(\bar{x})$ as a functional of J, order-by-order in λ. Then, we invert the equation in perturbation theory, and find $J(\bar{x})$ as a functional of $\varphi_{\rm cl}$. The result is

$$J(\bar{x}) = \left(\bar{\partial}^2 - m^2\right) \varphi_{\rm cl}(\bar{x}) - \frac{\lambda}{3!}\varphi_{\rm cl}^3(\bar{x}). \tag{3.119}$$

The remarkable thing is that there are no terms of higher order in λ in this equation. By comparing with (3.4.7), we conclude that

$$\varphi_{\rm cl}(\bar{x}) = \varphi_0(\bar{x}) + \mathcal{O}(\hbar) \ . \tag{3.120}$$

Integration of (3.3.9) gives us immediately the effective action to this order

$$\Gamma_E\left[\varphi_{\rm cl}\right] = -\int d^4\bar{x}\left[\frac{1}{2}\varphi_{\rm cl}\left(\bar{\partial}^2 - m^2\right)\varphi_{\rm cl} - \frac{\lambda}{4!}\varphi_{\rm cl}^4(\bar{x})\right] \ . \tag{3.121}$$

Thus, Γ_E is the classical action. We can therefore derive the expression

for the proper vertices. In this approximation, we see that $\tilde{\Gamma}^{(2)}$ is minus the inverse propagator and that $\tilde{\Gamma}^{(n)}$ for $n > 4$ vanish. The higher order diagrams do not appear. The reason is that the $\tilde{\Gamma}^{(N)}$ generate only Feynman graphs that cannot become disconnected by cutting off one of their internal lines. Such graphs are called *one-particle irreducible*. As we have seen, all the tree graphs are one-particle reducible except for the lowest one.

3.3.1 PROBLEMS

A. Show that the saddle point evaluation of the path integral corresponds to an asymptotic expansion in $\hbar$.

B. Solve the equation

$$\left(\partial^2 - m^2 - \frac{\lambda}{3!}\varphi^2\right)\varphi = -J$$

order-by-order about $\lambda = 0$. If we set $\varphi = \varphi^{(0)} + \lambda\varphi^{(1)} + \lambda^2\varphi^{(2)} + \cdots$, derive the explicit expressions for $\varphi^{(2)}$ and $\varphi^{(3)}$.

C. For $\lambda\varphi^4$ theory, find the effective classical action to order λ^3, and derive the classical Euclidean Green's functions to order λ^3, both in $\bar{x}$- and $\bar{p}$-space.

3.4 First Quantum Corrections ζ-Function Evaluation of Determinants

The $O(\hbar)$ correction to the effective action is computed by evaluating the determinant of (3.4.11). This determinant is to be interpreted as the product of the eigenvalues of the operator. In one possible procedure, the space is truncated (by, say, a box), resulting in discrete eigenvalues. Their product is computed and then the size of the box is let go to infinity. In the following, we want to make use of a powerful formal technique for computing the determinant of operators.

Consider an operator A with positive real discrete eigenvalues $a_1, \cdots a_n, \cdots$; call its eigenfunctions $f_n(x)$

$$Af_n(x) = a_n f_n(x) \quad . \tag{3.122}$$

We form the construct

$$\zeta_A(s) = \sum_n \frac{1}{a_n^s} \ , \tag{3.123}$$

called the ζ-function associated to A. [If A is the one-dimensional harmonic oscillator Hamiltonian, then ζ is, except for the zero-point energy, Riemann's ζ-function.] Then the sum extends over all the eigenvalues and A is a real variable. We note that

$$\left.\frac{d\zeta_A(s)}{ds}\right|_{s=0} = -\sum_n \ln a_n \, e^{-s\ln a_n}\Big|_{s=0} = -\ln\left(\prod_n a_n\right) \ , \tag{3.124}$$

leading to

$$\det A \equiv \prod_n a_n = e^{-\zeta_A'(0)} \ . \tag{3.125}$$

The advantage of this representation for $\det A$ is that for many operators of physical interest, ζ_A is not singular at $s = 0$. In fact, introduce the "Heat Function"

$$G\,(\bar{x},\,\bar{y},\,\tau) \equiv \sum_n e^{-a_n\tau} f_n(\bar{x}) f_n^*(\bar{y}) \ , \tag{3.126}$$

which obeys the differential equation (heat equation)

$$A_{\bar{x}} G\,(\bar{x},\,\bar{y},\,\tau) = -\frac{\partial}{\partial\tau} G\,(\bar{x},\,\bar{y},\tau) \ , \tag{3.127}$$

as can be seen by inspection. The ζ-function can now be expressed in terms of this "Heat Function" very easily:

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1} \int d^4\bar{x}\, G\,(\bar{x},\,\bar{x},\,\tau) \ , \tag{3.128}$$

using the orthogonality of the eigenfunctions and the well-known representation of the Γ-function. This equation is the desired analytic representation of $\zeta_A(s)$. Note that

$$G\,(\bar{x},\,\bar{y},\,\tau = 0) = \delta(\bar{x} - \bar{y}) \ , \tag{3.129}$$

using the orthonormality of the eigenfunctions. Thus, a possible way of computing $\det A$ emerges: 1) find the solution of Eq. (3.5.6) subject to the initial condition (3.5.8); 2) insert the solution into (3.5.7), to compute $\zeta_A(s)$, and use (3.5.4) to obtain $\det A$.

This procedure can be generalized to our problem. The operator is now

$\left[-\bar{\partial}^2 + m^2 + \frac{\lambda}{2}\varphi_0^2(\bar{x})\right]$, where $\varphi_0(\bar{x})$ is a solution of the classical equations with a source J.

It is easy to check that the solution of the equation

$$-\bar{\partial}_x^2 G_0\left(\bar{x},\, \bar{y},\, \tau\right) = -\frac{\partial G_0}{\partial \tau}\ , \tag{3.130}$$

with the boundary condition (3.5.8), is (in four dimensions only!)

$$G_0\left(\bar{x}, \bar{y}, \tau\right) = \frac{1}{16\pi^2\tau^2}\, e^{-\frac{1}{4\tau}(\bar{x}-\bar{y})^2}\ . \tag{3.131}$$

This does not yet solve our problem. In particular, the resulting $\zeta_{\bar{\partial}^2}(s)$ computed from (3.5.9) does not exist. We want to find $G(\bar{x}, \bar{y}, \tau)$ subject to (3.5.8) which obeys

$$\left[-\bar{\partial}_x^2 + m^2 + \frac{\lambda}{2}\varphi_0^2(\bar{x})\right] G(\bar{x}, \bar{y}, \tau) = -\frac{\partial G(\bar{x}, \bar{y}, \tau)}{\partial \tau}\ . \tag{3.132}$$

It is clear that for an arbitrary $\varphi_0(\bar{x})$ this equation is very hard to solve. Still, let us see what we can do. If we write the effective action in the form

$$\Gamma_E[\varphi_{\rm cl}] = \Gamma_E^{(0)}\varphi[\varphi_{\rm cl}] + \hbar\Gamma_E^{(1)}[\varphi_{\rm cl}] + \cdots\ , \tag{3.133}$$

we see that

$$\Gamma_E^{(1)}[\varphi_{\rm cl}] = -\frac{1}{2}\zeta'_{\left[-\bar{\partial}^2 + m^2 + \frac{\lambda}{2}\varphi_{\rm cl}^2(\bar{x})\right]}(0)\ , \tag{3.134}$$

where we have replaced φ_0 by $\varphi_{\rm cl}$ which does not induce any error up to $\mathcal{O}(\hbar)$, and used (3.5.4) and (3.4.11).

On the other hand, we can set

$$\Gamma_E[\varphi_{\rm cl}] = \int d^4\bar{x}\left[V\left(\varphi_{\rm cl}(\bar{x})\right) + F(\varphi_{\rm cl})\bar{\partial}_\mu\varphi_{\rm cl}(\bar{x})\bar{\partial}^\mu\varphi_{\rm cl}(\bar{x}) + \cdots\right]\ . \tag{3.135}$$

Hence, if we want to calculate the $\mathcal{O}(\hbar)$ contribution to $V(\varphi_{\rm cl})$, it suffices to consider a constant field configuration: suppose we set

$$\varphi_{\rm cl}(\bar{x}) = v\ , \tag{3.136}$$

where v is a constant independent of $\bar{x}$. Then

$$\Gamma_E[\varphi_{\rm cl}] = \int d^4\bar{x}V(v)\ , \tag{3.137}$$

and it is proportional to $\int d^4\bar{x}$, the infinite volume element, because the Euclidean space R_4 is not bounded. However, if we make believe we are on S_4, the surface of a sphere in five dimensions, we get a finite volume element (the surface of the sphere). This procedure avoids this infrared divergence. Later we can let the radius of the sphere go to infinity.

It follows that the $\mathcal{O}(\hbar)$ contribution to the potential is given by

$$V(v)\int d^4\bar{x} = -\frac{1}{2}\zeta'_{\left[-\bar{\partial}^2+m^2+\frac{\lambda}{2}v^2\right]}(0)\ . \tag{3.138}$$

With a constant v, (3.5.11) can be integrated very easily. We find

$$G(\bar{x},\bar{y},\tau) = \frac{\mu^4}{16\pi^2\tau^2}e^{\mu^2(\bar{x}-\bar{y})^2/4\tau}\, e^{-\left(m^2+\frac{\lambda}{2}v^2\right)\tau/\mu^2}\ , \tag{3.139}$$

where we have inserted an arbitrary factor μ with dimensions of mass to make τ dimensionless. Then, by using (3.5.7), we arrive at

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty d\tau\ \tau^{s-1}\int d^4\bar{x}\frac{\mu^4}{16\pi^2\tau^2}\, e^{-\left(m^2+\lambda^2v^2\right)\tau/\mu^2} \tag{3.140}$$

$$= \frac{\mu^4}{16\pi^2}\left(\frac{m^2+\frac{\lambda}{2}v^2}{\mu^2}\right)^{2-s}\frac{\Gamma(s-2)}{\Gamma(s)}\int d^4\bar{x} \tag{3.141}$$

where we have rescaled τ [the integration over τ is strictly valid only when $s-2>0$, but we define $\zeta(s)$ everywhere by analytic continuation]. Note the appearance of the volume factor $\int d^4\bar{x}$ which accounts for the one in (3.5.17). Comparison yields

$$V(v) = -\frac{\mu^4}{32\pi^2}\frac{d}{ds}\left\{\frac{1}{(s-2)(s-1)}\left(\frac{m^2+\frac{\lambda}{2}v^2}{\mu^2}\right)^{2-s}\right\}\Bigg|_{s=0} \tag{3.142}$$

$$= \frac{1}{64\pi^2}\left[m^2+\frac{\lambda}{2}v^2\right]^2\left(-\frac{3}{2}+\ln\frac{m^2+\frac{\lambda}{2}v^2}{\mu^2}\right)\ . \tag{3.143}$$

Now that we have the functional form of V, we can state that the effective potential of the theory is given by

$$V[\varphi_{\rm cl}] = \frac{1}{2}m^2\varphi_{\rm cl}^2(\bar{x})+\frac{\lambda}{4!}\varphi_{\rm cl}^4(\bar{x})+\frac{\hbar}{64\pi^2}\left(m^2+\frac{\lambda}{2}\varphi_{\rm cl}^2\right)^2 \tag{3.144}$$

$$\times \left[-\frac{3}{2} + \ln \frac{m^2 + \frac{\lambda}{2}\varphi_{\rm cl}^2}{\mu^2} \right] + \mathcal{O}(\hbar^2) \ . \tag{3.145}$$

This result is quite peculiar, because it seems to depend on the unknown scale μ^2, which was introduced arbitrarily. Does it mean that the potential thus obtained is arbitrary? Observe that V depends on the parameters m^2 and λ. These have not really been defined except as input parameters in the classical Lagrangian. For simplicity, take $m^2 = 0$ to start with. Then it is easy to see that automatically

$$\frac{d^2V}{d\varphi^2} = 0 \qquad \text{at } \varphi = 0 \ . \tag{3.146}$$

We define the mass squared as the coefficient of the φ^2 term in $\mathcal{L}$ evaluated at $\varphi = 0$; it is seen to be zero to $\mathcal{O}(\hbar)$, if it is classically zero. Next, what about λ Let us define it to be the coefficient of the fourth derivative of V evaluated at some constant point $\varphi = M$

$$\lambda \equiv \frac{d^4V}{d\varphi^4} \qquad \text{at } \varphi = M \ . \tag{3.147}$$

Note that we cannot take $\varphi = 0$ as in the previous case because of the divergence coming from the logarithm [infrared divergence]. This is typical of theories where $m^2 = 0$ classically.

The condition (3.5.25) requires

$$\ln \frac{\lambda M^2}{2\mu^2} = -\frac{8}{3} \ , \tag{3.148}$$

as seen by differentiating (3.5.23), setting $m^2 = 0$ and using (3.5.25). Thus we can eliminate $2\mu^2/\lambda$ in favor of M^2 and express the result as

$$V(\varphi_{\rm cl}) = \frac{\lambda}{4!}\varphi_{\rm cl}^4 + \frac{\lambda^2\varphi_{\rm cl}^4}{256\pi^2}\left[\ln\frac{\varphi_{\rm cl}^2}{M^2} - \frac{25}{6}\right] \ , \tag{3.149}$$

in accordance with the result of S. Coleman and E. Weinberg, *Phys. Rev.* **D7**, 1888 (1973). This little exercise shows that we must carefully define the input parameters in the Lagrangian in order to handle the quantum corrections. The result (3.5.27) still seems to depend on one arbitrary scale M^2 but it really does not, because, given the normalization condition, if we change the scale from M^2 to M'^2 we have to change at the same time λ to λ', where

$$\lambda' = \lambda + \frac{3\lambda^2}{16\pi^2} \ln \frac{M'}{M} \ , \tag{3.150}$$

using (3.5.25). We see that the potential

$$V(\varphi_{\rm cl}) = \frac{\lambda'}{4!}\varphi_{\rm cl}^4 + \frac{\lambda'^2\varphi_{\rm cl}^4}{256\pi^2}\left[\ln\frac{\varphi_{\rm cl}^2}{M'^2} - \frac{25}{6}\right] + \mathcal{O}(\lambda^3) \ , \tag{3.151}$$

is form invariant under this reparameterization:

$$V(\lambda', M') = V(\lambda, M) \ . \tag{3.152}$$

This shows that the physics does not change, only our way of interpreting the constants.

3.4.1 PROBLEMS

A. Suppose that the classical potential is given by $V_{\rm cl} = \frac{1}{6}f\varphi^3$, where f has dimension of mass. Find by the steepest descent method the first quantum correction to this potential. Interpret the resulting potential physically.

B. Repeat problem A, for $V_{\rm cl} = \frac{1}{6}f\varphi^3 + \frac{\lambda}{4!}\varphi^4$. Interpret physically.

**C. Find the solution of the heat equation

$$\begin{aligned}(\bar{\partial}^2 - m^2)G(\bar{x}, \bar{y}, \tau) &= \frac{\partial G}{\partial \tau} \\ G(\bar{x}, \bar{y}, 0) &= \delta(\bar{x} - \bar{y})\end{aligned}$$

in d-dimensions. Use your result to compute the effective potential for the theory defined by

$$\int d^6\bar{x}\left[\frac{1}{2}\bar{\partial}_\mu\varphi\bar{\partial}^\mu\varphi + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{3!}\varphi^3\right] \tag{3.153}$$

in six-dimensions. In particular, find the λ-rescaling necessary to provide invariance of the result under a scale transformation. Interpret the sign and plot the variation of λ with scale.

3.5 Scaling of Determinants: The Scale Dependent Coupling Constant

The ζ-function technique for evaluating determinants of operators makes it particularly simple to derive the scaling properties of these determinants. Under a scale change

$$A \to A' = e^{ad} A \ , \tag{3.154}$$

where d is the (natural) dimension of A. The definition of the ζ-function leads to

$$\zeta_{A'}(s) = e^{-sad} \zeta_A(s) \ , \tag{3.155}$$

from which

$$\det(e^{ad} A) = e^{ad\zeta_A(0)} \det(A) \ . \tag{3.156}$$

An illustrative application of this formula is obtained as follows: Under a dilatation

$$x_\mu \to x'_\mu = e^a x_\mu \quad . \qquad \varphi_{\rm cl} \to \varphi'_{\rm cl} = e^{-a} \varphi_{\rm cl} \ , \tag{3.157}$$

the classical action with $m^2 = 0$

$$S_E[\varphi_{\rm cl}] = -\int d^4\bar{x} \left[\frac{1}{2} \varphi_{\rm cl} \bar{\partial}^2 \varphi_{\rm cl} - \frac{\lambda}{4!} \varphi_{\rm cl}^4 \right] \tag{3.158}$$

suffers no change. On the other hand, the path integral for this action is not scale invariant. Indeed, in the steepest descent approximation, we find that the change in the effective action is to $\mathcal{O}(\hbar)$,

$$S_E^{\rm eff}[\varphi_{\rm cl}] \to S_E'^{\rm eff}[\varphi_{\rm cl}] = S_E^{\rm eff}[\varphi_{\rm cl}] - \hbar a \zeta_{\left[-\bar{\partial}^2 + \frac{\lambda}{2}\varphi_{\rm cl}^2\right]}(0) \ . \tag{3.159}$$

The ζ-function for the operator $-\bar{\partial}^2 + \frac{\lambda}{2}\varphi_{\rm cl}^2$ is calculated by assuming for $G(\bar{x}, \bar{y}, \tau)$ the asymptotic expansion (setting $\mu^2 = 1$)

$$G(\bar{x}, \bar{y}, \tau) = \frac{e^{-(\bar{x}-\bar{y})^2/4\tau}}{16\pi^2\tau^2} e^{-\epsilon\tau} \sum_{n=0}^{\infty} a_n(\bar{x}, \bar{y}) \tau^n \ , \tag{3.160}$$

where we have inserted an artificial convergence factor with $\epsilon > 0$. For the reader unhappy with this procedure, imagine that $m^2 \neq 0$ to start with. The boundary condition (3.5.8) requires that

$$a_0(\bar{x},\bar{x}) = 1 \ . \tag{3.161}$$

Furthermore, the differential equation (3.5.11) applied to the form (3.6.7) yields recursion relations for the $a_n(\bar{x},\bar{y})$ coefficients

$$(\bar{x}-\bar{y})_\mu \frac{\partial}{\partial x_\mu} a_0\left(\bar{x},\bar{y}\right) = 0 \ , \tag{3.162}$$

and for $n = 0,\ 1,\ 2, \cdots$

$$\left[(n+1) + (\bar{x}-\bar{y})_\mu \frac{\partial}{\partial x_\mu}\right] a_{n+1}(\bar{x}-\bar{y}) = \left(\bar{\partial}_x^2 - \frac{\lambda}{2}\varphi_{\rm cl}^2(\bar{x}) + \epsilon\right) a_n(\bar{x},\bar{y}) \ . \tag{3.163}$$

They can be solved, giving

$$a_1(\bar{x},\bar{x}) = -\frac{\lambda}{2}\varphi_{\rm cl}^2(\bar{x}) + \epsilon \ , \tag{3.164}$$

$$a_2(\bar{x},\bar{x}) = \frac{\lambda^2}{8}\varphi_{\rm cl}^4(\bar{x}) - \frac{\lambda}{6}\bar{\partial}^2\varphi_{\rm cl}(\bar{x}) + \epsilon\lambda\varphi_{\rm cl}^2(\bar{x}) \ . \tag{3.165}$$

The resulting ζ-functions, evaluated at $s = 0$, is now given by

$$\zeta(0) = \frac{1}{16\pi^2}\left[\frac{\epsilon^4}{2}\int d^4\bar{x} + \frac{\epsilon^2\lambda}{2}\int d^4\bar{x}\varphi_{\rm cl}^2(\bar{x}) + \int d^4\bar{x}\frac{\lambda^2}{8}\varphi_{\rm cl}^4(\bar{x})\right] \ , \tag{3.166}$$

where we have used the definition (3.5.7), (3.6.7) and (3.6.11). The $\bar{\partial}^2$ term in $a_2(\bar{x},\bar{x})$ has been integrated out. As we take ϵ to zero, we obtain the final result

$$S_E'^{\rm eff} = S_E^{\rm eff} - \hbar a \frac{\lambda^2}{8\cdot 16\pi^2}\int d^4\bar{x}\varphi_{\rm cl}^4(\bar{x}) \ . \tag{3.167}$$

Thus we see that the sole effect of the dilatation (to this order in $\hbar$) is to change the coupling constant λ by

$$\frac{\lambda}{4!} \to \frac{\lambda'}{4!} = \frac{\lambda}{4!} - \hbar a \frac{\lambda^2}{8\cdot 16\pi^2} \ , \tag{3.168}$$

i.e.

$$\lambda \to \lambda' = \lambda - \frac{3\lambda^2}{16\pi^2}\hbar a \ . \tag{3.169}$$

This very important formula tells us that the coupling constant, which is classically a dimensionless parameter, develops as a result of quantum effects a scale dependence. In this particular case, it tells us that at large scales the coupling constant decreases, which means that the non-interaction theory is in some sense a good approximation for asymptotic states. As the scale decreases, the coupling starts increasing, and even though we may have started from a small value of λ at an initial scale, λ may increase invalidating results obtained on the basis of perturbation in λ. Note that this scaling law is exactly the same as that obtained in the previous paragraph [recall that $a = -\ln \frac{M'}{M}$]. This result is exact to $\mathcal{O}(\hbar)$. It is customary to define the β-function

$$\beta = \frac{d\lambda(M^2)}{d\ln M^2} = \frac{3\lambda^2}{32\pi^2}\hbar + \cdots \ , \tag{3.170}$$

which in this case is positive.

Thus we have learned from a different point of view that in Quantum Field Theories, the coupling constants have to be defined at some scale because even though they may be classically scale independent, they develop quantum scale dependence.

3.5.1 PROBLEMS

**A. When $m^2 \neq 0$, the classical action with $V_{\rm cl} = \frac{1}{2}m^2\varphi_{\rm cl}^2 + \frac{\lambda}{4!}\varphi_{\rm cl}^4$ is no longer dilatation invariant. Find the changes in the effective action stemming from a dilatation. In particular, find the change in m^2, both classical and quantum (to $\mathcal{O}(\hbar)$).

**B. Introduce the new asymptotic expansion for $G(x, y, \tau)$

$$G(\bar{x}, \bar{y}, \tau) = \frac{e^{-(\bar{x}-\bar{y})^2/4\tau}}{16\pi^2\tau^2} e^{-\frac{\lambda}{2}\varphi^2(\bar{x})\tau} \sum_{n=0}^{\infty} b_n(\bar{x}, \bar{y})\tau^n \ , \tag{3.171}$$

corresponding to the operator $-\partial^2 + \frac{\lambda}{2}\varphi^2(\bar{x})$. Find the recursion relations for the b_n coefficients, and work out the form of $b_n(\bar{x}, \bar{x})$ for $n = 0, 1, 2, 3$.

3.6 Finite Temperature Field Theory

Path integral techniques can be readily applied to the description of dynamical systems at finite temperature, owing to a striking analogy between

the formulations of statistical mechanics and field theory. Indeed, given a physical system with degrees of freedom q_i, p_i, and Hamiltonian $H(p_i, q_i)$, the starting point of any calculation involving temperature effects is the evaluation of the partition function

$$Z = Tr[e^{-\beta H}] \ , \tag{3.172}$$

where

$$\beta = \frac{1}{kT} \ , \tag{3.173}$$

and the trace operation means to sum over all the possible configurations the system is allowed to take. In this description, time is clearly singled out. This starting point has to do with the fact that

$$\mathcal{P} = \frac{1}{Z} e^{-\beta E} \ , \tag{3.174}$$

is identified with the probability for the system to be in the state of energy E. Then the value of any function of the dynamical variable $f(p, q)$ is simply given by

$$< f >= Tr(f\mathcal{P}) = \frac{1}{Z} Tr(f e^{-\beta H}) \ . \tag{3.175}$$

Although the formal similarity with (zero temperature) quantum mechanics and quantum field theory is striking, it is not yet understood. Still, we can be pragmatic about it and make use of this analogy to compute the partition function. For simplicity we start with a quantum mechanical example which can be regarded as a field theory in zero space dimensions.

Take a quantum mechanical system with one degree of freedom q. Let p be its canonically conjugate momentum and $H(p, q)$ its Hamiltonian. At any given time t, the system is described in terms of the spectrum of H. Let us label these states by q as before. If the system at an initial time t_i is measured to be in the state $\mid q^i >$, then the probability that the system will be found in the state $\mid q^f >$ at a final time t_f is just

$$< q^f_{t_f} \mid q^i_{t_i} >=< q^f \mid e^{-i(t_f - t_i)H} \mid q^i > \ , \tag{3.176}$$

and it is expressed in terms of the path integral

$$< q^f \mid e^{-i(t_f - t_i)H} \mid q^i >= \int \mathcal{D}q \int \mathcal{D}p \ e^{i \int_{t_i}^{t_f} dt[p\dot{q} - H(p,q)]}, \ , \tag{3.177}$$

where the functional $\mathcal{D}q$ integration is taken between the initial and final configurations q^i and q^f; $\dot{q}$ denotes the derivative of q with respect to time. Compare this expression with the partition function for the same system at temperature β^{-1}:

$$Z = Tr\ e^{-\beta H} = \sum_q < q \mid e^{-\beta H} \mid q > \ . \tag{3.178}$$

It is apparent that (3.7.7) can be equated with (3.7.6) provided the following formal changes be made:

1. set $i(t_f - t_i) = \beta$ or set $t_i = 0 \quad it_f = \beta$, since the origin of time is arbitary.

2. set $q_i = q_f$ so that the initial and final configurations are the same, and, since they differ by a "time" β, require that the relevant configuration be periodic, *i.e.*

$$q(\beta) = q(0) \ , \tag{3.179}$$

in the functional integrations. Thus the functional integration $\mathcal{D}q$ is over the space of periodic functions. Then the sum over q in Eq. (3.7.6) is implicit. The result of this formal identification is to write

$$Z = Tr(e^{-\beta H}) = \int \mathcal{D}q \int \mathcal{D}p\ e^{\int_0^\beta d\tau [ip\frac{dq}{d\tau} - H]} \ , \tag{3.180}$$

where it is understood that $\mathcal{D}q$ is over periodic functions, *i.e.* obeying Eq. (3.7.8). For a well-behaved Hamiltonian,

$$H = \frac{1}{2}p^2 + V(q) \ , \tag{3.181}$$

it is convenient to scale the temperature dependence purely into the q integral. Let

$$\bar{\tau} = \frac{\tau}{\beta}, \quad \bar{p} = p\sqrt{\beta}, \quad \bar{q} = \frac{1}{\sqrt{\beta}}q \ , \tag{3.182}$$

then the exponent of the integrand becomes

$$\int_0^1 d\bar{\tau}[i\bar{p}\frac{d\bar{q}}{d\bar{\tau}} - \frac{\bar{p}^2}{2} - \beta V(\sqrt{\beta}\bar{q})] \ . \tag{3.183}$$

On the other hand the path integral measure is (at least formally) invariant

under the changes of variable (3.7.11), and since we are integrating over them, we can drop the bars and write

$$Z = \int \mathcal{D}q \int \mathcal{D}p \; e^{\int_0^1 d\tau [ip\dot{q} - \frac{1}{2}p^2 - \beta V(q\sqrt{\beta})]} \; . \tag{3.184}$$

Now, let

$$p' = p - i\dot{q} \; , \tag{3.185}$$

so that

$$\mathcal{D}p' = \mathcal{D}p \; , \tag{3.186}$$

which enables us to write, by completing the squares in the exponent,

$$Z = \int \mathcal{D}p' \; e^{-\int_0^1 d\tau \frac{1}{2}p'^2} \int \mathcal{D}q \; e^{-\int_0^1 d\tau [\frac{1}{2}\dot{q}^2 + \beta V(q\sqrt{\beta})]} \; . \tag{3.187}$$

The functional integral over p' is independent of β and thus of no interest to us. Call it N; the fact that it is infinite does not concern us either. Path integral practitioners are well acquainted with this phenomenon.

Putting these formal manipulations aside, let us evaluate Z for a simple system, the harmonic oscillator (of course) for which

$$V(q) = \frac{1}{2}\omega^2 q^2 \; . \tag{3.188}$$

The partition function is

$$Z = N \int \mathcal{D}q \; e^{-\int_0^1 d\tau [\frac{1}{2}\dot{q}^2 + \frac{1}{2}\beta^2\omega^2 q^2]} \; . \tag{3.189}$$

This path integral is of the only type we really know how to "path-integrate", since it is a Gaussian. We first note that

$$\frac{1}{2}\int_0^1 d\tau (\frac{dq}{d\tau})^2 = -\frac{1}{2}\int_0^1 d\tau \; q \frac{d^2}{d\tau^2} q \; , \tag{3.190}$$

since the extra surface term vanishes by periodicity; hence

$$Z = N \int \mathcal{D}q \; e^{-\frac{1}{2}\int_0^1 d\tau \; q(-\frac{d^2}{d\tau^2} + \omega^2\beta^2)q} \; . \tag{3.191}$$

By analogy with the discrete case,

$$\int \mathcal{D}q \; e^{-\frac{1}{2}(q,Aq)} = \frac{N'}{\sqrt{detA}} \; , \tag{3.192}$$

where N' is a constant, and A is an operator with positive definite eigenvalues [if A has zero eigenvalues, they will create infinities which have to be removed]. This formula can be proved by expressing $q(\tau)$ in terms of its Fourier components, then transform into the normal modes of A and integrate each one using

$$\int_0^\infty dq_n \; e^{-\frac{1}{2}a_n q_n^2} = \sqrt{\frac{2\pi}{a_n}} \; . \tag{3.193}$$

In our case, the operator is

$$A = (-\frac{d^2}{d\tau^2} + \omega^2\beta^2) \; , \tag{3.194}$$

operating on periodic functions with unit period, which can all be expanded in terms of the complete Fourier set $\{e^{i2\pi n\tau}\}$. On these functions, the eigenvalues of A are just

$$(4\pi^2 n^2 + \omega^2\beta^2) \quad ; \quad n = -\infty, ... -1, 0, +1, ..., +\infty \; . \tag{3.195}$$

Thus

$$\det \; A = \prod_{n=-\infty}^{+\infty} (4\pi^2 n^2 + \omega^2\beta^2) \; . \tag{3.196}$$

There are several tricks for computing this determinant. First we reproduce the standard treatment. Letting $x^2 = \omega^2\beta^2$, we note that

$$\frac{d}{dx^2} \ln \; \det \; A \; = \; \sum_{n=-\infty}^{+\infty} \frac{1}{4\pi^2 n^2 + x^2} \tag{3.197}$$

$$= \; \frac{1}{x^2} + 2\sum_{n=1}^{\infty} \frac{1}{4\pi^2 n^2 + x^2} \; . \tag{3.198}$$

Using the well-known formula (Gradshteyn and Ryzhik, p.36),

$$\coth \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2} \; , \tag{3.199}$$

we obtain

$$\frac{d}{dx^2}\ln\ \det\ A = \frac{1}{2x}\coth\frac{x}{2}\ , \tag{3.200}$$

from which

$$\ln(\frac{\det\ A}{C}) = \int_0^{\omega\beta} dx\coth\frac{x}{2} \tag{3.201}$$

$$= (\omega\beta + 2\ln[1-e^{-\omega\beta}])\ . \tag{3.202}$$

At last we obtain the desired formula

$$\det\ A = 4C\sinh^2\frac{\omega\beta}{2}\ , \tag{3.203}$$

where we have set $\det\ A\mid_{x=0}= C$, a constant. Hence putting it all together, we arrive at the standard thermodynamic potential used by physicists

$$F = -\frac{1}{\beta}\ln Z\ , \tag{3.204}$$

$$= -\frac{D}{\beta} + \frac{1}{2}\omega + \frac{1}{\beta}\ln[1-e^{-\omega\beta}]\ , \tag{3.205}$$

where D is another constant; note that the zero-point energy is clearly identified. One proceeds to define the entropy S, and energy U in the usual way

$$S = -\left(\frac{\partial F}{\partial T}\right), \qquad U = (1+\beta\frac{\partial}{\partial\beta})F\ . \tag{3.206}$$

The undetermined constant D drops out from the expression for the energy, but remains in the entropy; it is, however, taken to be zero by Nernst's theorem, since it is a measure of the entropy at zero temperature. The alert reader may have noticed the appearance of determinants; in this book this calls for the ζ-function techniques introduced earlier in this chapter. Although this technique is not ideal for discrete systems, let us apply it to our problem and see where it leads.

We first obtain the heat function associated with the operator (3.7.22); it is not hard to see that it is given by

$$G(\tau,\tau';\sigma) = \sum_{n=-\infty}^{+\infty} e^{i2\pi n(\tau-\tau')-(\omega^2\beta^2+4\pi^2n^2)\sigma}\ , \tag{3.207}$$

from which we obtain the ζ-function

$$\zeta_A[s] = \frac{1}{\Gamma(s)} \int_0^\infty d\sigma\ \sigma^{s-1} \int_0^1 d\tau \sum_{n=-\infty}^{+\infty} e^{-(\omega^2\beta^2+4\pi^2n^2)\sigma}\ . \tag{3.208}$$

We could at this point just scale σ by $(\omega^2\beta^2 + 4\pi^2n^2)$ and just recover the expected expression

$$\zeta_A[s] = \sum_{n=-\infty}^{+\infty} \frac{1}{(\omega^2\beta^2 + 4\pi^2n^2)^s}\ , \tag{3.209}$$

which will lead us around in a circle (without even picking up a phase!). Rather we do a little bit of (perverted) mathematical physics. We first expand in powers of $\omega\beta$, do the integral and then mess around with the sums. Thus we write

$$\sum_{n=-\infty}^{+\infty} e^{-4\pi^2n^2\sigma} = 1 + 2\sum_{n=1}^{\infty} e^{-4\pi^2n^2\sigma}\ , \tag{3.210}$$

so that upon integration

$$\zeta_A[s] = (\omega\beta)^{-2s} + \frac{2}{\Gamma(s)} \sum_{\ell=0}^{\infty} \frac{(\omega\beta)^{2\ell}}{\ell!}(-1)^\ell \sum_{n=1}^{\infty} \int_0^\infty d\sigma\ \sigma^{s+\ell-1} e^{-4\pi^2n^2\sigma}\ . \tag{3.211}$$

Now rescale σ by $4\pi^2n^2$ and identify the sum over n with Riemann's ζ-function

$$\zeta[2s] = \sum_{n=1}^{\infty} \frac{1}{n^{2s}}\ , \tag{3.212}$$

which finally yields

$$\zeta_A[s] = (\omega\beta)^{-2s} + \frac{2}{(4\pi^2)^s}\zeta[2s] + \frac{2}{\Gamma(s)} \sum_{\ell=1}^{\infty} \frac{(\omega\beta)^{2\ell}}{\ell!} \frac{(-1)^\ell}{(4\pi^2)^{s+\ell}} \Gamma(s+\ell)\zeta[2s+2\ell]\ . \tag{3.213}$$

It is now straightforward to find the derivative of ζ_A at s=0 by noting that the sum is well behaved at s=0 and that the only non zero term as $s \to 0$ comes from the derivative of $\Gamma^{-1}(s)$. Also using $\zeta[0] = -\frac{1}{2},\ \ \zeta'[0] = -\frac{1}{2}\ln 2\pi$, we get

$$\zeta_A'[0] = -2\ln(\omega\beta) - 2\ln 2\pi + 2\ln 2\pi + 2\sum_{\ell=1}^{\infty} \frac{(\omega\beta)^{2\ell}(-1)^\ell \zeta[2\ell]}{\ell(4\pi^2)^\ell} \ . \tag{3.214}$$

Now for a bit of real perversion. First of all [Bateman, vol. I, p. 35]

$$\zeta[2\ell] = \frac{(-1)^{\ell+1}(2\pi)^{2\ell}}{2(2\ell)!} B_{2\ell} \ , \tag{3.215}$$

where $B_{2\ell}$ are the Bernoulli numbers, and the sum over ℓ simplifies to give

$$\zeta_A'[0] = -2\ln(\omega\beta) - \sum_{\ell=1}^{\infty} \frac{(\omega\beta)^{2\ell}}{(2\ell)!} \frac{1}{\ell} B_{2\ell} \ . \tag{3.216}$$

Use [Gradshteyn and Ryzhik, p. 35 or Bateman, vol. I, p. 51]

$$\coth x = \frac{1}{x} + 2\sum_{\ell=1}^{\infty} \frac{(2x)^{2\ell-1}}{(2\ell)!} B_{2\ell} \ , \tag{3.217}$$

whence

$$\int dx \coth x = \ln x + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2x)^{2\ell}}{\ell(2\ell)!} B_{2\ell} \ , \tag{3.218}$$

so that by comparing with (3.7.42), and setting $x = \frac{1}{2}\omega\beta$, we get

$$\zeta_A'[0] = -2\ln(\omega\beta) + 2\ln(\frac{\omega\beta}{2}) - 2\ln\sinh(\frac{\omega\beta}{2}) \ , \tag{3.219}$$

$$= -\omega\beta - 2\ln(1 - e^{-\omega\beta}) \ , \tag{3.220}$$

or finally

$$\ln Z = \frac{1}{2}\zeta_A'[0] = -\frac{1}{2}\omega\beta - \ln(1 - e^{-\omega\beta}) \ , \tag{3.221}$$

in agreement with the previous result. At this point, this technique seems awfully contrived, but it is about to come into its own when we apply it to calculate the partition function of finite temperature field theory.

For simplicity we consider a scalar field $\varphi(t, \vec{x})$ interacting with itself. In terms of $\varphi(t, \vec{x})$ and its canonically conjugate variable field $\pi(t, \vec{x})$, we take the Hamiltonian to be of the form

$$H = \int d^3x \, [\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi)] \ , \tag{3.222}$$

$$= \int d^3x \, \mathcal{H} \ . \tag{3.223}$$

The partition function is now very easy to set up. In complete analogy with the quantum mechanical case it is given by

$$Z = \int \mathcal{D}\pi \int \mathcal{D}\varphi \ \exp\left\{\int_0^\beta d\tau \int d^3x [i\pi\frac{\partial\varphi}{\partial\tau} - \mathcal{H}]\right\} \ , \tag{3.224}$$

where the φ integral is taken over field configurations periodic in "time", *i.e.* those which obey

$$\varphi(\tau, \vec{x}) = \varphi(\tau + \beta, \vec{x}) \ , \tag{3.225}$$

while the space variables are unbounded. As before we can rescale the temperature dependence so that

$$Z = \int \mathcal{D}\pi \int \mathcal{D}\varphi \ \exp\left\{\int_0^1 d\tau \int d^3x [i\pi\frac{\partial\varphi}{\partial\tau} - \frac{1}{2}\pi^2 - \frac{1}{2}\beta^2(\vec{\nabla}\varphi)^2 - \beta V(\varphi\sqrt{\beta})]\right\} \ . \tag{3.226}$$

We introduce the new variable

$$\pi' = \pi - i\frac{\partial\varphi}{\partial\tau} \ , \tag{3.227}$$

and "complete the squares" to get

$$Z = N \int \mathcal{D}\varphi \ \exp\left\{-\int_0^1 d\tau \int d^3x [\frac{1}{2}(\frac{\partial\varphi}{\partial\tau})^2 + \frac{1}{2}\beta^2(\vec{\nabla}\varphi)^2 + \beta V(\varphi\sqrt{\beta})]\right\} \ , \tag{3.228}$$

after performing the π' integration. Note that the temperature dependence appears in the $(\vec{\nabla}\varphi)^2$ term as well: a new feature of field theory *vs* quantum mechanics. Needless to say, the evaluation of this integral for an arbitrary (even renormalizable) $V(\varphi)$ is beyond our ability, except perturbatively. To make life simpler, we can couple the original field φ to an external source obeying the same periodicity condition as φ. Then if the term in the original Lagrangian is of the form φJ, it is rescaled to $\beta\sqrt{\beta}\varphi J$.

We now proceed to evaluate Z by the saddle point method; as we have

seen, it corresponds to evaluating the one loop effect $\sim \mathcal{O}(\hbar)$ of the theory. Recalling section 4 in this chapter, we expand the exponent of the integrand around a solution of the classical theory, φ_0. If we call the exponent $S[\varphi, J]$, [in fact it is the action, with imaginary "time"], we can write, as in (3.4.6),

$$S[\varphi, J] = S[\varphi_0, J] + < (\varphi - \varphi_0) \frac{\delta S}{\delta \varphi}\bigg|_{\varphi_0} > + \frac{1}{2} < (\varphi - \varphi_0)^2 \frac{\delta^2 S}{\delta \varphi^2}\bigg|_{\varphi_0} > + \cdots \ , \tag{3.229}$$

where the $< \cdots >$ indicates integration(s) over τ and $\vec{x}$. Now if φ_0 obeys the classical equation of motion, clearly

$$\frac{\delta S}{\delta \varphi}\bigg|_{\varphi_0} = -\frac{\partial^2}{\partial \tau^2}\varphi_0 - \beta^2 \vec{\nabla}^2 \varphi_0 + m^2 \beta^2 \varphi_0 + \frac{\lambda}{3!}\beta^3 \varphi_0^3 + J\beta\sqrt{\beta} = 0 \ , \tag{3.230}$$

where we have assumed the usual renormalizable potential

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!}\varphi^4 \ . \tag{3.231}$$

Also

$$\frac{\delta^2 S}{\delta \varphi_0^2} = -\frac{\partial^2}{\partial \tau^2} - \beta^2 \vec{\nabla}^2 + m^2 \beta^2 + \frac{\lambda}{2}\beta^3 \varphi_0^2 \ . \tag{3.232}$$

Thus we have the saddle point approximation for Z

$$Z = N \ e^{-S[\varphi_0, J]} \times \tag{3.233}$$

$$\int \mathcal{D}\varphi \ \exp\left\{ -\frac{1}{2}\int_0^1 d\tau \int d^3x \ \varphi[-\frac{\partial^2}{\partial \tau^2} - \beta^2 \vec{\nabla}^2 + m^2 \beta^2 + \frac{\lambda}{2}\beta^3 \varphi_0^2]\varphi \right\} \tag{3.234}$$

where we have shifted our integration variable from φ to $\varphi - \varphi_0$. If we let

$$\mathbf{B} = -\frac{\partial^2}{\partial \tau^2} - \beta^2 \vec{\nabla}^2 + m^2 \beta^2 + \frac{\lambda}{2}\beta^3 \varphi_0^2 \ , \tag{3.235}$$

we find, at least formally,

$$Z = N' e^{-S[\varphi_0, J]} (\det \ \mathbf{B})^{-1/2} \ , \tag{3.236}$$

$$= N' e^{-S[\varphi_0, J] + \frac{1}{2}\zeta'_{\mathbf{B}}[0]} \ , \tag{3.237}$$

where N' is an unknown constant. Note that $\zeta'_{\mathbf{B}}[0]$ is a functional of φ_0 which

is itself temperature dependent. We see this by doing the scaling transformation backwards in the classical equation of motion (3.7.53). Indeed we can always write

$$\varphi_0(\tau, \vec{x}) = \sqrt{\beta}\bar{\varphi}_0(\beta\tau, \vec{x}) \ , \tag{3.238}$$

where $\bar{\varphi}_0$ does not have any dependence on β. Now in general, φ_0 can be a complicated function of τ and $\vec{x}$, which makes the evaluation of the determinant pretty hard. We will modestly restrict ourselves to a constant φ_0. This will nevertheless give us information about the part of the one loop correction which does not depend on derivatives of φ_0. We define

$$\mathbf{B} = -\frac{\partial^2}{\partial\tau^2} + \beta^2\mathbf{C} \ , \tag{3.239}$$

where $\mathbf{C}$ is the β-independent operator

$$\mathbf{C} = -\vec{\nabla}^2 + m^2 + \frac{\lambda}{2}\bar{\varphi}_0^2 \ . \tag{3.240}$$

We will proceed to evaluate the determinant of $\mathbf{B}$ by first finding the heat function for $\mathbf{C}$ and then using it to arrive at the heat function associated with $\mathbf{B}$. We look for the solution of

$$\mathbf{C}_x G_{\mathbf{C}}(\vec{x}, \vec{y}; \sigma) = -\frac{\partial}{\partial\sigma} G_{\mathbf{C}}(\vec{x}, \vec{y}; \sigma) \ , \tag{3.241}$$

with the boundary condition

$$G_{\mathbf{C}}(\vec{x}, \vec{y}; 0) = \delta(\vec{x} - \vec{y}) \ . \tag{3.242}$$

For d space dimensions, *i.e.* when $\vec{\nabla}^2 = \sum_1^d \partial_i^2$, we use the solution we have obtained in section 5 for zero temperature field theory,

$$G_{\mathbf{C}}(\vec{x}, \vec{y}; \sigma) = \frac{\mu^d}{(4\pi\sigma)^{d/2}} e^{-\frac{\mu^2}{4\sigma}(\vec{x}-\vec{y})^2 - (m^2 + \frac{\lambda}{2}\bar{\varphi}_0^2)\sigma/\mu^2} \ . \tag{3.243}$$

As before, we have inserted an arbitrary mass parameter μ in order to make σ dimensionless. Now, since the eigenvalues of the operator $-\frac{\partial^2}{\partial\tau^2}$ over periodic functions are just $4\pi^2 n^2$, it follows that the full heat function for the operator $\mathbf{B}$ is given by

$$G_{\mathbf{B}}(\tau,\vec{x},\tau',\vec{y};\sigma)=\frac{\mu^d}{(4\pi\sigma)^{d/2}}e^{-\frac{\mu^2}{4\sigma}(\vec{x}-\vec{y})^2-\frac{M^2\beta^2\sigma}{\mu^2}}\sum_{n=-\infty}^{+\infty}e^{-\frac{4\pi^2n^2}{\mu^2\beta^2}\sigma+2\pi in(\tau-\tau')}\ , \tag{3.244}$$

where we have set

$$M^2=m^2+\frac{1}{2}\lambda\bar{\varphi}_0^2\ . \tag{3.245}$$

The corresponding ζ-function is

$$\zeta_{\mathbf{B}}[s]=\frac{\mu^d}{\Gamma(s)}\int_0^\infty d\sigma\frac{\sigma^{s-1-d/2}}{(4\pi)^{d/2}}\int_0^1 d\tau\int d^dx\sum_{n=-\infty}^{+\infty}e^{-[4\pi^2n^2+\beta^2M^2]\frac{\sigma}{\mu^2\beta^2}}\ . \tag{3.246}$$

Note the appearance of the volume element $\int d^dx$ which can be regularized by putting the system in a finite box. We will call this factor V. Clearly when d=0, this reduces to the usual quantum mechanical result. We now scale the dimensionless $\mu^2\beta^2$ out of the exponent be redefining σ, thus obtaining

$$\zeta_{\mathbf{B}}[s]=\frac{V}{\Gamma(s)}\frac{(\mu\beta)^{2s}}{(4\pi\beta^2)^{d/2}}\int_0^\infty d\sigma\ \sigma^{s-1-d/2}\sum_{n=-\infty}^{+\infty}e^{-[4\pi^2n^2+\beta^2M^2]\sigma}\ . \tag{3.247}$$

We single out the $n=0$ term in the sum, to arrive at

$$\zeta_{\mathbf{B}}[s] = \frac{VM^d}{(4\pi)^{d/2}}(\frac{\mu}{M})^{2s}\frac{\Gamma(s-d/2)}{\Gamma(s)}+\frac{2V}{(4\pi\beta^2)^{d/2}}\frac{(\mu\beta)^{2s}}{\Gamma(s)}\ \times \tag{3.248}$$

$$\int_0^\infty d\sigma\ \sigma^{s-1-\frac{d}{2}}\sum_{n=1}^{\infty}e^{[-4\pi^2n^2+\beta^2M^2]\sigma}\ . \tag{3.249}$$

Let us specialize to the case of interest d=3 from now on. We first evaluate the contribution from the second part of (3.7.70) in the limit of high temperature ($\beta\to 0$). We expand the exponential to get

$$\zeta_{\mathbf{B}}[s] = \frac{VM^3}{8\pi^{3/2}}(\frac{\mu}{M})^{2s}\frac{\Gamma(s-\frac{3}{2})}{\Gamma(s)}+2V\frac{\pi}{\beta^2}^{\frac{3}{2}}\frac{(\mu\beta)^{2s}}{\Gamma(s)}\ \times \tag{3.250}$$

$$\sum_{\ell=0}^{\infty}\frac{(\beta M)^{2\ell}}{\ell!}\frac{(-1)^\ell}{(2\pi)^{2\ell+2s}}\Gamma(s-\frac{3}{2}+\ell)\zeta[2s+2\ell-3]\ , \tag{3.251}$$

and then expand the sum and note that the first three terms of the above

formula all vanish as $s \to 0$ because of thesimple zero of $\Gamma^{-1}(s)$. Hence their contributions to $\zeta'_{\mathbf{B}}[0]$ is easy to evaluate by setting s=0 to what multiplies Γ^{-1}. The next higher order term is a bit trickier to evaluate because $\zeta[1]$ actually diverges.

Given that $\Gamma(-\frac{3}{2}) = \frac{4}{3}\sqrt{\pi},\ \zeta[-3] = \frac{1}{120},\ \Gamma(-\frac{1}{2}) = -2\sqrt{\pi},\ \zeta[-1] = -\frac{1}{12}$, we find for the first three terms

$$\zeta'_{\mathcal{B}}[0] = V[\frac{M^3}{6\pi} + \frac{\pi^2}{45\beta^3} - \frac{M^2}{12\beta} + \cdots] \ . \tag{3.252}$$

We leave the evaluation of the fourth term to the reader. Its contribution to $\zeta'_{\mathbf{B}}[0]$ is

$$\frac{VM^4\beta}{8\pi^2}[\gamma + \ln(\frac{\mu\beta}{4\pi})] \ , \tag{3.253}$$

where γ is the Euler-Mascheroni constant. Putting everything together, we obtain our final result for the high temperature limit

$$\ln\ Z = \frac{1}{2}\zeta'_{\mathbf{B}}[0] \tag{3.254}$$

$$= V[\frac{M^3}{12\pi} + \frac{\pi^2}{90\beta^3} - \frac{M^2}{24\beta} + \frac{\beta M^4}{16\pi^2}[\gamma + \ln(\frac{\mu\beta}{4\pi})] + \cdots] \ , \tag{3.255}$$

which gives us the free energy per unit volume

$$\frac{F}{V} = -\frac{M^3}{12\pi\beta} - \frac{\pi^2}{90\beta^4} + \frac{M^2}{24\beta^2} - \frac{M^4}{16\pi^2}(\gamma + \ln\frac{\mu\beta}{4\pi}) + \cdots \ . \tag{3.256}$$

in the limit of high temperature $\beta \to 0$. We can also obtain a closed form expression for the free energy for any temperature as follows. The trick is to make use of the mathematical formula [Gradshteyn & Ryzhik, p 317]

$$\frac{1}{\sigma^{d/2}} = \frac{1}{\sqrt{\pi}}\frac{2^{\frac{d+1}{2}}}{1\cdot 2\cdot 5\cdots(d-2)}\int_0^\infty dx\ x^{d-1}e^{-\sigma x^2} \ , \tag{3.257}$$

allowing us to rewrite the ζ-function in the form

$$\zeta_{\mathbf{B}}[s] = \frac{V(4\pi\beta^2)^{-d/2}}{\Gamma(s)}\frac{2^{\frac{d+1}{2}}(\mu\beta)^{2s}}{\sqrt{\pi}\ 1\cdot 3\cdots(d-2)}\int_0^\infty d\sigma\ \sigma^{s-1}\ \times \tag{3.258}$$

$$\int_0^\infty dx\ x^{d-1}\sum_{n=+\infty}^{+\infty} e^{-[4\pi^2n^2+M^2\beta^2+x^2]\sigma} \tag{3.259}$$

$$= \frac{V(4\pi\beta^2)^{-d/2}2^{\frac{d+1}{2}}(\mu\beta)^{2s}}{\sqrt{\pi}\ 1\cdot 3\cdots(d-2)}\int_0^\infty dx\ x^{d-1}\times \tag{3.260}$$

$$\sum_{n=-\infty}^{+\infty}(4\pi^2n^2+M^2\beta^2+x^2)^{-s}\ . \tag{3.261}$$

We recognize the sum from the quantum mechanical example, except that now we have replaced $\omega^2\beta^2$ with $M^2\beta^2+x^2$. Thus it follows that

$$\zeta'_{\mathbf{B}}[0] = \frac{V(4\pi\beta^2)^{-d/2}}{\sqrt{\pi}\ 1\cdot 3\cdots(d-2)}2^{\frac{d+1}{2}}\int_0^\infty dx\ x^{d-1}\ \times \tag{3.262}$$

$$[-\sqrt{x^2+M^2\beta^2}-2\ln(1-e^{-\sqrt{x^2+M^2\beta^2}})]\ . \tag{3.263}$$

One can check that the $\ln\mu\beta$ term does not contribute. For $d=3$ this gives

$$\zeta'_{\mathbf{B}}[0] = -\frac{V}{\pi^2\beta^3}\int_0^\infty dx\ x^2(\frac{1}{2}\sqrt{x^2+M^2\beta^2}+\ln[1-e^{-\sqrt{x^2+M^2\beta^2}}])\ , \tag{3.264}$$

which leads to the free energy per unit volume

$$\frac{F}{V} = \frac{1}{2\pi^2\beta^3}\int_0^\infty dx\ x^2\left[\frac{1}{2}\sqrt{x^2+M^2\beta^2}+\ln(1-e^{-\sqrt{x^2+M^2\beta^2}})\right]\ . \tag{3.265}$$

The first term in the integrand comes from the zero point energy, and its β-dependence can be scaled away, leaving just

$$\frac{1}{4\pi^2}\int_0^\infty dx\ x^2\sqrt{x^2+M^2}\ , \tag{3.266}$$

Its significance becomes clear when it is realized that in the limit of zero temperature ($\beta\to\infty$), the second term vanishes since the argument of the ln term becomes arbitrarily close to 1 and it is furthermore multiplied by β^{-3}. Hence what remains as $\beta\to\infty$ is the usual field theory expression for the one-loop effective potential. Putting it all together, one obtains the one loop correction for the finite temperature field theory

$$\frac{F}{V} = \frac{1}{2}m^2\varphi_0^2+\frac{\lambda}{4!}\varphi_0^4+\frac{\hbar}{64\pi^2}[m^2+\frac{\lambda}{2}\varphi_0^2]\ln[\frac{m^2+\lambda/2\varphi_0^2}{\mu^2}-\frac{3}{2}] \tag{3.267}$$

$$+\frac{\hbar}{2\pi^2\beta^2}\int_0^\infty dx\ x^2\ln[1-e^{-\sqrt{x^2+\beta^2(m^2+\frac{\lambda}{2}\varphi_0^2)}}]\ . \tag{3.268}$$

At large temperatures, (β small) this expression has the correct expansion which coincides with (3.7.75).

3.6.1 PROBLEMS

A. Find the temperature dependence of the partition function for a quantum-mechanical particle in the potential $V(q) \sim \frac{1}{q^2}$. Discuss the significance of your result.

B. Evaluate the fourth term in the high temperature expansion of the free energy, starting from (3.7.71).