5

Path Integral Formulation with Fermions

5.1 Integration over Grassmann Numbers

In Chapter 1 we gave several examples of Action functionals involving Fermi fields, that is, fields transforming as half-integer spin representations of the Lorentz group. It was then pointed out that the Fermi fields should be taken to be anticommuting classical fields, and that this was a *classical* identification which did not imply any quantization. If we reason by analogy with the quantization of, say, the scalar field, we are led to considering a "path" integral over anticommuting fields. This can at best be a formal concept devoid of any direct physical meaning, but as is usual with such things, the final answer will be of interest, although the method of derivation might raise a few eyebrows! ,

To start with, consider the case of one Grassmann (anticommuting) "variable" θ. It satisfies [$\{\ ,\ \}$ denotes the anticommutator]

$$\{\theta, \theta\} = 0 \qquad \text{or} \qquad \theta^2 = 0 \ . \tag{5.1}$$

One defines the differential operator $\frac{d}{d\theta}$ by means of

$$\left\{ \frac{d}{d\theta}, \theta \right\} = 1 \ . \tag{5.2}$$

Any function of θ, $f(\theta)$, will have a simple expansion

$$f(\theta) = a + \beta\theta \ , \tag{5.3}$$

which terminates because of (5.1.1). For convenience take β to be of Grassmann type and a to be a real commuting number. [From now on Grassmann variables will be denoted by Greek letters.] Then it follows that

$$\frac{df}{d\theta} = -\beta \ , \tag{5.4}$$

so that

$$\frac{d^2 f}{d\theta^2} = 0 \tag{5.5}$$

where in the last two equations we have taken $\frac{d}{d\theta} a = \left\{ \frac{d}{d\theta}, \beta \right\} = 0$. From (5.1.5) we see that

$$\left\{ \frac{d}{d\theta}, \frac{d}{d\theta} \right\} = 0 \ , \tag{5.6}$$

which means that there is no inverse differentiation. This is awkward because one often likes to think of integration and differentiation as inverse operations. So we are warned that integration has to be introduced in formal terms. It is defined to be an operation denoted by $\int d\theta \cdots$ with the properties

$$\int d\theta = 0 \qquad \int d\theta\theta = 1 \ ; \tag{5.7}$$

it acts exactly like differentiation. This choice permits the integration to satisfy the criterion of invariance under a translation of the integration variable by a constant.

This world of one Grassmann variable is rather dull, so let us consider N Grassmann variables θ_i $i = 1, \cdots N$ which obey

$$\{\theta_i, \theta_j\} = 0 \qquad i, j = 1 \cdots N \ . \tag{5.8}$$

introduce their respective derivative operators $\frac{\partial}{\partial \theta_i}$ by means of

$$\left\{ \frac{\partial}{\partial \theta_i}, \theta_j \right\} = \delta_{ij} \ , \tag{5.9}$$

and

$$\left\{ \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j} \right\} = 0 \ . \tag{5.10}$$

Any normal (*i.e.*, non-Grassmann) function of the θ_i's can be written as

$$f(\theta_i) = a + \beta_i\theta_i + c_{ij}\theta_i\theta_j + \cdots + c\theta_1\theta_2\cdots\theta_n \ , \tag{5.11}$$

where the last coefficient is Grassmann or normal depending on N. Integration is defined in the same way as for one variable

$$\int d\theta_i = 0 \qquad \int d\theta_i\theta_i = 1 \qquad (i \text{ not summed}) \ . \tag{5.12}$$

When the measure of integration and integrand involve more than one variable we conventionally perform the integrations according to a nested procedure. Thus, for instance

$$\int d\theta_1 \int d\theta_2 \ \theta_1\theta_2 = -\int d\theta_1 \int (d\theta_2\theta_2)\theta_1 = -1 \ . \tag{5.13}$$

For instance, consider the integral

$$I_N(\mathbf{M}) = \int d\theta_1 \cdots d\theta_N \ e^{-\theta^T\mathbf{M}\theta} \ , \tag{5.14}$$

where $\mathbf{M}$ is an antisymmetric $N \times N$ matrix with normal elements, m_{ij}, and the exponential is defined according to its power series. When $N = 2$ we have

$$\begin{aligned} I_2(\mathbf{M}) &= \int d\theta_1 d\theta_2 \left[1 - 2m_{12}\theta_1\theta_2\right] && (5.15) \\ &= 2m_{12} = 2\sqrt{\det \mathbf{M}} \ . && (5.16) \end{aligned}$$

When N is odd one can show that I vanishes; this is consistent with identifying I with the square root of the determinant, since the determinant of odd-dimensional antisymmetric matrices vanishes. To guess at the general formula, let us examine the case $N = 4$. It is easy to see that the relevant terms in the expansion of the exponential are

$$\begin{aligned} e^{-\theta^T\mathbf{M}\theta} &= \cdots + \frac{1}{2!}\left(\theta^T\mathbf{M}\theta\right)^2 + \cdots && (5.17) \\ &= 4\theta_1\theta_2\theta_3\theta_4\left[m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}\right] + \cdots \ , && (5.18) \end{aligned}$$

leading to

$$\begin{aligned} I_4(M) &= 4\left[m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}\right] && (5.19) \\ &= 4\sqrt{\det \mathbf{M}} \ , && (5.20) \end{aligned}$$

so that the general formula turns out to be

$$I_N(\mathbf{M}) = (2)^{N/2}\sqrt{\det \mathbf{M}} \ . \tag{5.21}$$

This is the first formula of interest. Compare it with the equivalent formula for boson (normal) fields where the square root of the determinant appears in the denominator.

Secondly, let us consider

$$I_N(\mathbf{M};\vec{\chi}) \equiv \int d\theta_1 \cdots d\theta_N \ e^{-\theta^T \mathbf{M}\theta + \chi_i\theta_i} \ , \tag{5.22}$$

where the χ_i are Grassmann numbers

$$\{\chi_i, \chi_j\} = 0 \ , \qquad \{\chi_i, \theta_j\} = 0 \ . \tag{5.23}$$

To simplify matters, let us evaluate (5.1.21) directly when $N = 2$. There

$$e^{-\theta^T \mathbf{M}\theta + \chi^T\theta} = 1 - 2m_{12}\theta_1\theta_2 - \chi_1\chi_2\theta_1\theta_2 \ , \tag{5.24}$$

so that

$$I_2(\mathbf{M};\vec{\chi}) = 2\left(m_{12} + \frac{1}{2}\chi_1\chi_2\right) \ . \tag{5.25}$$

This result could have been obtained more easily by formally completing the squares in the exponent and shifting the variable of integration as if we were dealing with normal integration, *i.e.*, by letting

$$\theta' = \theta + \frac{1}{2}\mathbf{M}^{-1}\chi \ , \tag{5.26}$$

and rewriting

$$\begin{aligned} I_N(\mathbf{M};\vec{\chi}) &= \int d\theta_1 \cdots d\theta_N \ e^{-\theta'^T \mathbf{M}\theta' + \frac{1}{4}\chi^T \mathbf{M}^{-1}\chi} && (5.27) \\ &= e^{\frac{1}{4}\chi^T \mathbf{M}^{-1}\chi} I_N(\mathbf{M}) \ . && (5.28) \end{aligned}$$

Specializing to $N = 2$, we arrive at (5.1.24). The object of this little exercise is two-fold: to derive (5.1.27) and to show that shifting of variables is allowed for Grassmann integration because of the definition of (5.1.7).

These can be further generalized to integration over complex Grassmann variables. As an example, let

$$\eta = \frac{1}{\sqrt{2}}\,(\theta_1 + i\theta_2) \quad ; \qquad \eta^* = \frac{1}{\sqrt{2}}\,(\theta_1 - i\theta_2) \ , \tag{5.29}$$

such that

$$d\theta_1 d\theta_2 = d\eta^* d\eta \ , \tag{5.30}$$

and

$$\theta^T \mathbf{M}\theta = -2i\eta^* m_{12}\eta \ . \tag{5.31}$$

If we introduce a 1×1 matrix $2m_{12}$, application of (5.1.16)yields

$$\int d\eta^* d\eta \ e^{i\eta^* \mathbf{M}\eta} = \det \mathbf{M} \ ; \qquad \mathbf{M} = 2m_{12} \ . \tag{5.32}$$

This formula can be extended to N complex Grassmann numbers. In an analogous fashion, we can show that

$$\int d\eta^\dagger d\eta \ e^{i\eta^\dagger \mathbf{M}\eta + i\zeta^\dagger \eta + i\zeta^T \eta^*} = \det \mathbf{M} \ e^{-i\zeta^\dagger (\mathbf{M})^{-1}\zeta} \ , \tag{5.33}$$

where ζ are complex Grassmann numbers. Formulae (5.1.27) and (5.1.32) are very important for evaluating path integrals over fermions coupled to external Grassmann sources.

5.1.1 PROBLEMS

A. Develop a general proof for (5.1.20).

B. Prove (5.1.27) for $N = 4$ by explicit computation.

C. Prove (5.1.31) when $\mathbf{M}'$ is a 2×2 matrix.

D. Prove (5.1.32).

E. Show that

$$\int d\alpha d\beta \ e^{\alpha \mathbf{M}\beta} = \det \mathbf{M} \ , \tag{5.34}$$

where α and β are *independent* Grassmann variables.

5.2 Path Integral of Free Fermi Fields

In Minkowski space there are three ways to describe free spin 1/2 particles.

a) By means of the Weyl Lagrangian

$$\mathcal{L}_{\mathrm{W}} = \psi_L^\dagger \sigma \cdot \partial \psi_L \ , \tag{5.35}$$

containing a two component complex spinor ψ_L which describes a left-handed massless particle, together with its right-handed antiparticle (*e.g.*, the massless left-handed neutrinos and right-handed antineutrinos); both are related by the discrete CP transformation:

$$\mathrm{CP}: \quad \psi_L \to \sigma_2 \psi_L^* \ . \tag{5.36}$$

b) By means of the Majorana Lagrangian

$$\mathcal{L}_M = \psi_L^\dagger \sigma \cdot \partial \psi_L - \frac{im}{2} \left(\psi_L^T \sigma_2 \psi_L + \psi_L^\dagger \sigma_2 \psi_L^* \right) \ , \tag{5.37}$$

which describes a massive Weyl spinor. It is then interpreted as a spin 1/2 self-conjugate particle with spin up and spin down degrees of freedom. It can also be expressed in terms of the four component Majorana field

$$\Psi_M = \begin{pmatrix} \psi_L \\ -\sigma_2 \psi_L^* \end{pmatrix} \ , \tag{5.38}$$

in terms of which the Majorana Lagrangian becomes

$$\mathcal{L}_M = \frac{1}{2} \bar{\Psi}_M \gamma \cdot \partial \Psi_M + i \frac{m}{2} \bar{\Psi}_M \Psi_M \ . \tag{5.39}$$

c) Finally by means of the Dirac Lagrangian

$$\mathcal{L}_D = \psi_L^\dagger \sigma \cdot \partial \psi_L + \psi_R^\dagger \bar{\sigma} \cdot \partial \psi_R + im \left(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R \right) \tag{5.40}$$

which describes a particle with two degrees of freedom and its distinct antiparticle (*e.g.*, the electron and the positron). It has twice as many degrees of freedom as the Weyl or Majorana Lagrangian, and conserves P, in addition to CP. It can be conveniently expressed in terms of the four-component Dirac spinor

$$\Psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{5.41}$$

as

$$\mathcal{L}_D = \bar{\Psi}_D \left(\gamma \cdot \partial + im\right) \Psi_D \ . \tag{5.42}$$

For each of these Lagrangians we can build a generating functional when we add external sources. The Weyl fields ψ_L can be coupled to sources in the forms

$$\chi_L^T \sigma_2 \psi_L + \text{c.c.} \quad \text{and} \quad \chi_R^\dagger \psi_L + \text{c.c.} \ . \tag{5.43}$$

These two couplings are equivalent under the replacement $\chi_R = \sigma_2 \chi_L^*$. Thus it suffices to consider only one type of source coupling. We consider the functional

$$W[\chi_L, \chi_L^\dagger] = N \int \mathcal{D}\psi_L \mathcal{D}\psi_L^\dagger \ \exp\left\{ i \int d^4x [\mathcal{L}_W + i\chi_L^T \sigma_2 \psi_L + i\psi_L^\dagger \sigma_2 \chi_L^*] \right\} \ . \tag{5.44}$$

As in all free theories it can be readily evaluated. Introduce the Fourier transforms

$$\psi_L(x) = \int \frac{d^4p}{(2\pi)^2} \ e^{ip\cdot x} \tilde{\psi}_L(p) \ , \quad \text{etc.} \ , \tag{5.45}$$

as in Chapter III. The exponent now reads

$$iS_W = -\int d^4p \left[\tilde{\psi}_L^\dagger(p) \sigma \cdot p \tilde{\psi}_L(p) + \tilde{\chi}_L^T(-p) \sigma_2 \tilde{\psi}_L(p) + \tilde{\psi}_L^\dagger(p) \sigma_2 \tilde{\chi}_L^*(-p) \right] \ . \tag{5.46}$$

We rewrite it in the form

$$-\int d^4p \left\{ \left[\tilde{\psi}_L^\dagger(p) + \tilde{\phi}_L^\dagger(p) \right] \sigma \cdot p \left[\tilde{\psi}_L(p) + \tilde{\phi}_L(p) \right] + \tilde{\phi}_L^\dagger(p) \sigma \cdot p \tilde{\phi}_L(p) \right\} \ , \tag{5.47}$$

where $\tilde{\phi}_L(p)$ is the solution of the equations of motion

$$\tilde{\phi}_L(p) = \frac{\bar{\sigma} \cdot p}{p^2} \sigma_2 \tilde{\chi}_L^*(-p) \ . \tag{5.48}$$

In this form we see that integration over ψ_L can be readily performed after shifting the integration variable by $\tilde{\phi}_L$, resulting in a change in the arbitrary normalization:

$$W\left[\chi_L, \chi_L^\dagger\right] = N' \exp\left\{-\int d^4p \tilde{\chi}_L^\dagger(p) \frac{\sigma \cdot p}{p^2} \tilde{\chi}_L(p)\right\} \ , \tag{5.49}$$

where we have used [see (1.4.37)]

$$\bar{\sigma} \cdot p \ \sigma \cdot p = p^2 \ , \tag{5.50}$$

and

$$\sigma_2 \ \bar{\sigma}^T \cdot p \ \sigma_2 = \sigma \cdot p \ . \tag{5.51}$$

If we set as in the boson case

$$W = e^{iZ} \ , \tag{5.52}$$

where Z is the generator of connected Green's functions, we find that

$$Z\left[\chi_L, \chi_L^\dagger\right] = -i \int d^4x \, \chi_L^\dagger(x) \, (i\bar{\sigma} \cdot \partial)^{-1} \, \chi_L(x) \ . \tag{5.53}$$

Thus, we extract the two point connected Green's function

$$G^{(2)}(x_1, x_2) = -i \left(i\bar{\sigma}^\mu \frac{\partial}{\partial x_2^\mu}\right)^{-1} \delta^{(4)}(x_1 - x_2) \ , \tag{5.54}$$

or in momentum space

$$\tilde{G}^{(2)}(p) = -\frac{i}{\bar{\sigma} \cdot p} = -i \frac{\sigma \cdot p}{p^2} \ , \tag{5.55}$$

which is, of course, the inverse propagator. As it stands, this expression is meaningless until we have specified the pole prescriptions at $p^2 = 0$. We can interpret it in analogy with the boson case but it should be noted that in the fermion case we have no nice convergence argument to introduce the $-i\epsilon$ prescription since we are dealing with formal Grassmann integration. Thus it would seem that we have to set up the problem in Euclidean space in order to justify the same pole prescription as we used for bosons.

The other two cases are treated along similar lines. In the Majorana case we start with

$$W_M[X_M] = N \int \mathcal{D}\Psi_M \, e^{i\int d^4x[\mathcal{L}_M + \bar{\Psi}_M X_M]} \ , \tag{5.56}$$

and by completing the squares, arrive at the expression

$$W_M[X_M] = N' \, e^{-\frac{1}{2}\int d^4p \bar{X}_M (\not{p}+m)^{-1} X_M} \ , \tag{5.57}$$

where

$$\not{p} = \gamma^\mu p_\mu \tag{5.58}$$

leading to the propagator

$$\tilde{G}_M^{(2)}(p) = \frac{-i}{\not{p}+m} = -i\frac{\not{p}-m}{p^2-m^2} \ , \tag{5.59}$$

using $\not{p}\not{p} = p^2$. Again the pole prescription has to be added in explicitly. The Dirac case is treated the same way, starting with

$$W_D[\zeta,\bar{\zeta}] = N \int \mathcal{D}\Psi_D \mathcal{D}\bar{\Psi}_D \, \exp\left\{ i \int d^4x \left[\mathcal{L}_D + i\bar{\Psi}_D\zeta + i\bar{\zeta}\Psi_D \right] \right\} \ , \tag{5.60}$$

where now ζ and $\bar{\zeta}$ are four component Grassmann sources. A similar reasoning leads to

$$W_D = N' \, e^{-\int d^4p \bar{\zeta}_D (\not{p}+m)^{-1} \zeta_D} \ , \tag{5.61}$$

from which we extract the propagator

$$G_D^{(2)}(p) = \frac{-i}{\not{p}+m} = -i\frac{\not{p}-m}{p^2-m^2} \ , \tag{5.62}$$

where the $-i\epsilon$ prescription has to be added on.

As in the boson case, one might set up the generating functional directly in Euclidean space, and then continue the Green's functions into Minkowski space.

In Euclidean space the Lorentz group becomes compact, which means (see Chapter I) that it is composed of two truly inequivalent $SU(2)$ factors. However, the derivative operator still transforms as the$(\mathbf{1/2}, \mathbf{1/2})$ representation so that if we now want to make a Lorentz scalar linear in the derivative we need two different fields $\psi_L \sim (\mathbf{1/2}, \mathbf{0})$ and $\psi_R \sim (\mathbf{0}, \mathbf{1/2})$ to build vector quantities transforming as $(\mathbf{1/2}, \mathbf{1/2})$. We can build two such real vectors with components

$$\left(i\psi_L^\dagger\psi_R + i\psi_R^\dagger\psi_L, -\psi_L^\dagger\vec{\sigma}\psi_R + \psi_R^\dagger\vec{\sigma}\psi_L\right) \tag{5.63}$$

and

$$\left(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L, i\psi_L^\dagger\vec{\sigma}\psi_R + i\psi_R^\dagger\vec{\sigma}\psi_L\right) \ , \tag{5.64}$$

remembering that because ψ_L and ψ_R are Grassmann numbers

$$\left(\psi_L^\dagger\psi_R\right)^* = \psi_L^T\psi_R^* = -\psi_R^\dagger\psi_L \ . \tag{5.65}$$

If we introduce the four-component Euclidean Dirac spinor

$$\Psi_E = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \ , \tag{5.66}$$

we can rewrite the vectors in the form

$$\Psi_E^\dagger\bar{\gamma}_\mu\Psi_E \qquad \text{and} \qquad \Psi_E^\dagger\bar{\gamma}_5\bar{\gamma}_\mu\Psi_E \ , \tag{5.67}$$

respectively, where $\bar{\gamma}_\mu$ are the Euclidean γ-matrices

$$\bar{\gamma}_0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \ , \quad \bar{\gamma}_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{5.68}$$

which satisfy

$$\{\bar{\gamma}_\mu, \bar{\gamma}_\nu\} = -2\delta_{\mu\nu} \ , \tag{5.69}$$

and

$$\bar{\gamma}_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{5.70}$$

The possible mass terms are (in Euclidean space there is only type of mass)

$$\psi_L^\dagger\psi_L \qquad \text{and} \qquad \psi_R^\dagger\psi_R \ , \tag{5.71}$$

so that the Euclidean Lagrangian is given by

$$\mathcal{L}_E = \psi_E^\dagger \bar{\gamma}_\mu \bar{\partial}_\mu \psi_E + im\psi_E^\dagger \psi_E; \tag{5.72}$$

it is carefully chosen to be real

$$\mathcal{L}_E^* = \mathcal{L}_E \ . \tag{5.73}$$

The corresponding generating functional is

$$\begin{aligned} W_E\left[\zeta_E, \zeta_E^\dagger\right] &= N \int \mathcal{D}\psi_E^\dagger \mathcal{D}\psi_E \, \exp\left\{-\int d^4\bar{x}\left[\mathcal{L}_E + i\zeta_E^\dagger \psi_E + i\psi_E^\dagger \zeta_E\right]\right\} \\ &\quad (5.74) \\ &= N' \exp\left\{i\int d^4p \tilde{\zeta}_E^\dagger(p)\left[\bar{\not{p}} + m\right]^{-1} \tilde{\zeta}_E(p)\right\} \ , \end{aligned} \tag{5.75}$$

leading to the propagator

$$G_E(p) = \frac{-i}{\bar{\not{p}} + m} = i\frac{\bar{\not{p}} - m}{\bar{p}^2 + m^2} \ , \tag{5.76}$$

where we have used $\bar{\not{p}}\bar{\not{p}} = -\bar{p}^2$. We remark that it has the expected $\bar{p}^2 + m^2$ denominator. It is satisfying to see that it has the same structure as the Dirac propagator in Minkowski space.

There does not seem to be any such correspondence for Weyl fields: one cannot construct a first order equation for a field transforming as $(\mathbf{1/2}, \mathbf{0})$ in Euclidean space starting from an invariant Lagrangian that contains this field alone (as we just saw it can easily be done when two Weyl fields are considered). Should the FPI be well-defined only in Euclidean space, as axiomaticians would have it, then there seems to exist a very real problem when dealing with Weyl fields as in the theory of weak interactions or in its unification with QCD. It must be emphasized that there is nothing apparently wrong with the field theory of Weyl fields in Minkowski space (excepting possible anomalies) as far as perturbation theory is concerned — it could well be that a more complete treatment might yield surprises, requiring the doubling of the Weyl fermions, which would restore at some (large) scale a vector-like structure for the weak interactions. Or else, since this peculiarity occurs only in four dimensions, it could be used as an argument for the fundamental theory to reside in higher dimensions.

Alternatively, since Fermi fields appear only quadratically in renormalizable Lagrangians, their integration leads only to determinants. Thus one

might argue that any Euclidean functional that leads to the correct (as determined by its continuation in Minkowski space) determinant is all that is needed. This approach requires the doubling of the number of independent Grassmann fields (see S. Coleman's "The Uses of Instantons" in *Aspects of Symmetry*, *op. cit.*, and problem E).

5.2.1 PROBLEMS

A. Evaluate the generating functional for both Majorana and Dirac fields in Minkowski space

B. Show that in Euclidean space the derivative operator ∂_μ transforms according to the $(\mathbf{1/2}, \mathbf{1/2})$ representation.

C. Given in Euclidean space $\psi_L \sim (\mathbf{1/2}, \mathbf{0})$, build explicitly the quadratic form which behaves as $(\mathbf{1}, \mathbf{0})$.

D. Show that the Euclidean space spinor Lagrangian has the curious property that its mass is invariant under the so-called chiral transformations

$$\Psi_E \to e^{i\alpha\gamma_5}\Psi_E \ , \tag{5.77}$$

while the kinetic term is not!

E. Formally define

$$\mathcal{L} = \chi(\not{\partial} + im)\psi \ , \tag{5.78}$$

where χ and ψ are independent four-component Grassmann fields. Then show how to judiciously integrate in order to obtain the usual Dirac determinant. Discuss chiral invariance in this example.

5.3 Feynman Rules for Spinor Fields

The Feynman rules for free Fermi fields have already been discussed in the previous section. Here we derive the rules for interacting spinors. Spinors can interact in a variety of ways subject to the conservation of spin which requires that all interaction vertices include an even pair of spinor fields. We have given in Chapter I examples of interacting theories with fermions.

The number of possible fermion interactions is drastically reduced when we impose the constraint of renormalizability which demands as a necessary condition that the number of primitively divergent diagrams be finite. Let us therefore compute the superficial degree of divergence D of an arbitrary Feynman diagram with fermions.

Consider a diagram with L loops, I_b boson internal lines, I_f fermion internal lines, V vertices each with N_b boson and N_f fermion lines, E_b external boson lines and E_f external fermion lines. As we just remarked, N_f and E_f must be even. The number of loops L is given by

$$L = I - V + 1 = I_b + I_f - V + 1 \ . \tag{5.79}$$

The superficial degree of divergence in d dimensions is

$$D_d = dL - I_f - 2I_b \ , \tag{5.80}$$

since each internal spinor line contributes only one inverse power of momentum. Furthermore, the total number of fermion lines is given by

$$N_f V = E_f + 2I_f \tag{5.81}$$

with a similar relation for boson lines

$$N_b V = E_b + 2I_b \ . \tag{5.82}$$

These enable us to express D_d in the form

$$D_d = d - \frac{1}{2}(d-1)E_f - \frac{1}{2}(d-2)E_b - V\left[d - \frac{1}{2}(d-1)N_f - \frac{1}{2}(d-2)N_b\right] \ . \tag{5.83}$$

When $N_f = E_f = 0$ this reduces to the previously obtained expression with only bosons present. In two dimensions, it reduces to

$$D_2 = 2 - \frac{1}{2}E_f - V\left(2 - \frac{1}{2}N_f\right) \ , \quad \text{[two dimensions]} \ , \tag{5.84}$$

which shows that

$$N_f \leq 4 \ , \quad \text{[two dimensions]} \ ; \tag{5.85}$$

otherwise the divergence will grow with the number of vertices. Hence there is a restriction on the type of allowed fermion interaction, even in two dimensions: it must not be of a degree higher than $(\psi)^4$. We can understand this fact in another way: unlike boson fields which are dimensionless in two dimensions, spinor fields have dimension $-1/2$ so that ψ^4 is the highest

interaction which does not necessitate the introduction of a dimensionful coupling constant.

In four dimensions, we have

$$D_4 = 4 - \frac{3}{2}E_f - E_b - V\left[4 - \frac{3}{2}N_f - N_b\right] \ . \tag{5.86}$$

If we do not want the number of primitively divergent graphs to grow with the number of vertices, we must require

$$4 - \frac{3}{2}N_f - N_b \geq 0 \ , \tag{5.87}$$

where N_f is even. The possible solutions are

$$N_b = 0 \qquad N_f = 2 \ , \tag{5.88}$$

which is like a mass insertion and not an interaction vertex,

$$N_f = 0 \qquad N_b = 2,\, 3,\, 4 \ , \tag{5.89}$$

which give ϕ^2, ϕ^3, ϕ^4 interactions we have previously analyzed. The only new solution involving both fermions and bosons is

$$N_f = 2 \qquad N_b = 1 \ , \tag{5.90}$$

which gives

$$D_4 = 4 - \frac{3}{2}E_f - E_b \ . \tag{5.91}$$

This new solution which describes the only fermion interaction allows by renormalizability is incredibly restrictive: renormalizable fermion interactions must involve at most two spinor fields and one boson field. Thus, fermions in four-dimensions appear only quadratically in $\mathcal{L}$! This fact can also be understood in another way: in four dimensions fermions have dimension $-\frac{3}{2}$, bosons -1. Hence the only non-trivial coupling of dimension four is the one with two fermions and one boson:

$$mbox11 \tag{5.92}$$

This remarkable fact greatly simplifies our analysis of interacting theories with spinors. Given two spin $\frac{1}{2}$ fields we can form either a spin 0 or a spin 1 combination. The couplings to a spin 0 field are the Yukawa couplings

— they appear in many guises: in Minkowski space we have couplings for Dirac fields

$$i\bar{\Psi}_D\Psi_D\phi \ , \qquad \bar{\Psi}_D\gamma_5\Psi_D\phi' \ , \tag{5.93}$$

where ϕ and ϕ' are scalar and pseudoscalar fields, respectively. For Weyl fields we have

$$i\psi_L^T\sigma_2\psi_L\phi_1 \ , \qquad i\psi_L^\dagger\sigma_3\psi_L^*\phi_2 \ . \tag{5.94}$$

where ϕ_1 and ϕ_2 have no defined parity. In Euclidean space the possible couplings are

$$i\Psi_E^\dagger\Psi_E\phi \ , \qquad \Psi_E^\dagger\gamma_5\Psi_E\phi' \quad \text{[Euclidean space]} \ . \tag{5.95}$$

The couplings to spin 1 given for a Dirac particle in Minkowski space

$$i\bar{\Psi}_D\gamma_\mu\Psi_D A^\mu \ , \qquad \bar{\Psi}_D\gamma_\mu\gamma_5\Psi_D A'^\mu \ , \tag{5.96}$$

where A^μ is a vector field, A'^μ an axial vector: for Weyl fields we have

$$i\psi_L^\dagger\sigma_\mu\psi_L B^\mu \ , \qquad i\psi_R^\dagger\bar{\sigma}_\mu\psi_R B'^\mu \ , \tag{5.97}$$

Where B^μ and B'^μ have no well-defined parity properties. In Euclidean space the vector couplings are

$$i\Psi_E^\dagger\bar{\gamma}_\mu\Psi_E A_\mu \ , \qquad \Psi_E^\dagger\bar{\gamma}_\mu\bar{\gamma}_5\Psi_E A'_\mu \ . \tag{5.98}$$

Interaction of spinor and vector fields will be extensively discussed in later chapters. The Feynman rule for the Yukawa vertices is simply the dimensionless Yukawa couplings themselves

$$11 \qquad \leftrightarrow if: \quad if\bar{\Psi}\Psi\phi \tag{5.99}$$

$$11 \qquad \leftrightarrow f'\gamma_5: \quad f'\bar{\Psi}\gamma_5\Psi\phi' \ . \tag{5.100}$$

In the above, the dotted line denotes a boson field and the solid line a spinor field with the spinor indices suppressed.

The Grassmann nature of the spinor field is reflected in one crucial change in the Feynman rules: whenever a closed fermion line (loop) appears in a diagram, one should charge the diagram a minus sign, as the following example will illustrate. Consider the expression (say, in Euclidean space)

$$W = \frac{\det\left[\partial_\mu \partial_\mu + \lambda A(x)\right]}{\det\left[\partial_\mu \partial_\mu + \lambda A(x)\right]} = 1 \ , \tag{5.101}$$

where $A(x)$ is a scalar field and the determinants are to be understood as functional determinants. We can express them in terms of path integrals, the one in the denominator giving a path integral over boson fields and the one in the numerator a path integral over Grassmann fields. The result is

$$W = \int \mathcal{D}\phi \mathcal{D}\phi^* \mathcal{D}\Psi \mathcal{D}\Psi^* \, e^{i\left\langle \psi^* \left(\partial^2 + \lambda A(x)\right)\psi \right\rangle + \left\langle \phi^* \left(\partial^2 + \lambda A(x)\right)\phi \right\rangle} \ . \tag{5.102}$$

In this form it looks like a theory of a Grassmann field ψ interacting with a complex scalar field through the external field $A(x)$. The Feynman rules are

$$: \quad \frac{i}{p^2} \qquad \text{for the Grassmann line} \tag{5.103}$$

$$: \quad \frac{1}{p^2} \qquad \text{for the boson field line} \tag{5.104}$$

$$11 \quad : \lambda \quad ; \qquad 11 \quad : \lambda \ , \tag{5.105}$$

where A appears as the wavy line. In this theory, the A propagator will be corrected by the "vacuum polarization" diagrams

$$1.25.4 + 1.25.4 \ , \tag{5.106}$$

which, according to the previous Feynman rules are exactly the same. But, these diagrams cannot possible alter the A line because we know the theory to be trivial, starting from $W = 1$! Hence it must be that rather than adding, the two diagrams cancel: the closed Grassmann loop must acquire a minus sign relative to the closed boson loop.

Hence, wherever Grassmann (spinor) fields are encountered, the Feynman rules say that one must multiply a diagram with n distinct closed fermion loops by $(-1)^n$. A more physical way of understanding this fact is to remark that the cut fermion loop must by generalized Fermi statistics be antisymmetric under the interchange of its legs since it is related by unitarity to a physical amplitude.

Finally we note that the Yukawa coupling can induce renormalizable self-interactions among the scalar fields (see problem).

5.3.1 PROBLEMS

A. Find the dimensions for which there are renormalizable theories involving fermions.

B. Given the Lagrangian

$$\mathcal{L} = \bar{\Psi}_D \left(\gamma \cdot \partial + im\right) \Psi_D + if\bar{\Psi}_D \phi \Psi_D + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 \ . \tag{5.107}$$

–1) Derive the Feynman rules,

–2) Find the scale dependence of the Yukawa coupling constant at the one loop level.

–3) Discuss the renormalizability of this Lagrangian; in particular, analyze all the one loop diagrams and discuss the ensuing counterterm structure. Is this Lagrangian renormalizable as written? If not, amend it.

5.4 Evaluation and Scaling of Fermion Determinants

Let us start with the Euclidean space generating functional for the theory that describes a scalar field in interaction with a four-component spinor field

$$W_E\left[\zeta^\dagger, \zeta, J\right] = e^{-S_E} = \int \mathcal{D}\phi \mathcal{D}\Psi^\dagger \mathcal{D}\Psi \, e^{-S_E\left[\phi, \Psi^\dagger, \Psi, J, \zeta^\dagger, \zeta\right]} \ , \tag{5.108}$$

$$S_E = \int d^4\bar{x}[\frac{1}{2}\bar{\partial}_\mu \phi \bar{\partial}_\mu \phi \qquad + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4 \tag{5.109}$$

$$-J\phi + \Psi^\dagger(\not{\partial} + im' + if\phi)\Psi + i\zeta^\dagger \Psi - i\Psi^\dagger \zeta] \tag{5.110}$$

In this expression the spinor fields appear quadratically and can therefore be functionally integrated. This step leaves us with the expression ($S_E[\phi, J]$ is given by Eq. (3.4.5))

$$W_E\left[\zeta^\dagger, \zeta, J\right] = \int \mathcal{D}\phi \, e^{-S_E[\phi, J]} \det(\not{\partial} + im' + if\phi) \, e^{\left\langle \zeta^\dagger (\not{\partial} + im' + if\phi)^{-1} \zeta \right\rangle} \ , \tag{5.111}$$

where we have used (5.1.31) after completing the squares. We can rewrite it in the form

$$W_E = e^{\left\langle \zeta^\dagger \left(\not\partial + im' + if\frac{\delta}{\delta J}\right)^{-1}\zeta\right\rangle} \int \mathcal{D}\phi\, e^{-S_E[\phi,J]} \det(\not\partial + im' + if\phi)\ . \tag{5.112}$$

This is a pretty formula which is useful in the perturbative evaluation of W_E. In this section we would rather concentrate on the saddle point evaluation of W_E. To start with, let us expand S_E around a classical field configuration ϕ_0, Ψ_0, $\Psi_0^\dagger$. The functional expansion is of the form

$$\begin{aligned} S_E &= S_E\Big|_0 + \left\langle \eta^\dagger \frac{\delta S}{\delta \Psi^\dagger}\Big|_0 \right\rangle + \left\langle \frac{\delta S}{\delta \Psi}\Big|_0 \eta \right\rangle + \left\langle \frac{\delta S}{\delta \phi}\Big|_0 \rho \right\rangle && (5.113)\\ &+ \frac{1}{2}\left\langle \rho_1 \frac{\delta^2 S}{\delta\phi_1\delta\phi_2}\Big|_0 \rho_2 \right\rangle_{1,2,} + \left\langle \eta_1^\dagger \frac{\delta^2 S}{\delta\Psi_1^\dagger\delta\Psi_2}\eta_2 \right\rangle_{1,2} && (5.114)\\ &+ \left\langle \rho_1 \frac{\delta^2 S}{\delta\phi_1\delta\Psi_2}\Big|_0 \eta_2 \right\rangle_{1,2} + \left\langle \eta_1^\dagger \frac{\delta^2 S}{\delta\Psi_1^\dagger\delta\phi_2}\Big|_0 \rho_2 \right\rangle_{1,2} + \cdots && (5.115) \end{aligned}$$

where $\eta = \Psi - \Psi_0$, and $\rho = \phi - \phi_0$.

As in Section 4 of Chapter 3, we expand around field configurations that leave the Action stationary, *i.e.*, solve the classical equations of motion:

$$\begin{aligned} \frac{\delta S}{\delta \Psi^\dagger}\Big|_0 &= (\not\partial + im' + if\phi_0)\Psi_0 - i\zeta = 0 && (5.116)\\ \frac{\delta S}{\delta \Psi}\Big|_0 &= \Psi_0^\dagger(-\overleftrightarrow{\not\partial} + im' + if\phi_0) - i\zeta^\dagger = 0 && (5.117)\\ \frac{\delta S}{\delta \phi}\Big|_0 &= \left(-\partial^2 + m^2 + \frac{\lambda}{3!}\phi_0^2\right)\phi_0 + if\Psi_0^\dagger\Psi_0 - J = 0\ . && (5.118) \end{aligned}$$

By eliminating these linear terms we have an approximate expression for S_E that is quadratic in η, $\eta^\dagger$ and ρ, the differences of the fields away from their stationary values. Explicitly

$$\begin{aligned} S_E \simeq S_E\Big|_0 &= \left\langle \eta^\dagger(\not\partial + im' + if\phi_0)\eta \right\rangle + \frac{1}{2}\left\langle \rho\left(-\partial^2 + m^2 + \frac{\lambda}{2}\phi_0^2\right)\rho \right\rangle \\ &+ if\left\langle \eta^\dagger\rho\Psi_0 \right\rangle + if\left\langle \Psi_0^\dagger\rho\eta \right\rangle\ . && (5.119) \end{aligned}$$

Since we want to functionally integrate over η, $\eta^\dagger$ and ρ, we complete the squares, obtaining

$$S_E \simeq S_E\Big|_0 \quad + \left\langle \eta'^\dagger \left(\not\partial + im' + if\phi_0\right) \eta' \right\rangle$$
$$+ \frac{1}{2} \left\langle \rho \left[-\partial^2 + m^2 + \frac{\lambda}{2}\phi_0^2 + 2f^2 \Psi_0^\dagger \left(\not\partial + im' + if\phi_0\right)^{-1} \Psi_0 \right] \rho \right\rangle \ , \tag{5.120}$$

where

$$\eta' = \eta + if \left(\not\partial + im' + if\phi_0\right)^{-1} \rho \Psi_0 \ . \tag{5.121}$$

Moreover in this approximation

$$\mathcal{D}\Psi = \mathcal{D}\eta = \mathcal{D}\eta' \tag{5.122}$$
$$\mathcal{D}\phi = \mathcal{D}\rho \ , \tag{5.123}$$

since the Jacobian of the transformation is one. This allows us to functionally integrate (5.4.10) using the formulae of Appendix A and of Section 5.1. The result is

$$W_E \simeq e^{-S_E|_0} \det \left(\not\partial + im' + if\phi_0\right) \tag{5.124}$$
$$\times \left[\det \left(-\partial^2 + m^2 + \frac{\lambda}{2}\phi_0^2 + 2f\Psi_0^\dagger (\not\partial + im' + if\phi_0)^{-1} \Psi_0 \right) \right]^{-1/2} \tag{5.125}$$

where in the second determinant the inverse operator acts on and through Ψ_0.

As in the saddle point approximation for scalar field theory, $S_E|_0$ generates all the tree diagrams when viewed as a functional of the sources J, ζ and $\zeta^\dagger$, while the determinants give the one loop contributions which are of first order in $\hbar$.

Let us perform a functional Legendre transformation between the sources J, ζ and $\zeta^\dagger$ and the new classical sources

$$\phi_{\rm cl}(x) = -\frac{\delta Z_E}{\delta J(x)} \simeq -\frac{\delta S_E}{\delta J(x)} + \mathcal{O}(\hbar) \tag{5.126}$$
$$\Psi_{\rm cl}(x) = -\frac{\delta Z_E}{\delta \zeta^\dagger(x)} \simeq -\frac{\delta S_E}{\delta \zeta^\dagger(x)} + \mathcal{O}(\hbar) \ , \tag{5.127}$$

and introduce the effective Action

$$\Gamma_E\left[\phi_{\rm cl},\Psi_{\rm cl},\Psi^\dagger_{\rm cl}\right]\equiv Z_E\left[J,\zeta,\zeta^\dagger\right]-\langle J\phi_{\rm cl}\rangle-i\left\langle\zeta^\dagger\Psi_{\rm cl}\right\rangle-i\left\langle\Psi^\dagger_{\rm cl}\zeta\right\rangle\ , \quad (5.128)$$

which generates the one particle irreducible Green's functions. In the classical approximation, it is nothing but the classical Action with the classical sources of (5.4.15) and (5.4.16) playing the role of the fields:

$$\Gamma_E=\left\langle\Psi^\dagger_{\rm cl}\left(\not\partial+im+if\phi_{\rm cl}\right)\Psi_{\rm cl}\right\rangle+\frac{1}{2}\left\langle\phi_{\rm cl}\left(-\partial^2+m^2+\frac{\lambda}{12}\phi^2_{\rm cl}\right)\phi_{\rm cl}\right\rangle+\mathcal{O}(\hbar) \quad (5.129)$$

The first quantum mechanical corrections to (5.4.18) are given by the determinants of (5.4.14). We will evaluate some of their properties by using the ζ-function technique of Chapter III. These determinants are more complicated because of the spinor indices and of the inverse operator appearing in the second determinant of (5.4.14).

Let us specialize to *constant* field configurations, and neglect all masses. Then

$$\frac{1}{\not\partial+if\phi_0}=\left(\not\partial-if\phi_0\right)\frac{1}{\left[-\partial^2+f^2\phi_0^2\right]}\ , \quad (5.130)$$

so that the argument of the scalar field determinant becomes

$$\left[\left(-\partial^2+\frac{\lambda}{2}\phi_0^2\right)\left(-\partial^2+f^2\phi_0^2\right)+2f^2\Psi_0^\dagger\left(\not\partial-if\phi_0\right)\Psi_0\right]\frac{1}{\left(-\partial^2+f^2\phi_0^2\right)}\ . \quad (5.131)$$

In addition, for constant ϕ_0, the fermion determinant becomes (see problem)

$$\det\left(\not\partial+if\phi_0\right)=\left[\det\left(-\partial^2+f^2\phi_0^2\right)\right]^2\ . \quad (5.132)$$

Using the property of determinants on (5.4.20), we can rewrite the generating functional as

$$W_E\simeq e^{-S_E|_0}\qquad\left[\det\left(-\partial^2+f^2\phi_0^2\right)\right]^{5/2}\ \times \quad (5.133)$$

$$\left[\det\left\{\left(-\partial^2+\frac{\lambda}{2}\phi_0^2\right)\left(-\partial^2+f^2\phi_0^2\right)+2f^2\Psi_0^\dagger\left(\not\partial-if\phi_0\right)\Psi_0\right\}\right]^{-1/2} \quad (5.134)$$

which is valid only for constant ϕ_0 and Ψ_0. To simplify matters further, let us assume that Ψ_0 is chiral, *i.e.*,

$$Psi_0^\dagger \gamma_\mu \Psi_0 = 0 \qquad (\text{or } \psi_R = 0) \ ; \qquad \Psi^\dagger \Psi \neq 0 \ . \tag{5.135}$$

Then the argument of the second determinant can be rewritten in the form

$$\left(-\partial^2 + A\right)\left(-\partial^2 + B\right) \ , \tag{5.136}$$

where A and B are constants involving ϕ_0 and $\Psi_0^\dagger \Psi_0$ which satisfy

$$A + B = \left(f^2 + \frac{\lambda}{2}\right)\phi_0^2 \tag{5.137}$$

$$AB = \tfrac{1}{2}\lambda f^2 \phi_0^4 - 2if^3 \Psi_0^\dagger \phi_0 \Psi_0 \ . \tag{5.138}$$

We have thus reduced the problem to evaluating determinants of the form $\det(-\partial^2 + C)$ where C is a constant. Summarizing the results of Chapter 3, we know that

$$\det\left(-\partial^2 + C\right) = \exp\left\{-\zeta'_{[-\partial^2+C]}[0]\right\} \ , \tag{5.139}$$

with

$$\zeta_{[-\partial^2+C]}[s] = \frac{\mu^4}{16\pi^2}\left(\frac{C}{\mu^2}\right)^{2-s}\frac{\Gamma(s-2)}{\Gamma(s)}\int d^4x \tag{5.140}$$

$$\zeta'_{[-\partial^2+C]}[0] = -\frac{1}{32\pi^2}C^2\left(-\frac{3}{2} + \ln\frac{C}{\mu^2}\right) \ . \tag{5.141}$$

Thus, armed, it is easy to read off the one loop contributions to the effective potential [see problem]. Here we merely quote the result when $\Psi_0 = 0$:

$$V(\phi_0) = \frac{\lambda^2}{256\pi^2}\phi_0^4\left[-\frac{3}{2} + \ln\frac{\lambda\phi_0^2}{2\mu^2}\right] - \frac{f^4}{8\pi^2}\phi_0^4\left[-\frac{3}{2} + \ln\frac{f^2\phi_0^2}{\mu^2}\right] \ , \tag{5.142}$$

where the first term is the same as in the pure scalar case – it comes from the boson loops; the second term comes from the contribution of the closed fermion loops to the potential, and the relative minus sign comes from the closed fermion loop.

The scaling properties of these determinants are equally straightforward to evaluate. Recall from Chapter 3 that under a scale change

$$\det\left[e^{-2a}\left(-\partial^2 + C\right)\right] = e^{-2a\zeta_{[-\partial^2+C]}[0]}\det\left(-\partial^2 + C\right) \ ; \tag{5.143}$$

with ζ given by (5.4.28), so that

$$\det{}^n\left[e^{-2a}\left(-\partial^2+C\right)\right] = e^{-2na\int d^4x} \quad \frac{C^2}{8\cdot 16\pi^2}\det{}^n\left(-\partial^2+C\right) \ , \quad (5.144)$$

where we have treated the constant C as if is changed under a scale transformation with the same dimension as $-\partial^2$. For a more rigorous treatment, see Chapter 3, Section 6. Thus, the one loop scaling correction is

$$\Gamma_E \to \Gamma_E + \frac{\hbar a}{128\pi^2}\int d^4\bar{x}\left\{-2\left(\frac{5}{2}\right)\left(f^2\phi^2\right)^2 + (-2)\left(-\frac{1}{2}\right)\left[A^2+B^2\right]\right\} \ . \quad (5.145)$$

Rewriting

$$\begin{aligned} A^2+B^2 &= (A+B)^2 - 2AB & (5.146)\\ &= \left(f^2+\frac{\lambda}{2}\right)^2\phi_0^4 - \lambda f^2\phi_0^4 + 4if^3\Psi_0^\dagger\phi_0\Psi_0 \ , & (5.147)\end{aligned}$$

we see that the effect of a scale change on Γ_E is to generate terms of the same type as in the classical Lagrangian, which produces a change in the dimensionless coupling constants

$$\begin{aligned} \frac{\lambda}{4!}\to\frac{\lambda'}{4!} &= \frac{\lambda}{4!} - \frac{\hbar a}{128\pi^2}\left(\frac{\lambda^2}{4}-5f^4\right) \ , & (5.148)\\ f\to f' &= f - \frac{\hbar a}{32\pi^2}f^2 \ . & (5.149)\end{aligned}$$

This provides an example of the scale dependence of a theory with several coupling constants. The new feature is that the scale changes form a coupled system. This coupling phenomenon is easy to understand from a diagrammatic point of view since the closed fermion loop obviously contributes to the ϕ^4 coupling, while to $\mathcal{O}(\hbar)$ the fermion coupling is affected only by the presence of the original fermion vertex:

1.251.25 1.251.25 (5.150)

in these the dashed (solid) line represents a scalar (spinor) field.

It must be noted that a scale change will also generate additions to the fermion and scalar kinetic terms, but in our approximation of constant fields these did not show up in the changes (5.4.35) and (5.4.36). For instance, the fermion determinant

$$\det(\not{\partial}+if\phi) \tag{5.151}$$

cannot be written in the form (5.4.21) unless ϕ is independent of x. In general, its scaling will involve a kinetic term; it corresponds to the fact that the fields themselves acquire at the one-loop level anomalous dimensions coming from the diagram

$$1.5.7 \tag{5.152}$$

In the pure ϕ^4 theory, the scalar field acquired an anomalous dimension only at the two-loop level, and therefore the effect did not show up in the scaling of the determinant. Thus, Eq. (5.4.35) and (5.4.36) have to be corrected for wave function renormalization.

5.4.1 PROBLEMS

A. Show that $\Gamma_E\left[\phi_{\rm cl},\, \Psi_{\rm cl},\, \Psi^\dagger_{\rm cl}\right]$ is the classical Action when terms of $\mathcal{O}(\hbar)$ are neglected.

B. Show that in four-dimensions

$$\det(\not{\partial}+im) = \left[\det\left(-\partial^2+m^2\right)\right]^2 \ . \tag{5.153}$$

C. Find the one-loop contribution to the potential including the fermion contribution.

*D. Using diagrams, derive the expression for the scale dependence of λ and f (including wave function renormalization) at the one loop level, and compare with (5.4.35) and (5.4.36).