6

Gauge Symmetries: Yang–Mills and Gravity

6.1 Global and Local Symmetries

In the first chapter we gave examples of Lagrangians involving fields of spin 0 and 1/2 but refrained from presenting any theory involving higher spin fields in interaction. The reason for this omission stems from the fact that fields of spin 1, 3/2 and 2 can be introduced in a very beautiful way just be requiring that whatever symmetries present in the spin 1/2 – 0 system be generalized to vary arbitrarily from point to point in spacetime. Spin 1 fields correspond to generalizing internal (*i.e.*, non-Lorentz) symmetries; a spin 2 field occurs when spacetime symmetries (global Poincaré invariance) are made local in spacetime; spin 3/2 and 2 fields appear when globally supersymmetric theories are generalized to be locally supersymmetric in spacetime. We will defer till later the local generalization of spacetime symmetries, and start by building theories which are locally invariant under internal symmetries, following Yang and Mills, *Phys. Rev.* **96**, 191 (1954).

Maxwell's electrodynamics provides the earliest example of a theory with a local symmetry. It was E. Noether who first realized the generality of the concept of "gauging" (*i.e.*, "making local") symmetries [in *Nachr. Kgl. Ges. Wiss.*, Göttingen 235 (1918)]. The gauging procedure in its modern form was formulated by H. Weyl in the 1920's.

Consider the simplest possible Lagrangian involving a spinor field

$$\mathcal{L}_0 = \frac{1}{2}\psi_L^\dagger \sigma \cdot \overleftrightarrow{\partial}\psi_L = \psi_L^\dagger \sigma^\mu \partial_\mu \psi_L + \text{surface term}\ , \tag{6.1}$$

which we know to be invariant under the phase transformation

$$\psi_L(x) \to e^{i\alpha}\psi_L(x)\ , \tag{6.2}$$

where α is a constant. The basic idea behind "gauging" this phase symmetry is to make our Lagrangian invariant under phase transformations just like (6.1.2) with α depending arbitrarily on x_μ, *i.e.*,

$$\psi_L(x) \to e^{i\alpha(x)}\psi_L(x) \ . \tag{6.3}$$

The Lagrangian $\mathcal{L}_0$ is no longer invariant under this local phase transformation due to the presence of the derivative operator ∂_μ; indeed under (6.1.3) we have

$$\partial_\mu\psi_L(x) \to \partial_\mu\, e^{i\alpha(x)}\psi_L(x) = i\ e^{i\alpha(x)}\left[\partial_\mu + i\partial_\mu\alpha(x)\right]\psi_L(x) \ , \tag{6.4}$$

so that

$$\mathcal{L}_0 \to \mathcal{L}_0 + i\psi_L^\dagger\sigma^\mu\psi_L\partial_\mu\alpha(x) \ . \tag{6.5}$$

The trick behind building an invariant $\mathcal{L}$ is the invention of a new operator which generalizes ∂_μ, call it D_μ, with the property that, under a local phase transformation,

$$D_\mu\psi_L \to e^{i\alpha(x)}D_\mu\psi_L \ , \tag{6.6}$$

or in operator language

$$D_\mu \to e^{i\alpha(x)}D_\mu\, e^{-i\alpha(x)} \ . \tag{6.7}$$

This new derivative operator is called the *covariant derivative.* Then it trivially follows that the new Lagrangian

$$\mathcal{L} \equiv \psi_L^\dagger\sigma^\mu D_\mu\psi_L \ , \tag{6.8}$$

is invariant under (6.1.3). This is all fine but we have to build this covariant derivative. We look for an expression of the form

$$D_\mu = \partial_\mu + iA_\mu(x) \ , \tag{6.9}$$

where $A_\mu(x)$ is a function of x. Then the covariance requirement

$$D_\mu \to D'_\mu = \partial_\mu + iA'_\mu(x) = e^{i\alpha(x)}\left(\partial_\mu + iA_\mu(x)\right)\, e^{-i\alpha(x)} \ , \tag{6.10}$$

becomes a transformation property of A_μ

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu\alpha(x) \ . \tag{6.11}$$

The new Lagrangian

$$\mathcal{L} = \psi_L^\dagger \sigma^\mu \left(\partial_\mu + iA_\mu(x)\right) \psi_L = \mathcal{L}_0 + i\psi_L^\dagger \sigma^\mu \psi_L A_\mu(x) \tag{6.12}$$

is then invariant under the simultaneous local transformations

$$\psi_L(x) \quad \to e^{i\alpha(x)}\psi_L \tag{6.13}$$

$$A_\mu(x) \quad \to \quad A_\mu(x) - \partial_\mu \alpha(x) \ . \tag{6.14}$$

The global symmetry of $\mathcal{L}_0$ is generalized to a local symmetry or gauged at the price of introducing a new vector field $A_\mu(x)$ which interacts with the conserved current. We can see that the new field $A_\mu(x)$ has the same dimensions as ∂_μ; it can therefore be identified with a canonical field in four dimensions [in other number of dimensions, one has to understand $A_\mu(x)$ as being multiplied by a dimensionful coupling before it can be thus identified]. Furthermore, $A_\mu(x)$ is real since $i\partial_\mu$ is Hermitean.

It is therefore easy to write a kinetic term for the field $A_\mu(x)$ in such a way that preserves the gauge invariance (6.1.13), by noting that the combination

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6.15}$$

is invariant. It has dimension -2 and therefore we can build out of it a new Lagrangian

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \ , \tag{6.16}$$

where we have introduced a dimensionless constant g which can be absorbed by letting $A_\mu = gA'_\mu$, so that in terms of A'_μ it appears in the coupling between A'_μ and the current $\psi_L^\dagger \sigma^\mu \psi_L$ in (6.1.12). The factor of $-1/4$ corresponds to the conventional definition of g. Of course, as you might have guessed (6.1.15) is the Maxwell Lagrangian. We now have a fully interacting theory of spin 1 and spin 1/2 fields, described by

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \psi_L^\dagger \sigma^\mu \left(\partial_\mu + iA_\mu(x)\right) \psi_L \ . \tag{6.17}$$

Although pretty, this theory is not renormalizable (as we shall see later) because of a tricky complication appropriately called the (Adler-Bell-Jackiw) anomaly, which has to do with the left-handed nature of ψ_L. It causes no problems if we couple A_μ gauge invariantly to a Dirac four component field, leading to

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}\gamma^{\mu}\left(\partial_{\mu} + iA_{\mu}(x)\right)\Psi \tag{6.18}$$

which describes QED when $A_\mu(x)$ is identified with the photon, Ψ with the electron and e with the electric charge. $\mathcal{L}_{\text{QED}}$ is invariant under the local symmetry

$$\Psi(x) \quad\rightarrow\quad e^{i\alpha(x)}\Psi(x) \tag{6.19}$$
$$A_\mu(x) \quad\rightarrow\quad A_\mu(x) - \partial_\mu\alpha(x)\ , \tag{6.20}$$

and in the absence of a mass term for Ψ under the global chiral transformation

$$\Psi(x) \rightarrow e^{i\beta\gamma_5}\Psi(x)\ . \tag{6.21}$$

[This chiral symmetry is not exact (the anomaly again) in quantum field theory even in the absence of the electron mass but it does not cause any problem since no gauge field is coupled to it.] Gauge invariance forbids any mass term for A_μ.

Before generalizing this construction to more complicated symmetries, let us review the different types of global symmetries.

The Lagrangian for n real scalar fields $\phi_1, \cdots, \phi_n$

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n} \partial_\mu\phi_i\partial^\mu\phi_i = \frac{1}{2}\partial_\mu\Phi^T\partial^\mu\Phi\ , \tag{6.22}$$

is invariant under global rotations in n dimension, $O(n)$ under which the n-dimensional column vector Φ changes as

$$\Phi \rightarrow \Phi' = \mathbf{R}\Phi\ , \tag{6.23}$$

where $\mathbf{R}$ is a rotation matrix (proper and improper). Since $\Phi^T\Phi$ (the length of Φ) is $O(n)$ invariant, the matrix $\mathbf{R}$ obeys

$$\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = 1\ . \tag{6.24}$$

Proper rotation matrices can be written in the form

$$\mathbf{R} = e^{\frac{i}{2}\omega^{ij}\Sigma_{ij}}\ , \tag{6.25}$$

where $\omega^{ij} = -\omega^{ji}$ are the real $\frac{n(n-1)}{2}$ parameters of the rotation group, and the Σ_{ij} are the $\frac{n(n-1)}{2}$ generators of the rotation group. By considering the infinitesimal change in Φ and by requiring that the group properties be satisfied

$$\delta\Phi = \frac{i}{2}\omega^{ij}\Sigma_{ij}\Phi \ , \tag{6.26}$$

one can prove that the Σ_{ij} satisfy a Lie algebra

$$[\Sigma_{ij}, \Sigma_{k\ell}] = i\delta_{ik}\Sigma_{j\ell} + i\delta_{j\ell}\Sigma_{ik} - i\delta_{i\ell}\Sigma_{jk} - i\delta_{jk}\Sigma_{i\ell} \ . \tag{6.27}$$

In the above we have derived the Lie algebra for $SO(n)$ by using the $n \times n$ Σ_{ij} matrices which act on the n-dimensional vector Φ. It is easy to see from (6.1.21) and (6.1.22) that they are real and antisymmetric. However, one can build many kinds of matrices which satisfy (6.1.24). This is because there are many kinds of ways of representing $SO(n)$. We have chosen to do it in the n-dimensional representation but we could as well have described it in the adjoint representation which has the same number of dimensions as there are parameters in the group. In the case of $SO(n)$ it can be represented by an antisymmetric second rank tensor $A_{ij} = -A_{ji}$. For the adjoint, it is convenient to treat the $A_{ij}(x)$ as matrix elements of an antisymmetric matrix $\mathbf{A}(x)$. Then the rotations take the form

$$\mathbf{A} \to \mathbf{A}' = \mathbf{R}\mathbf{A}\mathbf{R}^T \quad ; \qquad \mathbf{A}^T = -\mathbf{A} \ , \tag{6.28}$$

where $\mathbf{R}$ is the $n \times n$ matrix given by (6.1.22). Then it is easy to build an invariant Lagrangian with $\mathbf{A}$ as scalar fields

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}\left(\partial_\mu \mathbf{A}^T \partial^\mu \mathbf{A}\right) \ . \tag{6.29}$$

The symmetric "quadrupole" representation $S_{ij} = +S_{ji}$ can be handled in a similar way when the trace of $\mathbf{S}$ is recognized to be $SO(n)$ invariant. Starting from the $\mathbf{n}$ representation of $SO(n)$, one can construct more complicated representations described by higher rank tensors. An arbitrarily high rank tensor is in general a combination of tensors that transform irreducibly (among themselves) under the group. For instance, consider a third rank tensor T_{ijk} and take i, j, $k = 1, \cdots, 10$ for convenience. It is decomposed into irreducible representations as follows: • the totally antisymmetric part $T_{[ijk]}$ with $\frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3} = 120$ components. • the totally symmetric part $T_{(ijk)}$ has $\frac{10\cdot 11\cdot 12}{1\cdot 2\cdot 3} = 220$ components, but it contains, by contracting two indices, a vector T_{iij} with 10 dimensions. Hence $T_{(ijk)}$ is decomposed as an irreducible 210 dimensional representation plus a 10 dimensional representation of SO_{10}. •

tensors with mixed symmetry among the indices: antisymmetric under the interchange of two indices with 320 components ($= 45 \times 10 - 10 - 120$), and symmetric in two indices only, with $320 + 10$ components.

Thus in summary we have obtained the $SO(10)$ decomposition of the third rank reducible tensor with 1,000 components into its irreducible parts

$$1000 = 120 + 220 + 10 + 10 + 320 + 320 \ . \tag{6.30}$$

This type of construction is straightforward (if tedious); the only subtlety occurs when n is even in which case one can use the Levi-Civita tensor $\epsilon_{ij\cdots k}$ with n entries to split the $n/2$-rank totally antisymmetric tensor in half. Furthermore, when $n/2$ is even, this results in splitting the $n/2$-rank anti-symmetrized tensor into two real and inequivalent representations. When $n/2$ is odd the procedure results in two representations which are the conjugate of one another. For instance, in $SO(10)$, the fifth rank totally anti-symmetrized tensor has 252 components, which split into the 126-dimensional representation and its conjugate, the ϵ-symbol acting as the conjugate.

The procedure of taking tensor products of vectors does not generate all representations because $SO(n)$ has in addition spinor representations (*e.g.*, $SO(3)$ has half-integer spin representation). When $n = 2m + 1$, $m = 1, 2, \cdots, SO(n)$ has only one real fundamental spinor representation of dimension 2^m, *e.g.*, $SO(3)$ has a two-dimensional real representation, $SO(5)$ a real four-dimensional spinor representation, etc., out of which all representation can be built. When $n = 2m$, $m = 2, 4, 6$, $SO(n)$ has two real and inequivalent fundamental spinor representations each with 2^{m-1} dimensions. Finally, for $n = 2m$, $m = 3, 5, \cdots$, $SO(n)$ has two fundamental complex spinor representations conjugate to one another. For instance $SO(6)$ has a $\mathbf{4}$ and a $\overline{\mathbf{4}}$ conjugate to one another, etc. All representations can be constructed from these spinor representations, which means that they are in this sense more fundamental than the vector representation.

Consider now the kinetic term for n two component spinor fields

$$\mathcal{L}_F = \frac{1}{2}\psi_L^{\dagger a}\sigma^\mu \overleftrightarrow{\partial_\mu}\psi_{La} \ , \tag{6.31}$$

where a runs from 1 to n and sum over a is implied. For $a = 1$ we have seen that (6.1.27) is invariant under a phase transformation. When $a > 1$, $\mathcal{L}$ is invariant under a much larger symmetry: consider the change (suppressing the a index)

$$\psi_L \to \mathbf{U}\psi_L \ , \tag{6.32}$$

where $\mathbf{U}$ is an $n \times n$ matrix; and then

$$\psi_L^\dagger \to \psi_L^\dagger \mathbf{U}^\dagger \ . \tag{6.33}$$

Clearly, if $\mathbf{U}$ is x-independent and unitary

$$\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger \mathbf{U} = 1 \ . \tag{6.34}$$

$\mathcal{L}_F$ is invariant under the transformation (6.1.28). The unitarity condition implies that $\mathbf{U}$ can be expressed in terms of a Hermitean $n \times n$ matrix

$$\mathbf{U} = e^{i\mathbf{H}} \quad ; \qquad \mathbf{H} = \mathbf{H}^\dagger \ . \tag{6.35}$$

This Hermitean matrix depends on n^2 real parameters. Note that by taking $\mathbf{H}$ proportional to the identity matrix we recover the earlier phase invariance. The extra new transformations are then generated by the traceless part of $\mathbf{H}$, expressed in terms of $n^2 - 1$ parameters by

$$\mathbf{H} = \sum_{A=1}^{n^2-1} \omega^A \mathbf{T}^A \ , \qquad \mathbf{T}^{A\dagger} = \mathbf{T}^A \ , \tag{6.36}$$

where the ω^A are real parameters and the $\mathbf{T}^A$ are Hermitean traceless $n \times n$ matrices. They generate $SU(n)$, the unitary group in n dimensions, and satisfy the appropriate Lie algebra

$$\left[\mathbf{T}^A, \mathbf{T}^B\right] = i f^{ABC} \mathbf{T}^C \ , \tag{6.37}$$

where f^{ABC} are real totally antisymmetric coefficients called the structure constants of the algebra [this relation is similar to (6.1.24), but has different f's]. Some celebrated examples are

$$\begin{aligned} &n = 2: \quad \mathbf{T}^A = \tfrac{1}{2}\sigma^A \ , \ \sigma^A: \ \text{Pauli spin matrices} \quad A = 1, 2, 3 \\ &n = 3: \quad \mathbf{T}^A = \tfrac{1}{2}\lambda^A \ , \ \lambda^A: \ \text{Gell–Mann matrices} \quad A = 1, \cdots, 8 \ . \end{aligned} \tag{6.38}$$

Both Pauli and Gell-Mann matrices satisfy the normalization condition

$$\mathrm{Tr}\left(\sigma^A \sigma^B\right) = \mathrm{Tr}\left(\lambda^A \lambda^B\right) = 2\delta^{AB} \ , \tag{6.39}$$

and the latter are given by

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \qquad (6.40)
\end{aligned}$$

Under SU_n, then, we have the following fundamental representations

$$\psi \sim \mathbf{n} \text{ means that } \delta\psi = i\omega^A \mathbf{T}^A \psi \qquad (6.41)$$

$$\chi \sim \bar{\mathbf{n}} \text{ means that } \delta\chi = -i\omega^A \mathbf{T}^{A*} \chi\ , \qquad (6.42)$$

where the last transformation property is obtained by requiring that $\chi^T\psi$ be invariant. The tensor structure of $SU(n)$ is simpler than that of $SO(n)$: associate a lower (upper) index a to a quantity transforming as the $\mathbf{n}(\bar{\mathbf{n}})$ representation of $SU(n)$, *i.e.*, $\psi_a \sim \mathbf{n}$, $\psi^a \sim \bar{\mathbf{n}}$. Starting from these as building blocks we can generate all the representations of $SU(n)$ by taking tensor products. One representation of interest is the adjoint representation M^a_b where $\mathbf{M}$ is traceless Hermitean and contains n^2-1 components; as its indices indicate it is constructed out of the product of $\mathbf{n}$ and $\bar{\mathbf{n}}$: $\mathbf{n} \otimes \bar{\mathbf{n}} = (\mathbf{n^2} - \mathbf{1}) \oplus \mathbf{1}$. It is convenient to express it as a matrix $\mathbf{M}$ which then transforms as

$$\mathbf{M} \to \mathbf{U}\mathbf{M}\mathbf{U}^\dagger\ , \qquad (6.43)$$

or

$$\delta\mathbf{M} = i\omega^A[\mathbf{T}^A, \mathbf{M}]\ . \qquad (6.44)$$

Alternatively, we could have represented the adjoint representation by an n^2-1 dimensional real vector in which case the representation matrices $\mathbf{T}^A$ would have been $(n^2-1)\times(n^2-1)$ dimensional.

Other types of representations can be built as tensors with arbitrary numbers of upper and lower indices. Upper and lower indices can be contracted to make singlets but not lower or upper indices among themselves. Thus for instance T_{ab} can be broken down to its irreducible components just by the symmetry scheme of the ab indices: symmetric and antisymmetric. For example, take $SU(5)$:

$$
\begin{aligned}
T_{ab} &= T_{(ab)} + T_{[ab]} & (6.45)\\
\mathbf{5} \otimes \mathbf{5} &= \mathbf{15} \oplus \mathbf{10}\ . & (6.46)
\end{aligned}
$$

Here $(\cdots)$ means total symmetry, $[\cdots]$ antisymmetry.

We have seen that by considering the kinetic terms of fermion and scalar fields, we can generate Lagrangians invariant under unitary and orthogonal transformations. It is also possible to generate symplectic group invariance by means of a kinetic term. We note that for a Grassmann Majorana field the possible candidate for a kinetic term

$$
\frac{1}{4}\bar{\Psi}_M \gamma_5 \gamma^\mu \overleftrightarrow{\partial_\mu} \Psi_M\ , \tag{6.47}
$$

is identically zero when there is only one Majorana field. However, consider the case of an even number of Majorana fields, Ψ_{Mi} $i = 1 \cdots 2n$, coupled by means of an antisymmetric numerical matrix $E_{ij} = -E_{ji}$ with entries

$$
E_{ij} = \begin{cases} +1 & i > j \\ 0 & i = j \\ -1 & i < j \end{cases}\ . \tag{6.48}
$$

In this case we can form invariant expressions of the form (6.1.40) and obtain a non-zero result provided we antisymmetrize in the running i indices. In this way we arrive at the non-vanishing kinetic term

$$
\mathcal{L} = \frac{1}{4}\bar{\Psi}_{Mi} E_{ij} \gamma_5 \gamma^\mu \overleftarrow{\partial_\mu} \Psi_{Mj}\ . \tag{6.49}
$$

It is invariant under transformations that leave an antisymmetric quadratic expression invariant. These transformations form the symplectic group $Sp(2n)$, with the Ψ_{Mi} transforming as the real $2n$ dimensional defining representation. As the presence of the antisymmetric tensor E_{ij} indicates, the $Sp(2n)$ singlet resides in the antisymmetric product of two $\mathbf{2n}$ representations. [As a consequence, one cannot form a kinetic term for scalar fields transforming as the $\mathbf{2n}$ of $Sp(2n)$.] In fact, for symplectic groups irreducible representations appear in the symmetric tensor product, and reducible representations in the antisymmetrized product [exactly the opposite of rotation groups]. In particular, the adjoint representation is given by $(\mathbf{2n}\otimes\mathbf{2n})$sym. and therefore contains $n(2n+1)$ elements. A case of interest arises when $n=1$. Then the Lagrangian (6.1.42) is seen to be invariant under $SU(2)$ because the E-matrix can be identified with the Levi-Cività symbol ϵ_{ij}. This is no accident: the Lie algebras of $Sp(2)$, $SU(2)$ and $SO(3)$ are the same. In fact, we note that $Sp(2n)$ has the same number of dimensions as $SO(2n+1)$. By matching representations, we see that we have another identification

$$SO(5) \sim Sp(4) \; , \tag{6.50}$$

since the $\mathbf{4}$ of $Sp(4)$ has the same dimensions as the spinor of $SO(5)$. However, $SO(7)$ is not the same as $Sp(6)$ since $SO(7)$ has no six-dimensional representation. [Moreover, even when the Lie algebras match, their global properties may be different.] In conclusion we note that by building different

types of kinetic terms, we have been able to generate Lagrangians invariant under $O(n)$, $U(n)$ and $Sp(2n)$. Other Lie groups called exceptional groups do not appear in our list because they are not defined in terms of quadratic invariants only, such as kinetic and mass terms; their specification appears at the level of higher order invariants. These can appear in the interaction part of the Lagrangian. So exceptional symmetries are interaction symmetries. Here we do not give examples of Lagrangians invariant under exceptional groups but merely list them: G_2 with 14 generators, rank 2 and only real representations generated by the fundamental seven-dimensional representation; F_4 with 52 generators, rank 4, only real representations generated by the fundamental 26-dimensional representation; E_6 with 78 generators, rank 6, and real and complex representations generated by the fundamental $\mathbf{27}$ or $\overline{\mathbf{27}}$ representation; E_7 with 133 generators, rank 7, only real representations generated by the $\mathbf{56}$; and finally, E_8 with 248 generators has the unique feature of having the adjoint 248-dimensional representation as its fundamental.

6.1.1 PROBLEMS

A. Given a Lie algebra

$$[\mathbf{T}^A, \mathbf{T}^B] = if^{ABC}\mathbf{T}^C \qquad A,\, B,\, C = 1, \cdots, K \tag{6.51}$$

where the f^{ABC} are totally antisymmetric real coefficients. The f^{ABC} can be regarded as K $(K\times K)$ matrices $\mathbf{f}^A = \left(f^A\right)^{BC}$. Using the Jacobi identity show that these $K \times K$ matrices obey the same Lie algebra as the $\mathbf{T}^A$ when a proper factor of -1 is provided.

B. Given a complex third rank tensor T_{abc} a, b, $c = 1, \cdots 5$, decompose it into its $SU(5)$ irreducible components.

C. For $SU(n)$, given $n-1$ different fields $\phi_a^1, \phi_a^2, \cdots \phi_a^{n-1}$, each transforming as the $\mathbf{n}$, show that it is always possible to build out of their product a field transforming as the $\bar{\mathbf{n}}$ representation (*i.e.*, with an upper index).

D. For $SU(3)$ express the product of two Gell–Mann matrices in terms of the Gell–Mann matrices.

E. Show that the Lie algebras of $SU(2)$ and $SO(3)$, $SU(4)$ and $SO(6)$ are isomorphic.

F. Given $\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^T \partial^\mu \Phi$ where Φ is a column vector with n real scalar fields. Find the Noether currents and charges. What conditions must be imposed on the fields if the Noether charges are to satisfy the Lie algebra of $SO(n)$?

G. Show explicitly that the Lagrangian (6.1.42) is $SU(2)$ invariant when $n = 2$.

6.2 Construction of Locally Symmetric Lagrangians

In the previous section we have shown how to build *local* phase invariance into a Lagrangian. Now we show how to do the same for the more complicated non-Abelian Lie symmetries we have just discussed.

In the following we will use the Lagrangian for N complex two component spinor fields to illustrate the construction. However, the reader must be cautioned that the "gauging" of unitary symmetries with left-handed fields results in a nonrenormalizable theory because of the Adler-Bell-Jackiw anomaly. Since we concern ourselves at this stage only with classical considerations, we temporarily ignore this subtlety. The uneasy reader can carry out the same construction with n Dirac four component spinors if he or she wishes.

As we just saw, the Lagrangian

$$\mathcal{L} = \psi_L^{\dagger a} \sigma^\mu \partial_\mu \psi_{La} \tag{6.52}$$

where a is summed from 1 to n, is invariant under global $U(n)$ transformations; suppressing the $SU(n)$ indices, they are given by

$$\psi_L(x) \to \mathbf{U}\psi_L(x) \quad ; \qquad \mathbf{U}^\dagger \mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = 1 \tag{6.53}$$

with

$$\mathbf{U} = e^{i\alpha}\, e^{i\omega^A \mathbf{T}^A} \ , \tag{6.54}$$

in which the $\mathbf{T}^A$ are the $n^2 - 1$ traceless Hermitean matrices generating $SU(n)$. We now want to extend (6.2.1) to incorporate invariance under local transformations of the form (6.2.2), *i.e.*,

$$\psi_L(x) \to \mathbf{U}(x)\psi_L(x) \tag{6.55}$$

where now

$$\mathbf{U} = e^{i\alpha(x)}\, e^{i\omega^A(x)\mathbf{T}^A} \ . \tag{6.56}$$

Note that it is pretty much a matter of choice how much of the global symmetry one wants to gauge. For instance, we could have limited ourselves only to gauging any subgroup of $SU(n)$. Here we gauge the whole thing! When $\mathbf{U}$ depends on x, the derivative term $\partial_\mu \psi_L$ no longer transforms as it should: indeed

$$\begin{aligned} \partial_\mu \psi_L(x) \to \partial_\mu \mathbf{U}(x)\psi_L(x) \ &= \ \left[\partial_\mu \mathbf{U}(x)\right] \psi_L(x) + \mathbf{U}(x)\partial_\mu \psi_L(x) && (6.57) \\ &\not= \mathbf{U}\partial_\mu \psi_L(x) && (6.58) \end{aligned}$$

So we look for a generalization of the derivative which does not spoil the invariance of $\mathcal{L}$. We define accordingly the *covariant derivative* $\mathbf{D}_\mu$ by demanding that

$$\mathbf{D}_\mu \psi_L(x) \to \mathbf{U}(x)\mathbf{D}_\mu \psi_L(x) \tag{6.59}$$

or in operator form

$$\mathbf{D}_\mu \to \mathbf{D}'_\mu = \mathbf{U}(x)\mathbf{D}_\mu \mathbf{U}^\dagger(x) \ . \tag{6.60}$$

We emphasize that $\mathbf{D}_\mu$ is, in this case, an $n \times n$ matrix so if we wanted to show all indices we would write (6.2.8) as

$$[\mathbf{D}_\mu \psi_L(x)]_a \to [\mathbf{U}(x)]_a^b \, (\mathbf{D}_\mu)_b^c \, \psi_{Lc}(c) \ . \tag{6.61}$$

Then if we can find such a $\mathbf{D}_\mu$, the new Lagrangian

$$\mathcal{L}' = \frac{1}{2}\psi_L^\dagger \sigma^\mu \mathbf{D}_\mu \psi_L \tag{6.62}$$

is locally invariant under $U(n)$. Since $\mathbf{D}_\mu$ is to generalize ∂_μ, let us try the *Ansatz*

$$\mathbf{D}_\mu = \partial_\mu \mathbf{1} + i\mathbf{A}_\mu(x). \tag{6.63}$$

$\mathbf{A}_\mu(x)$ is an $n \times n$ Hermitean matrix with vector elements since $i\partial_\mu$ is itself a Hermitean vector:

$$\mathbf{A}_\mu(x) = B_\mu(x)\mathbf{1} + A_\mu^C(x)\mathbf{T}^C \ , \tag{6.64}$$

the $\mathbf{T}^C$ being the $n^2 - 1$ Hermitean generators of $SU(n)$. The transformation requirement (6.2.9) implies that

$$\begin{aligned}
\partial_\mu \mathbf{1} + i\mathbf{A}'_\mu(x) &= \mathbf{U}(x)\left[\partial_\mu \mathbf{1} + i\mathbf{A}_\mu(x)\right]\mathbf{U}^\dagger(x) && (6.65)\\
&= \partial_\mu \mathbf{1} + \mathbf{U}(x)\left[\partial_\mu \mathbf{U}^\dagger(x)\right] + i\mathbf{U}(x)\mathbf{A}_\mu(x)\mathbf{U}^\dagger(x) \ , && (6.66)
\end{aligned}$$

or

$$\mathbf{A}'_\mu(x) = -i\mathbf{U}(x)\left[\partial_\mu \mathbf{U}^\dagger(x)\right] + \mathbf{U}(x)\mathbf{A}_\mu(x)\mathbf{U}^\dagger(x) \ . \tag{6.67}$$

It is easy to see that the fields $B_\mu(x)$ and $A_\mu^C(x)$ transform separately. Indeed taking the trace of (6.2.15), we find

$$B'_\mu(x) = -\frac{i}{n}\mathrm{Tr}\left(\mathbf{U}(x)\left[\partial_\mu \mathbf{U}^\dagger(x)\right]\right) + B_\mu(x) \ . \tag{6.68}$$

It can be shown (see problem) that

$$\mathrm{Tr}\left(\mathbf{U}(x)\left[\partial_\mu \mathbf{U}^\dagger(x)\right]\right) = -in\partial_\mu \alpha(x) \ , \tag{6.69}$$

leading to

$$B'_\mu(x) = -\partial_\mu \alpha(x) + B_\mu(x) \tag{6.70}$$

which is the transformation previously obtained. Now by multiplying (6.2.15) by $\mathbf{T}^C$ and taking the trace, we obtain the change in the n^2-1 fields A_μ^C using the trace properties

$$\mathrm{Tr}\mathbf{T}^A = 0 \qquad \mathrm{Tr}\left(\mathbf{T}^A\mathbf{T}^B\right) = \frac{1}{2}\delta^{AB} \quad , \tag{6.71}$$

where we have used a conventional normalization. It is perhaps easier to consider infinitesimal "gauge transformations" (6.2.15). Setting

$$\mathbf{U}(x) = 1 + i\omega^A\mathbf{T}^A + \cdots \ , \tag{6.72}$$

we arrive at

$$\delta\mathbf{A}_\mu(x) = \mathbf{A}'_\mu(x) - \mathbf{A}_\mu(x) = -\mathbf{T}^B\partial_\mu\omega^B(x) + i\omega^B(x)\left[\mathbf{T}^B, \mathbf{A}_\mu(x)\right] + \mathcal{O}(\omega^2) \ . \tag{6.73}$$

Multiply by $\mathbf{T}^C$ and take the trace, using (6.2.19) and (6.2.13) we find

$$\delta A_\mu^C(x) = -\partial_\mu\omega^C(x) + 2i\omega^B(x)\mathrm{Tr}\left(\left[\mathbf{T}^B, \mathbf{A}_\mu(x)\right]\mathbf{T}^C\right) + \mathcal{O}(\omega^2) \ . \tag{6.74}$$

The $\mathbf{T}^A$'s obey the Lie algebra of $SU(n)$

$$\left[\mathbf{T}^A, \mathbf{T}^B\right] = if^{ABC}\mathbf{T}^C \ , \tag{6.75}$$

from which we finally obtain

$$\delta A_\mu^C(x) = -\partial_\mu\omega^C(x) - \omega^B(x)A_\mu^D(x)f^{BDC} + \mathcal{O}(\omega^2) \ . \tag{6.76}$$

The remarkable thing about the gauge transformation (6.2.24) is that it is expressed in a way that does not depend on the representation of the fermion fields we started with.

The variation (6.2.21) can be rewritten very elegantly in terms of the covariant derivative: under an $SU(n)$ transformation,

$$\omega^A\mathbf{T}^A \equiv \omega \to \mathbf{U}\omega\mathbf{U}^\dagger \ . \tag{6.77}$$

Hence the covariant derivative acting on ω is given by (see problem)

$$\mathbf{D}_\mu\omega = \partial_\mu\omega + i\left[\mathbf{A}_\mu, \omega\right] \ . \tag{6.78}$$

Comparison with (6.2.21) yields

$$\delta\mathbf{A}_\mu(x) = -\mathbf{D}_\mu\omega \ , \tag{6.79}$$

which shows that even if $\mathbf{A}_\mu(x)$ does not transform under $SU(n)$ because of the $\mathbf{U}\partial_\mu\mathbf{U}^\dagger$ term, its infinitesimal change does since it can be expressed in terms of a covariant derivative.

So far we have enlarged our Lagrangian in order to have local $U(n)$ symmetry. The price has been the introduction of n^2 vector fields to build the covariant derivative. In order to give these new fields an existence of their own, we should include their kinetic terms in a way that hopefully does not break the original local symmetry. For the $B_\mu(x)$ field corresponding to the overall phase transformations, we just repeat the steps of the previous paragraph. So let us rather concentrate on the n^2-1 fields which come with local $SU(n)$ invariance. The trick in constructing a kinetic term invariant under (6.2.15) is in building things out of the covariant derivative $\mathbf{D}_\mu$.

Consider the Hermitean quantity

$$\mathbf{F}_{\mu\nu} \equiv -i\,[\mathbf{D}_\mu, \mathbf{D}_\nu] \quad . \tag{6.80}$$

It is assured to transform covariantly since $\mathbf{D}_\mu$ does, that is

$$\mathbf{F}_{\mu\nu}(x) \to \mathbf{U}(x)\mathbf{F}_{\mu\nu}(x)\mathbf{U}^\dagger(x) \ . \tag{6.81}$$

Using the expression for $\mathbf{D}_\mu$ in the fundamental representation (6.2.12) and omitting the B_μ field, we obtain

$$\begin{aligned} \mathbf{F}_{\mu\nu} &= -i\,[\partial_\mu \mathbf{1} + i\mathbf{A}_\mu, \partial_\nu \mathbf{1} + i\mathbf{A}_\nu] && (6.82)\\ &= \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + i\,[\mathbf{A}_\mu, \mathbf{A}_\nu] \ . && (6.83) \end{aligned}$$

Since $\mathbf{F}_{\mu\nu}(x)$ is a Hermitean $n \times n$ matrix, we can expand it

$$\mathbf{F}_{\mu\nu}(x) = F^B_{\mu\nu}\mathbf{T}^B \ , \tag{6.84}$$

with

$$F^B_{\mu\nu}(x) = \partial_\mu A^B_\nu(x) - \partial_\nu A^B_\mu(x) - f^{BCD}A^C_\mu(x)A^D_\nu(x) \ , \tag{6.85}$$

where we have used (6.2.13) without B_μ. These $\mathbf{F}_{\mu\nu}$'s are, of course, the Yang-Mills generalization of the field strengths of electromagnetism. They are not all independent for they obey the Bianchi identities

$$\mathbf{D}_\mu \mathbf{F}_{\rho\sigma} + \mathbf{D}_\rho \mathbf{F}_{\sigma\mu} + \mathbf{D}_\sigma \mathbf{F}_{\mu\rho} = 0 \ , \tag{6.86}$$

where the $\mathbf{D}_\mu$'s acting on the $\mathbf{F}_{\mu\nu}$'s are to be understood in the sense of (6.2.25) since the $\mathbf{F}_{\mu\nu}$'s transform as members of the adjoint of $SU(n)$. These identities are a direct consequence of the Jacobi identity for the covariant derivative

$$[\mathbf{D}_\mu, [\mathbf{D}_\rho, \mathbf{D}_\sigma]] + [\mathbf{D}_\rho, [\mathbf{D}_\sigma, \mathbf{D}_\mu]] + [\mathbf{D}_\sigma, [\mathbf{D}_\mu, \mathbf{D}_\rho]] = 0 \ . \tag{6.87}$$

These are just kinematic constraints which are trivially satisfied by the field strengths.

It is now easy to build an invariant kinetic term. It is given by

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2g^2}\mathrm{Tr}\left(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right) \ , \tag{6.88}$$

with the normalization (6.2.19) for the T-matrices; it generalizes the Maxwell Lagrangian, and can be seen to have the proper dimensions – g is a dimensionless coupling.

We remark that $\mathcal{L}_{\mathrm{YM}}$ does not depend on the representation of the fermions, and therefore stands on its own as a highly nontrivial theory. Furthermore, by taking the f^{ABC} structure functions to be those of the other Lie groups, we can obtain the corresponding Yang-Mills theories for these other Lie groups.

The discerning reader may have wondered why we did not consider the other invariant

$$I = \mathrm{Tr}\epsilon^{\mu\nu\rho\sigma}\mathbf{F}_{\mu\nu}\mathbf{F}_{\rho\sigma} \tag{6.89}$$

as a candidate for the kinetic term. After all it is Lorentz and gauge invariant and has the proper dimension. The answer is that it can be expressed as a pure divergence. To see this we write

$$\begin{aligned} I &= 4\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(\left[\partial_\mu\mathbf{A}_\nu + i\mathbf{A}_\mu\mathbf{A}_\nu\right]\left[\partial_\rho\mathbf{A}_\sigma + i\mathbf{A}_\rho\mathbf{A}_\sigma\right]\right) & (6.90)\\ &= 4\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left[\partial_\mu\mathbf{A}_\nu\partial_\rho\mathbf{A}_\sigma + 2i\mathbf{A}_\mu\mathbf{A}_\nu\partial_\rho\mathbf{A}_\sigma\right] \ , & (6.91)\end{aligned}$$

where we have eliminated the $\mathbf{AAAA}$ term using the cyclic property of the trace. Now

$$\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(\mathbf{A}_\mu\mathbf{A}_\nu\partial_\rho\mathbf{A}_\sigma\right) = \frac{1}{3}\partial_\rho\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(\mathbf{A}_\mu\mathbf{A}_\nu\mathbf{A}_\sigma\right) \ , \tag{6.92}$$

so that

$$I = 4\partial_\rho\left\{\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left[\mathbf{A}_\sigma\partial_\mu\mathbf{A}_\nu + \frac{2i}{3}\mathbf{A}_\sigma\mathbf{A}_\mu\mathbf{A}_\nu\right]\right\} \ , \tag{6.93}$$

using $\epsilon^{\mu\nu\rho\sigma}\partial_\rho\partial_\mu\mathbf{A}_\nu = 0$. Thus we arrive at

$$\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(\mathbf{F}_{\mu\nu}\mathbf{F}_{\rho\sigma}\right) = 4\partial_\rho W^\rho \tag{6.94}$$

with

$$W^\rho = \epsilon^{\rho\sigma\mu\nu}\mathrm{Tr}\left[\mathbf{A}_\sigma\partial_\mu\mathbf{A}_\nu + \frac{2i}{3}\mathbf{A}_\sigma\mathbf{A}_\mu\mathbf{A}_\nu\right] \ . \tag{6.95}$$

It means that by taking I as the kinetic Lagrangian, we could not generate any equation of motion for the vector potential since it would only affect the Action at its end points. We can, however, add it to $\mathcal{L}_{\rm YM}$, resulting in a canonical transformation on A_μ.

6.2.1 PROBLEMS

A. Given a Weyl field transforming as the **6** of $SU(3)$, build the $SU(3)$ covariant derivative acting on it in terms of the Gell-Mann matrices.

B. Show that

$$\mathrm{Tr}\left[\mathbf{U}^\dagger(x)\partial_\mu\mathbf{U}(x)\right] = in\partial_\mu\alpha(x)\ . \tag{6.96}$$

C. If ω transforms as the adjoint representation of $SU(n)$, show that its covariant derivative is given by $\mathbf{D}_\mu\omega = \partial_\mu\omega + i\left[\mathbf{A}_\mu, \omega\right]$ and transforms in the same way as ω where $\mathbf{A}_\mu$ is the matrix of gauge fields.

D. Show from the gauge transformation properties of $\mathbf{A}_\mu$ that the field strength $\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + i\left[\mathbf{A}_\mu, \mathbf{A}_\nu\right]$ does indeed transform as the adjoint of $SU(n)$.

E. Starting from $\mathcal{L} = \frac{1}{2}\partial_\mu\Phi^T\partial^\mu\Phi$, where $\Phi(x)$ is an n column vector of real scalar fields, generalize it to be locally invariant under $SO(n)$, duplicating the procedure in the text. How many vector fields must be introduced? Show that their infinitesimal change under an $SO(n)$ transformation can also be expressed in terms of the covariant derivative acting on the gauge parameters.

6.3 The Pure Yang-Mills Theory

In this section we study some of the classical properties of the Yang-Mills Action given by

$$S^{\rm YM} = -\frac{1}{2g^2}\int d^4x\,\mathrm{Tr}\left(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right)\ , \tag{6.97}$$

where

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + i\left[\mathbf{A}_\mu, \mathbf{A}_\nu\right] \ , \tag{6.98}$$

and

$$\mathbf{A}_\mu(x) = A^B_\mu(x)\mathbf{T}^B \ . \tag{6.99}$$

The $\mathbf{T}^B$ matrices generate any one of the Lie algebras

$$\left[\mathbf{T}^B, \mathbf{T}^C\right] = i f^{BCD}\mathbf{T}^D \ , \tag{6.100}$$

with the indices B, C, D running from 1 to K, the dimension of the Lie algebra, itself defined by the totally antisymmetric structure constants f^{BCD}. As a consequence of (6.3.4), the $\mathbf{T}^B$ matrices are traceless; they are normalized to satisfy

$$\mathrm{Tr}\left(\mathbf{T}^B\mathbf{T}^C\right) = \frac{1}{2}\delta^{BC} \ . \tag{6.101}$$

The possible Lie algebras have been classified in E. Cartan's thesis; they are the classical algebras $SU(n)$, dimension $n^2-1,\ n \geq 2$; $SO(n)$, dimension $n(n{-}1)/2\ n > 2$; $Sp(2n)$, dimension $n(2n{+}1)$, $n > 1$; and the exceptional Lie algebras $G_2(14)$, $F_4(52)$, $E_6(78)$, $E_7(133)$ and $E_8(248)$ with their dimensions indicated in parenthesis.

One can also write the Yang-Mills Action independently of the $\mathbf{T}^B$ matrices,

$$S^{\mathrm{YM}} = -\frac{1}{4g^2}\int d^4x \left(F^B_{\mu\nu}F^{\mu\nu B}\right) \ , \tag{6.102}$$

where now

$$F^B_{\mu\nu} = \partial_\mu A^B_\nu - \partial_\nu A^B_\mu - f^{BCD}A^C_\mu A^D_\nu \ . \tag{6.103}$$

It follows that

$$\begin{aligned} g^2 S^{\mathrm{YM}} &= \int d^4x \left[-\frac{1}{2}\partial_\mu A^B_\nu \partial^\mu A^{\nu B} + \frac{1}{2}\partial_\mu A^B_\nu \partial^\nu A^{\mu B}\right. && (6.104) \\ &+ \left. g f^{BCD}A^C_\mu A^D_\nu \partial^\mu A^{\nu B} - \frac{g^2}{4} f^{BCD} f^{BEF} A^C_\mu A^D_\nu A^{\mu E} A^{\nu F}\right] && (6.105) \end{aligned}$$

The first two terms are recognized to be of the same type as in Maxwell's Lagrangian (except for the summation). However, the next two show that the vector fields have highly nontrivial cubic and quartic interactions among themselves.

The derivation of the equations of motion proceeds most easily in the matrix form. We start by varying the Action

$$\delta S = -\frac{1}{g^2} \int d^4x \mathrm{Tr} \left(\mathbf{F}_{\mu\nu} \delta \mathbf{F}^{\mu\nu} \right) \ , \tag{6.106}$$

where

$$\delta \mathbf{F}^{\mu\nu} = \partial^\mu \delta \mathbf{A}^\nu + i \delta \mathbf{A}^\mu \mathbf{A}^\nu + i \mathbf{A}^\mu \delta \mathbf{A}^\nu - (\mu \leftrightarrow \nu) \ . \tag{6.107}$$

Hence, using the antisymmetry of $\mathbf{F}_{\mu\nu}$

$$\delta S = -\frac{2}{g^2} \int d^4x \mathrm{Tr} \left[\mathbf{F}_{\mu\nu} \left(\partial^\mu \delta \mathbf{A}^\nu + i \delta \mathbf{A}^\mu \mathbf{A}^\nu + i \mathbf{A}^\mu \delta \mathbf{A}^\nu \right) \right] \ . \tag{6.108}$$

Next we integrate the first term by parts, throwing away the surface term due to the vanishing of the variation at the boundaries. Using the cyclic properties of the trace, we arrive at

$$\delta S = \frac{2}{g^2} \int d^4x \mathrm{Tr} \left[\left(\partial^\mu \mathbf{F}_{\mu\nu} + i \left[\mathbf{A}^\mu, \mathbf{F}_{\mu\nu} \right] \right) \delta \mathbf{A}^\nu \right] \tag{6.109}$$

from which we read off the equation of motion in matrix form

$$\partial^\mu \mathbf{F}_{\mu\nu} + i \left[\mathbf{A}^\mu, \mathbf{F}_{\mu\nu} \right] = 0 \ . \tag{6.110}$$

Since $\mathbf{F}_{\mu\nu}$ transforms according to the adjoint representation, this equation can be expressed directly in terms of the covariant derivative

$$\mathbf{D}^\mu \mathbf{F}_{\mu\nu} = 0 \ , \tag{6.111}$$

which shows that it is itself covariant. In addition, the $\mathbf{F}_{\mu\nu}$ fields satisfy the kinematic (Bianchi) constraints (as do the $F_{\mu\nu}$ of electromagnetism)

$$\mathbf{D}^\mu \tilde{\mathbf{F}}_{\mu\nu} = 0 \ , \tag{6.112}$$

where

$$\tilde{\mathbf{F}}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \mathbf{F}^{\rho\sigma} \tag{6.113}$$

is the dual of $\mathbf{F}_{\mu\nu}$. We emphasize that (6.3.15) is not an equation of motion since it is trivially solved by expressing $\mathbf{F}_{\mu\nu}$ in terms of the potentials.

From the equation of motion (6.3.13) it is clear that one can define a current $\mathbf{j}_\nu$ which is conserved; indeed the expression

$$\mathbf{j}_\nu = -\partial^\mu \mathbf{F}_{\mu\nu} = i\,[\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}] \tag{6.114}$$

does satisfy

$$\partial^\nu \mathbf{j}_\nu = 0\ , \tag{6.115}$$

leading to the conserved charges [here in matrix form $Q^A\mathbf{T}^A$]

$$\mathbf{Q} \equiv \int d^3x \mathbf{j}_o \tag{6.116}$$

$$= -\int d^3x \partial^i \mathbf{F}_{io} \tag{6.117}$$

$$= -\oint d^2\sigma^i \mathbf{F}_{io}\ , \tag{6.118}$$

where the last integral is over the surface at spatial infinity. Now the current $\mathbf{j}_\nu$ has atrocious transformation properties under a change of gauge, but the charges $\mathbf{Q}$, as can be seen from (6.3.21), transform nicely under a very large class of gauge transformations. From (6.3.21) we have

$$\mathbf{Q} \to \mathbf{Q}' = -\oint d^2\sigma^i \mathbf{U}\mathbf{F}_{io}\mathbf{U}^\dagger\ , \tag{6.119}$$

where the $\mathbf{U}$'s are on the bounding surface at infinity. Thus by requiring that we limit ourselves to $\mathbf{U}$'s which are constant in space at spatial infinity, we can take them out of the surface integral and obtain a covariant transformation for the conserved charges. We should add that this current is the Noether current obtained by canonical methods.

We can couple the Yang-Mills system by adding to S^{YM} a term of the form

$$\frac{2}{g}\int d^4 \mathrm{Tr}\left(\mathbf{A}^\mu \mathbf{J}_\mu\right)\ , \tag{6.120}$$

where $\mathbf{J}_\mu(x)$ is an external source written here in matrix form:

$$\mathbf{J}_\mu(x) = J_\mu^B(x)\mathbf{T}^B\ . \tag{6.121}$$

Then the equations of motion read

$$\mathbf{D}_\mu \mathbf{F}_{\mu\nu} = \mathbf{J}_\nu \ . \tag{6.122}$$

From this equation, we can require that $\mathbf{J}_\nu$ transform covariantly in order to preserve the covariance of the equation of motion:

$$\mathbf{J}^\mu \to \mathbf{U}\mathbf{J}^\mu\mathbf{U}^\dagger \ . \tag{6.123}$$

Furthermore, it is not hard to see that $\mathbf{J}^\mu$ must be covariantly conserved as a consequence of the equation of motion (see problem)

$$\mathbf{D}^\mu \mathbf{J}_\mu = \partial^\mu \mathbf{J}_\mu + i\,[\mathbf{A}^\mu, \mathbf{J}_\mu] = 0 \quad . \tag{6.124}$$

We remark that the Noether current is *not* $\mathbf{J}_\mu$ but rather

$$\mathbf{j}_\mu = -\partial^\rho \mathbf{F}_{\rho\mu} + \mathbf{J}_\mu \ . \tag{6.125}$$

Now if we go back to the extra term (6.3.23) we see that it is not invariant under a gauge transformation. Assuming that $\mathbf{J}^\mu$ transforms covariantly, we find that

$$\delta \int d^4x \mathrm{Tr}\,(\mathbf{A}_\mu \mathbf{J}^\mu) \;=\; \int d^4x \mathrm{Tr}\,(\mathbf{J}^\mu \partial_\mu \omega) \tag{6.126}$$

$$\;=\; -\int d^4x \mathrm{Tr}\,(\omega \partial^\mu \mathbf{J}_\mu) \ , \tag{6.127}$$

which means that we can restore invariance if the external source $\mathbf{J}^\mu$ is conserved. In Maxwell's theory this is no problem since $\mathbf{J}^\mu$ does not transform under a change of gauge. But in Yang-Mills the statement $\partial_\mu \mathbf{J}^\mu = 0$ is not covariant. This means that coupling to sources in this way breaks gauge invariance. This should not be too surprising. After all, we have seen that by reversing our earlier construction, the way to couple $\mathbf{A}_\mu$ in a gauge invariant way is to add a kinetic term for the fields which make up the source $\mathbf{J}^\mu$. Having an external nondynamical source will not do.

Nevertheless, one is free to examine the solutions of the classical equations (6.3.25) together with the constraint (6.3.27), but remember that coupling Yang-Mills to nondynamical external sources is a shady business.

Let us now return to the sourceless equations of motion (6.3.14). There are in Minkowski space many solutions of this equation. Just as in electrodynamics, there are plane wave solutions to this equation (see problem). They have infinite energy (but finite energy density). However, unlike in

Maxwell's theory, they cannot be superimposed to produce finite energy solutions because of the nonlinear nature of this theory, unless they move in the same direction.

There exist many other very interesting finite energy solutions to this equation but they involve some sort of singularities and therefore imply the existence of singular sources (see problem).

In Euclidean space, the Yang-Mills equation of motion has an exceedingly rich structure. Euclidean space can be regarded as Minkowski space with an imaginary time, and in Quantum Mechanics, processes with imaginary time evolution formally correspond to tunneling which happens instantly in real time. Hence ‘t Hooft called the nonsingular solutions of the sourceless Yang-Mills equation in Euclidean space *instantons*. On the other hand, we have seen earlier that the Feynman Path Integral may be better defined in Euclidean space. Hence, the study of Euclidean space solutions is doubly interesting.

Let us concentrate on Euclidean space solutions which have finite action following Belavin, Polyakov, Schwartz and Tyupkin, *Phys. Lett.* **59B**, 85 (1975). In Euclidean space we have

$$\mathrm{Tr}\left[\left(\mathbf{F}_{\mu\nu} - \tilde{\mathbf{F}}_{\mu\nu}\right)\left(\mathbf{F}_{\mu\nu} - \tilde{\mathbf{F}}_{\mu\nu}\right)\right] \geq 0 \ , \tag{6.128}$$

since it is the sum of squares. It follows that

$$\mathrm{Tr}\left(\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu} + \tilde{\mathbf{F}}_{\mu\nu}\tilde{\mathbf{F}}_{\mu\nu}\right) \geq 2\mathrm{Tr}\left(\mathbf{F}_{\mu\nu}\tilde{\mathbf{F}}_{\mu\nu}\right) \ , \tag{6.129}$$

or, using

$$\mathrm{Tr}\left(\tilde{\mathbf{F}}_{\mu\nu}\tilde{\mathbf{F}}_{\mu\nu}\right) = \mathrm{Tr}\left(\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu}\right) \ , \tag{6.130}$$

and that

$$\mathrm{Tr}\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu} \geq \mathrm{Tr}\mathbf{F}_{\mu\nu}\tilde{\mathbf{F}}_{\mu\nu} \ , \tag{6.131}$$

which establishes upon integration a lower bound for the value of the Yang-Mills Euclidean Action. Clearly equality is achieved when

$$\mathbf{F}_{\mu\nu} = \tilde{\mathbf{F}}_{\mu\nu} \ , \tag{6.132}$$

corresponding to self-dual solutions. Antiself-dual solutions also correspond to a lower bound. It is easy to see that self-dual or antiself-dual solutions have zero Euclidean energy momentum tensor (see problem). The integral of the right-hand side of the inequality (6.3.33) can be rewritten as the integral of a divergence (see Eq. (6.2.41))

$$\int d^4x \mathrm{Tr}\left(\mathbf{F}_{\mu\nu}\tilde{\mathbf{F}}_{\mu\nu}\right) = 2\int d^4x \partial_\mu W_\mu \ , \tag{6.133}$$

where

$$W_\mu = \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[\mathbf{A}_\nu\partial_\rho\mathbf{A}_\sigma + \frac{2i}{3}\mathbf{A}_\nu\mathbf{A}_\rho\mathbf{A}_\sigma\right] \ , \tag{6.134}$$

so that

$$S_E^{\mathrm{YM}} = \frac{1}{2g^2}\int d^4x \mathrm{Tr}\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu} \geq \frac{2}{g^2}\oint_S d^3\sigma_\mu W_\mu \ , \tag{6.135}$$

where the last term is integrated over the bounding surface at Euclidean infinity. Hence the minimum value of the action will depend on the properties of the gauge fields at infinity.

Now in order for S_E^{YM} to be finite, it must be that $F^B_{\mu\nu}$ decreases sufficiently fast at Euclidean infinity

Underarrow needs work which means in general that $\mathbf{A}_\mu$ tends to a configuration

$$\mathbf{A}_\mu = -i\mathbf{U}\partial_\mu\mathbf{U}^\dagger \ , \qquad \text{for} \ \ x^2 \to \infty \tag{6.136}$$

which is obtained from $\mathbf{A}_\mu = 0$ by a gauge transformation; it therefore gives $\mathbf{F}_{\mu\nu} = 0$.

Now recall that S^{YM} is bounded from below by a quantity which depends entirely on the behavior of the potentials at Euclidean infinity. In fact, substituting (6.3.39) into (6.3.36) we see that on S

$$W_\mu = \frac{1}{3}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[\mathbf{U}\partial_\nu\mathbf{U}^\dagger\mathbf{U}\partial_\rho\mathbf{U}^\dagger\mathbf{U}\partial_\sigma\mathbf{U}^\dagger\right] \ , \tag{6.137}$$

where we have used the antisymmetry of ρ and σ and $\mathbf{U}\mathbf{U}^\dagger = 1$. Thus

$$S_E^{\mathrm{YM}} \geq \frac{2}{3g^2}\oint_S d^3\sigma_\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[\mathbf{U}\partial_\nu\mathbf{U}^\dagger\mathbf{U}\partial_\rho\mathbf{U}^\dagger\mathbf{U}\partial_\sigma\mathbf{U}^\dagger\right] \ , \tag{6.138}$$

which depends entirely on the group element $\mathbf{U}(x)$! We have the remarkable result that the (minimum) value of the Euclidean Action depends on the properties of $\mathbf{U}(x)$ only and not on the details of the field configuration at finite x.

Let us specialize to the case of $SU(2)$. There the group elements $\mathbf{U}(x)$ depend on three parameters, call them ϕ_1, ϕ_2, ϕ_3, which are themselves x-dependent. On the other hand, the surface of integration S is the surface of a sphere with very large ($\sim$ infinite) radius. Thus, we can think of $\mathbf{U}$ as a mapping between the three group parameters and the three coordinates which label the surface of our sphere, that is of a three-sphere onto a three-sphere. Such mappings are characterized by the *homotopy* class. It roughly corresponds to the number of times one sphere is mapped onto the other. For instance, homotopy class 1 means that the surface of the sphere S_3^∞ at Euclidean infinity is mapped only once on the surface of the sphere S_3 of the group manifold labeled by the angles ϕ_i. In general, homotopy class n means that n points of S_3^∞ are mapped into one point of S_3, etc.

If we set

$$\partial_\mu \mathbf{U}^\dagger = \sum_{a=1}^{3} \frac{\partial \phi_a}{\partial x^\mu} \frac{\partial}{\partial \phi_a} \mathbf{U}^\dagger = \partial_\mu \phi^a \partial_a \mathbf{U}^\dagger \ , \tag{6.139}$$

we arrive at

$$S_E^{\mathrm{YM}} \geq \frac{2}{3g^2} \oint_S d^3\sigma_\mu \epsilon_{\mu\nu\rho\sigma} \partial_\nu \phi^a \partial_\rho \phi^b \partial_\sigma \phi^c \mathrm{Tr}(\mathbf{U}\partial_a \mathbf{U}^\dagger \mathbf{U} \partial_b \mathbf{U}^\dagger \mathbf{U} \partial_c \mathbf{U}^\dagger) \ , \tag{6.140}$$

or, using the antisymmetry of the ϵ symbol,

$$S_E^{\mathrm{YM}} \geq \frac{4}{g^2} \oint_S d^3\sigma_\mu \epsilon_{\mu\nu\rho\sigma} \partial_\nu \phi^1 \partial_\rho \phi^2 \partial_\sigma \phi^3 \mathrm{Tr}(\mathbf{U}\partial_1 \mathbf{U}^\dagger \mathbf{U} \partial_2 \mathbf{U}^\dagger \mathbf{U} \partial_3 \mathbf{U}^\dagger) \ . \tag{6.141}$$

In this form we see clearly the Jacobian of the transformation between variables that label the surface S and the angle ϕ_a. But, as we have just discussed, this map is characterized by its homotopy class n, when S_3^∞ is mapped n times onto the group manifold of SU_2. By parametrizing $\mathbf{U}$ in terms of, say, Euler angles, it is straightforward to arrive at

$$S_E^{\mathrm{YM}} \geq \frac{8\pi^2}{g^2} n \ , \tag{6.142}$$

where n is an integer, given by

$$n = \frac{1}{16\pi^2} \int d^4x \mathrm{Tr}\left(\mathbf{F}_{\mu\nu} \tilde{\mathbf{F}}_{\mu\nu}\right) \quad ; \tag{6.143}$$

it is called the Pontryagin index.

Thus Euclidean solutions with finite action are labeled by their homotopy class which gives the lower bound for the (Euclidean) Action. The lower bound is attained when the field configurations are either dual or antiself dual, *i.e.*, when

$$\mathbf{F}_{\mu\nu} = \pm\tilde{\mathbf{F}}_{\mu\nu} \; . \tag{6.144}$$

As an example, consider the original instanton solution; there the Euclidean $SU(2)$ potential is given by

$$\mathbf{A}_\mu(x) = \frac{-ix^2}{x^2+\lambda^2}\mathbf{U}\partial_\mu\mathbf{U}, \tag{6.145}$$

where

$$\mathbf{U} = \frac{1}{\sqrt{x^2}}\left(x_0 - i\vec{x}\cdot\vec{\sigma}\right) \quad , \tag{6.146}$$

where the $\vec{\sigma}$ matrices act in the $SU(2)$ space, and

$$x^2 = x_0^2 + \vec{x}\cdot\vec{x} \; . \tag{6.147}$$

It satisfies the requirement (6.3.39) for finite action [λ^2 is a constant]. It can be shown that it is self-dual and that the form of $\mathbf{U}$ implies Pontryagin index $+1$.

Finally, let us mention that in Yang-Mills theories, functions which transform under gauge transformations cannot in general be taken to be constant because they can become x-dependent through a gauge transformation. The closest one can define is a covariant constant which satisfies

$$\mathbf{D}_\mu\phi = \left(\partial_\mu\mathbf{1} + i\mathbf{A}_\mu\right)\phi = 0 \quad , \tag{6.148}$$

where we have suppressed all group indices. In solving for ϕ, we are going to unearth a very interesting object: the path ordered integral. To see it we note that

$$\phi(x+dx) = \phi(x) + dx^\mu\partial_\mu\phi + \cdots \quad , \tag{6.149}$$

where dx_μ is an arbitrarily small displacement. Using (6.3.51), we obtain

$$\begin{aligned}\phi(x+dx) &= \phi(x) - idx^\mu\mathbf{A}_\mu\phi(x) + \cdots & (6.150)\\ &= e^{-idx^\mu\mathbf{A}_\mu}\phi(x) + \mathcal{O}(dx)^2 \; . & (6.151)\end{aligned}$$

Since under a gauge transformation

$$\phi(x) \to \mathbf{U}(x)\phi(x) \ , \tag{6.152}$$

it follows from (6.3.53) that

$$e^{-idx^\mu \mathbf{A}_\mu} \to \mathbf{U}(x+dx)\, e^{-idx^\mu \mathbf{A}_\mu(x)} \mathbf{U}^\dagger(x) \ , \tag{6.153}$$

which is the fundamental relation we sought to obtain. Now, (6.3.51) can be integrated by iterating on the displacement: $\phi(y)$ can be obtained from $\phi(x)$ by taking small displacements along a curve that begins at x and ends at y, thus obtaining

$$\phi(y) = \left(P\, e^{-i\oint_x^y dx\cdot\mathbf{A}}\right) \phi(x) \ , \tag{6.154}$$

where the path ordered exponential is defined by

$$P\, e^{-i\int dx\cdot\mathbf{A}} \equiv \prod_k \left(1 - idx_k^c \cdot \mathbf{A}(x_k)\right) \ , \tag{6.155}$$

dx_k being the displacement centered around x_k on the curve C:

$$1.51 \tag{6.156}$$

From (6.3.55) it follows that

$$P\, e^{-i\oint_x^y dx\cdot\mathbf{A}} \to \mathbf{U}(y) P\, e^{-i\oint_z^y dx\cdot\mathbf{A}} \mathbf{U}^\dagger(x) \ , \tag{6.157}$$

and in particular the path ordered exponential along a closed path transforms like a local covariant quantity:

$$P\, e^{-i\oint dx\cdot\mathbf{A}} \to \mathbf{U}(x) P\, e^{-i\oint dx\cdot\mathbf{A}} \mathbf{U}^\dagger(x) \ , \tag{6.158}$$

so that its trace is gauge invariant. It is a functional of the path. We emphasize that, although we have motivated the construction of the covariant functional starting from (6.3.51), which implies that the field strengths are zero, it should be clear that the path ordered exponential can be built for any field configuration $\mathbf{A}_\mu$.

There are many more aspects of the classical Yang-Mills theory we have not touched on, such as monopole solutions, generalization of instanton solutions, meron solutions with infinite Euclidean action (but finite Minkowski action and singular sources), etc. Alas it is time to go on and start thinking about how to define the quantum Yang-Mills theory.

6.3.1 PROBLEMS

A. Show that the field configuration [S. Coleman, *Phys. Lett.* **70B**, 59 (1977)]

$$\begin{aligned} A_1^B &= A_2^B = 0 \\ A_0^B &= A_3^B = x_1 F_1^B(x^0 + x^3) + x_2 F_2^B(x^0 + x^3) \ . \end{aligned}$$

is a solution of the Yang-Mills equations of motion, where $F_{1,2}^B$ are arbitrary functions. Compare these solutions with the plane wave solutions of Maxwell's theory.

*B. Analyze the Wu-Yang *Ansatz* for $SU(2)$ Yang-Mills

$$A_0^C = x^C \frac{g(r)}{r^2} \quad ; \qquad A_i^C = \epsilon_i^{cj} x^j \frac{f(r)}{r^2} \ ,$$

where C is the $SU(2)$ index $C = 1, 2, 3$, r is the length of the position vector x. [Recall that for $SU(2)$ $f^{ABC} = \epsilon^{ABC}$ the Levi-Cività tensor.] Derive the equations that f and g must satisfy. Show that they are satisfied by $f = 1$, $g =$ constant. For this solution describe the potential and field configurations and find the energy density and energy.

C. For an $SU(2)$ gauge theory, show that the 't Hooft–Corrigan–Fairlie–Wilczek *Ansatz* for the potentials in terms of one scalar field ϕ

$$A_0^C = \frac{1}{\phi}\partial^C \phi \quad ; \qquad A_i^C = \frac{1}{\phi}\left[\delta_i^C \partial_0 \phi - \epsilon_i{}^{Cj} \partial_j \phi\right] \ ,$$

implies that ϕ obeys the equation of motion for the $\lambda\phi^4$ theory where λ is an arbitrary constant.

D. Show that the Noether energy momentum tensor for the Euclidean Yang-Mills theory can be written in the form

$$\theta_{\mu\nu} = \frac{1}{2g^2}\left(\mathbf{F}_{\mu\rho}^B + \tilde{\mathbf{F}}_{\mu\rho}^B\right)\left(\mathbf{F}_{\nu\rho}^B - \tilde{\mathbf{F}}_{\nu\rho}^B\right) \ .$$

E. Find the change in W_μ under a gauge transformation, and verify that $\partial_\mu W_\mu$ is gauge invariant.

F. Evaluate the trace of the path ordered exponential around a closed loop for the instanton solution described in the text. Choose a simple path at your convenience.

6.4 Gravity as a Gauge Theory

It is a fact that in the absence of gravity the laws of Physics are invariant under *global* Lorentz transformations and translations; these give rise to the well- known conservation laws of Special Relativity. In order to incorporate gravity into this framework, Einstein seized on the Equivalence Principle as the centerpiece of his conceptual leap from Special to General Relativity. This principle was known to many previous generations of physicists, but its significance unappreciated. In fact, according to Newton, any external force on a particle is to be equated to its acceleration times the intrinsic mass of the particle, called the *inertial* mass. However any external gravitational force is proportional to a parameter with the dimensions of mass, called the *gravitational* mass of the particle. Although both masses are in this framework logically different from one another, coming as they are from different sides of Newton's equation, they have always been measured to be numerically equal to the impressive accuracy of twelve significant figures.

To be specific consider Newton's equation for a particle in a constant gravitational field

$$m_I \frac{d^2\vec{r}}{dt^2} = m_G \vec{g} \ , \tag{6.159}$$

where m_I is the inertial mass of the particle, $\vec{r}(t)$ its position vector, m_G its gravitational mass and $\vec{g}$ the acceleration due to the external constant gravitational field. If the inertial and gravitational masses are one and the same, one can rewrite this equation in the form ($m \equiv m_I = m_G$)

$$m \frac{d^2}{dt^2} \left[\vec{r}(t) - \frac{1}{2} \vec{g} t^2 \right] = 0, \tag{6.160}$$

leading to the interpretation that the whole right hand-side, *i.e.* the external gravitational field, can be generated by a change of frame of reference

$$\vec{r}(t) \to \vec{r}\,'(t) = \vec{r}(t) - \frac{1}{2} \vec{g}\ t^2 \ . \tag{6.161}$$

Physically this means that when viewed from a freely falling frame of reference the particle experiences no gravity.

We were able to do this because the external gravitational field was constant, but in general, gravitational fields are not constant. To account for this, Einstein formulated the following principle: gravitational fields are of such a nature that, at each point in space-time, they allow themselves to be transformed away by choosing an appropriate set of coordinates. Of course this set of coordinate axes will vary from space-time point to space-time point, but there will always be a set of coordinates in terms of which it looks like there is no gravitation! Thus the recipe for including gravitation is very straightforward: 1) - take any *local* quantity, such as a Lagrangian density, $\mathcal{L}$, or an infinitesimal volume element, written in a Lorentz invariant way so as to satisfy the laws of Special Relativity; 2) - identify the coordinates appearing in the local quantities with the "freely falling" coordinates; when expressed in terms of arbitrary space-time variables, the interaction with gravity will be magically generated.

This recipe clearly insures general coordinate invariance: given the preferred coordinate system ξ^m at the space-time point P, we can re-express it in terms of any arbitrary coordinate label of P, with the result that the Physics, having been expressed in the ξ^m system, is independent of the labelling of P. This invariance is called gauge invariance by General Relativists. From here on we use Latin tensor indices $m, n, p, q, \cdots$ in the freely falling frame, and Greek tensor indices $\mu, \nu, \rho, \sigma, \cdots$ in arbitrary coordinates.

To illustrate the procedure, consider a self-interacting scalar field $\phi(x)$. Its behavior, **in the absence of gravity**, is described by the action

$$S[\phi] = \int d^4x[\frac{1}{2}\partial_\mu\phi\ \partial^\mu\phi - V(\phi)]\ . \tag{6.162}$$

Recall that $x^\mu = (t, x^i) = (t, \vec{x})\ ,\ i = 1, 2, 3$ are the coordinates, and

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t}, \vec{\nabla})\ , \tag{6.163}$$

are the derivative operators, and $V(\phi)$ is the potential density. Also we have

$$\partial^\mu = \eta^{\mu\nu}\partial_\nu = (\frac{\partial}{\partial t}, -\vec{\nabla}), \tag{6.164}$$

where $\eta^{\mu\nu}$ is the inverse metric of Special Relativity: $\eta_{oo} = -\eta_{ii} = 1; \eta_{ij} = 0$ for $i \neq j$. In the above action, the local quantities are the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi) \ , \tag{6.165}$$

and the volume element

$$d^4x = dx^0 dx^1 dx^2 dx^3. \tag{6.166}$$

The equivalence principle tells us that in order to immerse this scalar field in a gravitational field, we just have to reinterpret the variable x^μ and its derivative as being the "free fall" coordinates (or else imagine that we are in the preferred frame at that point). Thus we identify

$$\{x^\mu\} \to \{\xi^m\} \quad , \ m = 0, 1, 2, 3 \ , \tag{6.167}$$

as the "flat" coordinates. In this flat system, the line element is

$$ds^2 = \eta_{mn} d\xi^m d\xi^n \ , \tag{6.168}$$

where η_{mn} is the metric of Special Relativity,

$$\eta_{mn}\eta^{np} = \delta_m^p, \tag{6.169}$$

and δ_m^p is the Kronecker delta function. The new Lagrangian is then

$$\mathcal{L} \to \frac{1}{2}\eta^{mn}\partial_m \phi \partial_n \phi - V(\phi) \ , \tag{6.170}$$

where

$$\partial_m \equiv \frac{\partial}{\partial \xi^m} \ , \tag{6.171}$$

are the derivative operators with respect to the flat coordinates, ξ^m. Since we are dealing with a scalar field we do not have to make any changes in its description; all we are doing so far is to write $\mathcal{L}$ in a very special coordinate system. The volume element d^4x becomes the volume in terms of the flat coordinates

$$d^4x \to d\xi^0 d\xi^1 d\xi^2 d\xi^3 \ . \tag{6.172}$$

The action, generalized to include the effects of gravitation, is now given by

$$S[\phi] = \int d^4\xi [\frac{1}{2}\eta^{mn}\partial_m\phi\partial_n\phi - V(\phi)] \ . \tag{6.173}$$

The reader should not be confused by the fact that this expression looks a lot like (6.4.4). The difference lies in the integration: here one integrates over the *manifold* which is labeled by some arbitrary coordinate system $\{x^\mu\}$. The labels $\{\xi^m\}$ vary from point to point, and should they coincide with the coordinate labels, then gravity would be absent. The information about the gravitational field is in fact contained in the change of the flat coordinates from point to point. Thus we can express ξ^m as a local function of any non-inertial coordinates x^μ or equivalently we can write

$$d\xi^m = \frac{\partial \xi^m}{\partial x^\mu} dx^\mu \ , \tag{6.174}$$

where the derivatives are evaluated at the point of interest. The transformation matrix between the flat and arbitrary coordinates is called the *vierbein*

$$e^m_\mu(x) \equiv \frac{\partial \xi^m}{\partial x^\mu} \ , \tag{6.175}$$

and it depends on x^μ (or ξ^m); it has a "flat" index m and a "curvy" index μ. We can also define the inverse operation

$$dx^\mu = \frac{\partial x^\mu}{\partial \xi^m} d\xi^m \equiv e^\mu_m d\xi^m \ , \tag{6.176}$$

where $e^\mu_m(x)$ are the inverse vierbeins. They are so named because from

$$d\xi^m = e^m_\mu dx^\mu = e^m_\mu e^\mu_n d\xi^n, \tag{6.177}$$

we deduce

$$e^m_\mu e^\mu_n = \delta^m_n \ ; \tag{6.178}$$

also

$$e^\mu_m e^m_\nu = \delta^\mu_\nu \ . \tag{6.179}$$

We can also express the derivative operators by means of the vierbeins

$$\frac{\partial}{\partial \xi^m} = \frac{\partial x^\mu}{\partial \xi^m}\frac{\partial}{\partial x^\mu} = e^\mu_m \partial_\mu \ , \tag{6.180}$$

in an arbitrary coordinate system. Our Lagrangian rewritten in the $\{x^\mu\}$ system becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\eta^{mn} e^\mu_m e^\nu_n \partial_\mu\phi\partial_\nu\phi - V(\phi), & (6.181)\\ \mathcal{L} &= \frac{1}{2} g^{\mu\nu}(x)\partial_\mu\phi\partial_\nu\phi - V(\phi), & (6.182)\end{aligned}$$

where we identify the inverse metric

$$g^{\mu\nu}(x) = \eta^{mn} e^\mu_m(x) e^\nu_n(x) \ . \tag{6.183}$$

The metric appears in the line element expressed in an arbitrary system of coordinates

$$\begin{aligned} ds^2 &= \eta_{mn} d\xi^m d\xi^n = \eta_{mn} e^m_\mu(x) e^n_\nu(x) dx^\mu dx^\nu & (6.184)\\ &\equiv g_{\mu\nu}(x) dx^\mu dx^\nu, & (6.185)\end{aligned}$$

thus defining the metric tensor

$$g_{\mu\nu}(x) = \eta_{mn} e^m_\mu(x) e^n_\nu(x) \ . \tag{6.186}$$

It is straightforward to verify that

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho \ . \tag{6.187}$$

The volume element must also be expressed in an arbitrary system. We have

$$d^4\xi = J(\xi, x) d^4x \ , \tag{6.188}$$

where $J(\xi, x)$ is the Jacobian of the transformation. It is easy to show that this quantity reduces to (see problem)

$$d^4\xi = \sqrt{-\det g_{\mu\nu}}\ d^4x \ , \tag{6.189}$$

or alternatively

$$
\begin{aligned}
d^4\xi &= (\det e_\mu^m) d^4x, && (6.190)\\
&\equiv E\ d^4x\ , && (6.191)
\end{aligned}
$$

where E is the determinant of the vierbein regarded as a 4 x 4 matrix. Hence the action for a scalar field in a gravitational field is given by

$$S = \int d^4x \sqrt{-\det g_{\mu\nu}}\ [\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)]\ , \tag{6.192}$$

and in this case the effect of the gravitational fields resides fully in the metric $g_{\mu\nu}(x)$ and its inverse. In the absence of gravity, the vierbein becomes trivial

$$e_\mu^m \to \delta_\mu^m \text{ (no gravity)}\ , \tag{6.193}$$

and the action reduces to the original one.

We note at this stage that the derivatives ∂_m obey a non trivial algebra, namely

$$[\partial_m, \partial_n] = [\partial_m e_n^\mu - \partial_n e_m^\mu] e_\mu^p \partial_p\ , \tag{6.194}$$

where we have used the fact that ∂_μ and ∂_ν commute. This shows that the notation ∂_m is slightly misleading: one would expect ∂_m and ∂_n to commute. Why don't they? We now proceed to apply the Equivalence Principle to generate the gravitational interaction of a Dirac spinor. In the absence of gravity, a free Dirac spinor is described by the Lagrangian

$$\mathcal{L}_D = \frac{1}{2} \bar{\Psi} \gamma^\mu \overleftrightarrow{\partial}_\mu \Psi\ \equiv \frac{1}{2} \bar{\Psi} \gamma^\mu \partial_\mu \Psi - \frac{1}{2} (\partial_\mu \bar{\Psi}) \gamma^\mu \Psi\ , \tag{6.195}$$

where Ψ is a four component Dirac field. The difference between the previous case of a scalar field and the present case is that the Dirac field transforms as a spinor under a Lorentz transformation (suppressing spinor indices)

$$\Psi \to \exp\left\{ \frac{i}{2} \epsilon^{\mu\nu} \sigma_{\mu\nu} \right\} \Psi\ , \tag{6.196}$$

where $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ are the parameters of the transformation and $\sigma_{\mu\nu}$ are the six matrices representing the generators of Lorentz transformations on a spinor; in terms of the Dirac matrices

$$\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]\ . \tag{6.197}$$

The flat space Dirac Lagrangian (6.4.36) is clearly invariant under these transformations provided that the parameters of the transformation be space-time independent; it was built that way.

The Equivalence Principle says that at each space-time point P, gravitational fields are such that there is a favored coordinate system in which things look Special Relativistic, *i.e.* the invariances of the Dirac equation of Special Relativity are exactly reproduced in that coordinate system. These include of course the coordinate transformations of the Poincaré group, *i.e.* translations and Lorentz transformations on the coordinates, but also the transformation (6.4.37) of the Dirac spinor field itself. This must be true at any space time point with the favored coordinate system varying from point to point. Hence at P, with favored coordinates ξ^m, the invariance group is

$$\xi^m \rightarrow \xi^m + \epsilon^m \quad \text{(translations)}, \tag{6.198}$$

$$\xi^m \rightarrow \xi^m + \epsilon^m{}_n \xi^n \quad \text{(Lorentz transformations)}, \tag{6.199}$$

$$\Psi \rightarrow \exp(\frac{i}{2}\epsilon^{mn}\sigma_{mn})\Psi \equiv U(x)\Psi \quad \text{(Lorentz transformations)} \tag{6.200}$$

where the parameters $\epsilon^m, \epsilon^{mn}$ must depend on the point P, and therefore on its coordinate label x^μ. Thus, in order to generalize the Dirac equation in a gravitational field, we must preserve *local* invariance under Lorentz transformations as well. This is the first subtlety in applying the Equivalence Principle.

Fortunately for us, a long acquaintance (of ten pages!) with Yang-Mills theories enables us to quickly meet this challenge.

The invariance of Special Relativity

$$\Psi \rightarrow U\Psi, \tag{6.201}$$

$$\partial_\mu \Psi \rightarrow \Lambda_\mu{}^\nu U \partial_\nu \Psi , \tag{6.202}$$

where U does not depend on x, can be easily generalized to include x-dependent Lorentz transformations. We define a new derivative operator

$$D_m \equiv e^\mu_m(\partial_\mu + i\omega_\mu) , \tag{6.203}$$

and require that under the local Lorentz transformations (6.4.39) and (6.4.40), its action on the spinor field transform the same way as the old derivative did in the absence of gravity

$$D_m\Psi \rightarrow \Lambda_m{}^n U(x) D_n \Psi \tag{6.204}$$

If we can find such an object (and we will), the construct

$$\mathcal{L} = \bar{\Psi}\gamma^m D_m \Psi \ , \tag{6.205}$$

will serve as the desired extension of the Dirac Lagrangian to include gravity.

In operator form, (6.4.44) reads

$$D_m \to D'_m = \Lambda_m{}^n U(x) D_n U^\dagger(x) \ , \tag{6.206}$$

with D_m regarded as a matrix in the 4 x 4 space of the Dirac spinor. This transformation requirement can be satisfied if the 4 x 4 matrix $\omega_\mu(x)$ itself undergoes the transformation

$$\omega_\mu \to {\omega_\mu}' = -iU(\partial_\mu U^\dagger) + U\omega_\mu U^\dagger \ , \tag{6.207}$$

and also

$$e^\mu_m \to e^{\mu\prime}_m = \Lambda_m{}^n e^\mu_n \ . \tag{6.208}$$

We can expand $\omega_\mu(x)$ in terms of the matrices that represent the Lorentz group in the Dirac spinor space:

$$\omega_\mu(x) = \frac{1}{2}\omega_\mu{}^{mn}(x)\sigma_{mn} \ , \tag{6.209}$$

where we have (again!) suppressed the spinor indices. Thus the fully covariant derivative acting on a Dirac spinor is

$$D_p = e^\mu_p(x)[\partial_\mu + \frac{1}{2}\omega_\mu{}^{mn}(x)\sigma_{mn}] \ , \tag{6.210}$$

where the fields $\omega_\mu{}^{mn}(x)$ are the exact analogues of the Yang-Mills fields, and the matrices σ_{mn} generate the action of SO(3,1) on Dirac spinors. The covariant derivative acting on an arbitrary representation of SO(3,1) is given by

$$D_p = e^\mu_p(x)[\partial_\mu + \frac{1}{2}\omega_\mu{}^{mn} X_{mn}] \ . \tag{6.211}$$

Here X_{mn} are the generators of SO(3,1) in the representation of interest (whose indices have been suppressed), depending on what D_p acts on. These matrices obey the SO(3,1) commutation rules

$$\lfloor X_{mn}, X_{pq} \rceil = -i\eta_{mp} X_{nq} + i\eta_{np} X_{mq} - i\eta_{nq} X_{mp} + i\eta_{mq} X_{np} \ . \qquad (6.212)$$

The Dirac Lagrangian in a gravitational field is now

$$\mathcal{L}_D = \frac{1}{2} \bar{\Psi} \gamma^p e_p^\mu (\partial_\mu + \frac{i}{2} \omega_\mu{}^{mn} \sigma_{mn}) \Psi + \text{c.c.} \quad . \qquad (6.213)$$

The covariant derivatives now obey the more general algebra

$$[D_m, D_n] = S_{mn}{}^p D_p + \frac{i}{2} R_{mn}{}^{pq} X_{pq} \ , \qquad (6.214)$$

where $S_{mn}{}^p$ are called the torsion coefficients, and $R_{mn}{}^{pq}$ the curvature coefficients. They have honest transformation properties under Lorentz Transformations, *i.e.* they transform as indicated by their Latin index structure.

The expressions for $S_{mn}{}^p$ and $R_{mn}{}^{pq}$ are a bit tricky to obtain since the matrix X_{mn} will attack anything standing to its right that has a Latin or spinor index on it. Specifically one finds (see problem)

$$[D_m, D_n] = e_m^\mu (D_\mu e_n^\rho) D_\rho + e_m^\mu e_n^\rho D_\mu D_\rho - (m \leftrightarrow n) \ , \qquad (6.215)$$

but

$$D_\mu e_n^\rho = \partial_\mu e_n^\rho + \frac{1}{2} \omega_\mu{}^{pq} (\eta_{pn} e_q^\rho - \eta_{qn} e_p^\rho) \ , \qquad (6.216)$$

where we have used the action of the generator on the Latin index n:

$$X_{pq} \bullet e_n^\rho = i\eta_{qn} e_p^\rho - i\eta_{pn} e_q^\rho \ . \qquad (6.217)$$

Also

$$D_\mu D_\rho = \frac{1}{2} \partial_\mu \omega_\rho{}^{mn} X_{mn} + \frac{1}{4} \omega_\mu{}^{mn} \omega_\rho{}^{pq} X_{mn} X_{pq} \ . \qquad (6.218)$$

Using the commutation relations we find that the torsion is given by

$$\begin{aligned} S_{mn}{}^p &= e_\rho^p (e_m^\mu D_\mu e_n^\rho - e_n^\mu D_\mu e_m^\rho) && (6.219) \\ &= [e_m^\mu (\partial_\mu e_n^\rho + \omega_{\mu n}{}^q e_q^\rho) - (m \leftrightarrow n)] e_\rho^p \ , && (6.220) \\ &= e_\rho^p (e_m^\mu \partial_\mu e_n^\rho - e_n^\mu \partial_\mu e_m^\rho) + e_m^\mu \omega_{\mu n}{}^p - e_n^\mu \omega_{\mu m}{}^p \ . && (6.221) \end{aligned}$$

The *Riemann* curvature tensor is itself given by

$$
\begin{aligned}
R_{mn}{}^{pq} &= e^\mu_m e^\rho_n [\partial_\mu \omega_\rho{}^{pq} - \partial_\rho \omega_\mu{}^{pq} - \omega_\mu{}^{rp}\omega_{\rho r}{}^q + \omega_\rho{}^{rp}\omega_{\mu r}{}^q] \quad (6.222)\\
&= (e^\mu_m e^\rho_n - e^\mu_n e^\rho_m)(\partial_\mu \omega_\rho{}^{pq} - \omega_\mu{}^{rp}\omega_{\rho r}{}^q) \ . \quad (6.223)
\end{aligned}
$$

These two quantities, having only Latin indices, will transform covariantly.

It is instructive at this point to identify their various components. Let us decompose them in terms of SO(3,1) irreducible tensors. This is easiest done in the $SU(2) \otimes SU(2)$ language where a Latin tensor index transforms as $(\mathbf{2}, \mathbf{2})$. Given the SU(2) product, $\mathbf{2} \otimes \mathbf{2} = \mathbf{1}_A \oplus \mathbf{3}_S$ where the subscripts $S(A)$ denotes the symmetric (antisymmetric) parts, we see that an antisymmetric pair of indices transforms as $[mn]$: $[(\mathbf{2}, \mathbf{2}) \otimes (\mathbf{2}, \mathbf{2})]_{\mathbf{A}} = (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{3}, \mathbf{1})$ etc...

Thus the torsion, having one antisymmetric pair and another index, transforms as [using the SU(2) Kronecker product $\mathbf{2} \otimes \mathbf{3} = \mathbf{4} \oplus \mathbf{2}$]

$$
[(\mathbf{1}, \mathbf{3}) \oplus (\mathbf{3}, \mathbf{1})] \otimes [(\mathbf{2}, \mathbf{2})] = (\mathbf{2}, \mathbf{4}) \oplus (\mathbf{2}, \mathbf{2}) \oplus (\mathbf{4}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{2}) \ , \qquad (6.224)
$$

so it contains two vectors $(\mathbf{2}, \mathbf{2})$ and another representation $(\mathbf{2}, \mathbf{4}) \oplus (\mathbf{4}, \mathbf{2})$. The vectors can be easily built, one by using the totally antisymmetric Levi-Cività symbol

$$
\epsilon_{mnpq} = \begin{cases} +1 & \text{for even permutations of } 0123 \ , \\ -1 & \text{for odd permutations of } 0123 \ , \end{cases} \qquad (6.225)
$$

giving

$$
V_q = S_{mn}{}^p \epsilon^{mn}{}_{pq} \ , \qquad (6.226)
$$

where we have raised indices by means of the metric η^{rs}, while the other vector is just

$$
T_q = S_{qp}{}^p \ . \qquad (6.227)
$$

One can do exactly the same for the curvature tensor which is made up of two pairs of antisymmetric indices. Thus it transforms like $[(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})] \otimes [(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})]$. Using the Kronecker products $\mathbf{3} \otimes \mathbf{3} = (\mathbf{5} \oplus \mathbf{1})_S \oplus \mathbf{3}_A$ we find that it contains

$$
(\mathbf{3}, \mathbf{3}) \oplus (\mathbf{5}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{5}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}) \ , \qquad (6.228)
$$

in the part that is symmetric under the interchange of the two pairs of indices, and

$$(\mathbf{3},\mathbf{3}) \oplus (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \ , \tag{6.229}$$

in the antisymmetric part.

We can see that two invariants appear in the symmetric product. It is easy to see that they correspond to

$$R^* = \epsilon^{mnpq} R_{mnpq} \ , \tag{6.230}$$

and to the scalar curvature

$$R = R_{mn}{}^{mn} \ . \tag{6.231}$$

The tensor transforming as $(\mathbf{3},\mathbf{3})$ in the symmetric part is a symmetric second rank traceless tensor; it is the traceless part of the Ricci tensor, R_{mn}. The tensor transforming as $(\mathbf{1},\mathbf{5}) \oplus (\mathbf{5},\mathbf{1})$ is a fourth rank tensor $C_{mn}{}^{pq}$ satisfying all the cross trace conditions

$$C_{mn}{}^{nq} = C_{mn}{}^{qn} = 0 \ ; \tag{6.232}$$

it is called the *Weyl* or conformal tensor.

The tensors appearing in the antisymmetric part are not nearly as famous since they automatically vanish in Einstein's theory.

The torsion and curvature tensors, being built by commuting two covariant derivatives, obey additional structural identities, called the **Bianchi** identities and they can all be derived from the Jacobi identities of the commutators, namely

$$[D_m,[D_n,D_p]] + [D_n,[D_p,D_m]] + [D_p,[D_m,D_n]] \equiv 0 \ . \tag{6.233}$$

Thus in order to describe the interaction of matter with gravity, along the lines of the Equivalence Principle, we have introduced 16 vierbeins $e^{\mu}_{m}(x)$, and 24 connections $\omega_{\mu}{}^{mn}(x)$. In order to complete the picture we must indicate the dynamics obeyed by these new degrees of freedom.

The action must have the invariance group implied by the equivalence principle. Thus we have to build it out of the Lorentz invariants we have just constructed. Analogy with Yang-Mills theory would suggest an action of the form

$$\int d^4\xi R_{mn}{}^{pq} R^{mn}{}_{pq} \ . \tag{6.234}$$

It satisfies all invariance criteria but does not lead to the right answer; it has a naive dimension -4, just like Yang-Mills theories, and therefore introduces no dimensionful parameters in the theory . However the theory of gravitation, unlike Yang-Mills theories, has a fundamental dimensionful constant, Newton's constant G with cgs value

$$G = 6.6720 \times 10^{-8}\ cm^3/(g - sec^2)\ .$$

Thus the desirable action should leave room for the introduction of G , and thus must have the "wrong" dimensions. The simplest possibilities are then

$$\int d^4\xi\ \epsilon^{mnpq} R_{mnpq} \quad \text{and} \quad \int d^4\xi\ R_{mn}{}^{mn}\ . \tag{6.235}$$

The Einstein Action is the second one, given by

$$S_E = \frac{1}{16\pi G}\int d^4x E\ R_{mn}{}^{mn}\ , \tag{6.236}$$

where $E = det\ (e^m_\mu)$. It is a functional of both the vierbeins e^m_μ and the connections $\omega_\mu{}^{mn}$, and it corresponds to the first order formulation (*Palatini* formalism) of the equations of General Relativity.

In order to get the equations of motion, we have to vary with respect to both connections and vierbeins. We first vary with respect to the connections:

$$\delta_\omega S_E = \frac{1}{16\pi G}\int d^4x\ E\ \delta R \tag{6.237}$$

$$= \frac{1}{16\pi G}\int d^4x\ \delta\omega_\rho{}^{mn}[\partial_\mu\{E(e^\mu_m e^\rho_n - e^\mu_m e^\rho_n)\} - E(e^\rho_m e^\mu_q - e^\mu_m e^\rho_q)\omega_\mu{}^q{}_n \tag{6.238}$$

$$+ E(e^\mu_q e^\rho_n - e^\rho_q e^\mu_n)\omega_\mu{}^q{}_m] + \text{surface terms}\ , \tag{6.239}$$

giving the equations of motion (in the absence of matter)

$$D_\mu[E(e^\mu_m e^\rho_n - e^\mu_n e^\rho_m)] = 0\ . \tag{6.240}$$

This equation can be solved for the connections, with the result

$$\omega_\mu{}^{mn} = \frac{1}{2}e^q_\mu[T_q{}^{mn} - T^{mn}{}_q - T^n{}_q{}^m]\ , \tag{6.241}$$

where

$$T^q{}_{mn} = (e^\mu_m e^\rho_n - e^\mu_n e^\rho_m)\partial_\rho e^q_\mu\ . \tag{6.242}$$

Thus we see that as a result of the equations of motion, the connections are just auxiliary fields. One can even show that the equation of motion (6.4.78) implies that the torsion coefficients given by (6.4.59) vanish identically. A little bit of index shuffling (see problem) shows that (6.4.78) can be rewritten in the form

$$e^{\rho}_{q} S_{mn}{}^{q} - e^{\rho}_{m} S_{qn}{}^{q} + e^{\rho}_{n} S_{qm}{}^{q} = 0 \ . \tag{6.243}$$

Multiplication by e^{r}_{ρ} and contraction of m with r yield

$$S_{nq}{}^{q} = 0 \ , \tag{6.244}$$

which, by comparing with (6.4.81) implies

$$S_{mn}{}^{q} = 0 \ . \tag{6.245}$$

This is still true when gravity is minimally coupled to a scalar field, but the coupling to a spinor field yields a non-zero value for the torsion.

Furthermore, the curvature tensor suffers great simplifications as well; in particular the part of R_{mnpq} that is antisymmetric under the interchange of the pairs (mn) and (pq) vanishes identically, as does R^{*}.

The variation with respect to the vierbein is simplified by the absence of derivatives. Thus we write

$$\delta S_E = \frac{1}{16\pi G} \int d^4x [\delta E R_{mn}{}^{mn} + E(\partial_{\mu}\omega_{\rho}{}^{mn} - \omega_{\mu}{}^{rm}\omega_{\rho r}{}^{n})\delta(e^{\mu}_{m}e^{\rho}_{n} - e^{\mu}_{n}e^{\rho}_{m})] \ . \tag{6.246}$$

Now the variation of the determinant of any matrix $\mathbf{M}$ is given by

$$\begin{aligned} \delta \det \mathbf{M} &= \det(\mathbf{M} + \delta\mathbf{M}) - \det \mathbf{M} & (6.247) \\ &= e^{\mathrm{Tr}\ln(\mathbf{M}+\delta\mathbf{M})} - e^{\mathrm{Tr}\ln\mathbf{M}} & (6.248) \\ &\simeq e^{\mathrm{Tr}\ln\mathbf{M}} \mathrm{Tr}(\mathbf{M}^{-1}\delta\mathbf{M}) \ , & (6.249) \end{aligned}$$

so that

$$\delta E = E \ e^{\mu}_{m} \delta e^{m}_{\mu} \ . \tag{6.250}$$

Also, it is useful to note that

$$\delta e^{\rho}_{m} = -e^{\mu}_{m} e^{\rho}_{n} \delta e^{n}_{\mu} \ . \tag{6.251}$$

It is then straightforward, using (6.4.86) and (6.4.87), to obtain the equations of motion

$$R_{mn}{}^{pn} - \frac{1}{2}\delta^p_m R = 0 \; . \tag{6.252}$$

When the $\omega_\mu{}^{mn}$ are expressed in terms of the vierbeins and their derivatives, these are Einstein's equations of motion for General Relativity, in the absence of matter.

If we write the matter part of the Action S_M in the form

$$\delta S_M \equiv \int d^4x \; E[\frac{1}{2}\delta e^m_\mu T^\mu_m + \delta\omega_\mu{}^{mn} C_{mn}{}^\mu] \; , \tag{6.253}$$

thereby defining the sources T^μ_m and $C_{mn}{}^\mu$, we obtain the full Einstein equations

$$R_{mn}{}^{pn} - \frac{1}{2}\delta^p_m R = 8\pi G \; T^p_m \; ; \tag{6.254}$$

of course, T^p_m is the energy-momentum tensor of the matter. Similarly the variation with respect to the connections gives

$$D_\mu[E(e^\mu_m e^\rho_n - e^\mu_n e^\rho_m)] = 16\pi G \; C_{mn}{}^\rho \; . \tag{6.255}$$

These equations can still be solved, expressing the connections as the sum of two expressions, one involving the $C_{mn}{}^\rho$, the other being given by (6.4.79).

It might be necessary at this point to try to make contact with the more conventional treatments of General Relativity. We will see that the theory we have just obtained is exactly Einstein's theory, except for a technicality involving fermions. We will proceed algebraically. Consider the expression

$$e^m_\mu D_\rho V_m \; , \tag{6.256}$$

where V is any vector expressed in the favored frame. Explicitly

$$e^m_\mu D_\rho V_m = e^m_\mu[\partial_\rho V_m - \omega_{\rho m}{}^n V_n] \; . \tag{6.257}$$

But since $V_m = e^\mu_m V_\mu$, we can write

$$e^m_\mu \partial_\rho (e^\sigma_m V_\sigma) = e^m_\mu e^\sigma_m \partial_\rho V_\sigma + e^m_\mu (\partial_\rho e^\sigma_m) V_\sigma \; , \tag{6.258}$$

leading to

$$e^m_\mu D_\rho V_m = \partial_\rho V_\mu + [e^m_\mu \partial_\rho e^\sigma_m + \omega_{\rho m}{}^n e^\sigma_n] V_\sigma \tag{6.259}$$

$$\equiv \partial_\rho V_\mu + \Gamma^\sigma{}_{\mu\rho} V_\sigma \ , \tag{6.260}$$

where we have introduced the quantity

$$\Gamma^\sigma{}_{\mu\rho} \equiv e^m_\mu D_\rho e^\sigma_m \ . \tag{6.261}$$

It is to be identified with a connection (for the Greek indices), but it is not manifestly symmetric under the interchange of μ and ρ. However, we can still define a new covariant derivative as

$$\nabla_\rho V_\mu \equiv \partial_\rho V_\mu + \Gamma^\sigma{}_{\mu\rho} V_\sigma \ . \tag{6.262}$$

It acts exclusively on Greek indices. One can also define the generalization of the operator ∇_ρ acting on tensors with both Latin and Greek indices. Given any tensor T_{mp}, starting from

$$e^m_\mu D_\rho T_{mp} = e^m_\mu [\partial_\rho T_{mp} - \omega_{\rho m}{}^q T_{qp} - \omega_{\rho p}{}^q T_{mq}], \ , \tag{6.263}$$

and using

$$T_{mn} = e^\sigma_m T_{\sigma n} \ , \tag{6.264}$$

we arrive at

$$\partial_\rho T_{\mu p} + \omega_{\rho p}{}^q T_{\mu q} + \Gamma^\sigma{}_{\mu\rho} T_{\sigma q} \ , \tag{6.265}$$

which we identify with $\nabla_\rho T_{\mu p}$. Now if we set $T_{\mu p} = \eta_{pq} e^q_\mu$, we see immediately, from the definition of $\Gamma^\sigma{}_{\mu\rho}$, that

$$\nabla_\rho e_{\mu p} = \partial_\rho e_{\mu p} + \omega_{\rho p \mu} + e_{\sigma p} e^m_\mu D_\rho e^\sigma_m \tag{6.266}$$

$$= \partial_\rho e_{\mu p} - \partial_\rho e_{\mu p} - \omega_{\rho p \mu} - \omega_{\rho \mu p} \tag{6.267}$$

$$= 0 \ . \tag{6.268}$$

Similarly

$$\nabla_\rho e^{\mu p} = 0 \ . \tag{6.269}$$

The operator ∇_ρ is to be identified with the usual covariant derivative of the geometric formulation, except that the connection coefficients $\Gamma^\sigma{}_{\mu\rho}$ are not symmetric under $\mu \leftrightarrow \rho$; hence they cannot yet be identified with the Christoffel symbols.

Thus it is inherent to our formulation that the metric we have constructed satisfies the 40 equations

$$\nabla_\rho g_{\mu\nu} = 0 \ . \tag{6.270}$$

This should not be too surprising. We are only considering spaces which can be mapped at each point in space, by means of the vierbein, into the flat space of special relativity, and vice-versa. There are therefore some implicit restrictions made in our formalism, but they are motivated by the underlying physics of the Equivalence Principle.

We can now form the commutator of the Greek covariant derivative acting on quantities with only Greek indices; for instance

$$[\nabla_\nu, \nabla_\mu] T_\rho = S_{\nu\mu}{}^\sigma \nabla_\sigma T_\rho + R_{\nu\mu}{}^\lambda{}_\rho T_\lambda \ , \tag{6.271}$$

where the new torsion coefficients are just expressible in terms of the old torsion as

$$S_{\nu\mu}{}^\sigma = e_\nu^m e_\mu^n e_q^\sigma S_{mn}{}^q \ , \tag{6.272}$$

as well as by

$$S_{\nu\mu}{}^\sigma = \Gamma^\sigma{}_{\mu\nu} - \Gamma^\sigma{}_{\nu\mu} \ , \tag{6.273}$$

so that the torsion is seen to be proportional to the antisymmetric part of the Greek connections. We also get from (6.4.103) the Greek curvature tensor

$$R_{\nu\mu}{}^\lambda{}_\rho = \partial_\nu \Gamma^\lambda{}_{\rho\mu} + \Gamma^\sigma{}_{\rho\nu}\Gamma^\lambda{}_{\sigma\mu} - (\mu \leftrightarrow \nu) \ . \tag{6.274}$$

So, it looks exactly like the usual formalism, except that the Γ's are not symmetric. (In order to be able to identify the operation ∇_ρ with a derivative, certain integrability conditions must be met; it is easy to see that they are the Jacobi identities.)

In the absence of types of matter which can contribute to the torsion, the antisymmetric part of the Γ's vanishes, allowing for their identification with the Christoffel symbols. The theory we have described is then exactly the same as Einstein's. In the case of matter made up of scalar (or pseudoscalar) fields, there is no torsion, because the minimal matter action (6.4.33) does not contain the connections $\omega_\mu{}^{mn}(x)$.

The same is true for the gauge invariant electromagnetic interaction, although it may not appear to be, at first sight, torsionless. In this case the field strength tensor is constructed by forming the commutator of two gauge-covariant derivatives, but the covariant derivatives must be simultaneously covariant under both local internal gauge and Lorentz transformations, thus it reads

$$D_m = e^\mu_m(\partial_\mu + \frac{i}{2}\omega_\mu{}^{mn}X_{mn} + iA_\mu) \ , \tag{6.275}$$

so that the field strengths F_{mn} are read off from

$$[D_m, D_n] = S_{mn}{}^q D_q + \frac{i}{2}R_{mn}{}^{pq}X_{pq} + iF_{mn} \ , \tag{6.276}$$

giving in the Maxwell case,

$$F_{mn} = e^\mu_m e^\rho_n(\partial_\mu A_\rho - \partial_\rho A_\mu) \ , \tag{6.277}$$

as compared to the naive guess $D_mA_n - D_nA_m$, which is not correct. The reason is that gauge invariance must be maintained, $D_rA_s - D_sA_r$ is simply not gauge invariant. Thus we must apply the Equivalence Principle to the gauge covariant (invariant in the Maxwell case) field strengths. This is the second subtlety in applying the Equivalence Principle. Fermions on the other hand do give a non-zero contribution to the torsion. In the case of the Dirac Lagrangian, we have

$$S_D = \frac{1}{2}\int d^4x \ E\bar{\Psi}\gamma^p e^\mu_p(\partial_\mu + \frac{i}{2}\omega_\mu{}^{mn}\sigma_{mn})\Psi \ , \tag{6.278}$$

leading to a non-vanishing torsion coefficient

$$C_{mn}{}^\mu = \frac{i}{4}E \ e^\mu_p\bar{\Psi}\gamma^p\sigma_{mn}\Psi \ . \tag{6.279}$$

A spin 3/2 fermion is described in the absence of gravity by the Rarita-Schwinger Lagrangian

$$\mathcal{L}_{3/2} = \frac{1}{2}\bar{\Psi}_\mu\gamma_5\gamma_\rho\partial_\sigma\Psi_\nu \ \epsilon^{\mu\rho\sigma\nu} \ , \tag{6.280}$$

which involves a four compent Majorana vector-spinor Ψ_μ (the spinor index has been suppressed). The inclusion of a gravitational field, according to our recipe, reads

$$\mathcal{L}_{3/2} \to \frac{1}{2}\epsilon^{mnpq}\bar{\Psi}_m\gamma_5\gamma_n e^\mu_p e^\rho_q D_\mu\Psi_\rho + \text{c.c.} \quad , \tag{6.281}$$

where

$$D_\mu\Psi_\rho = (\partial_\mu + \frac{i}{2}\omega_\mu{}^{mn}\sigma_{mn})\Psi_\rho \ . \tag{6.282}$$

Note the subtlety: we have not blindly replaced the indices on the fields by Latin indices. The reason (again) is gauge invariance. The flat space Rarita-Schwinger Action is invariant under the gauge transformation

$$\Psi_\rho \to \Psi_\rho + \partial_\rho\chi \ , \tag{6.283}$$

and it forces us to apply the equivalence principle vierbein construction to the gauge invariant combination $\partial_\mu\Psi_\rho - \partial_\rho\Psi_\mu$. As an aside, we note that the Action

$$S = \frac{1}{16\pi G}\int d^4x \ ER + \int d^4x \ E \ \frac{1}{2}\epsilon^{mnpq}\bar{\Psi}_m\gamma_5\gamma_n e^\mu_p e^\rho_q D_\mu\Psi_\rho \ , \tag{6.284}$$

for a spin 3/2 field in interaction with gravity has an additional invariance – that of supersymmetry! This is in fact the action of (N=1) *supergravity*. This Action is invariant under the transformations

$$\delta e^m_\mu = \bar{\alpha}\gamma^m\Psi_\mu \tag{6.285}$$

$$\delta\Psi_\mu = D_\mu\alpha = \partial_\mu\alpha + \tfrac{i}{2}\omega_\mu{}^{mn}(\sigma_{mn}\alpha), \tag{6.286}$$

$$\delta\omega_\mu{}^{mn} = -\frac{1}{4}\bar{\alpha}\gamma_5\gamma_\mu(D_\rho\Psi_\sigma - D_\sigma\Psi_\rho)e^\rho_p e^\sigma_q\epsilon^{mnpq} \tag{6.287}$$

$$+ \frac{1}{4}\bar{\alpha}\gamma_5\gamma_\lambda(\epsilon^{\lambda m\rho\sigma}D_\rho\Psi_\sigma e^n_\mu - \epsilon^{\lambda n\rho\sigma}D_\rho\Psi_\sigma e^m_\mu) \tag{6.288}$$

Here $\alpha(x)$ is the infinitesimal parameter of a supersymmetry transformation; it is a Majorana four-component spinor.

Let us conclude this long section by touching upon the treatment of symmetries of the gravitational field. Consider some space-time point P. Let $\{x^\mu\}$and $\{\bar{x}^\mu\}$ represent two ways to label P. If the locally flat system is written as $\{\xi^m\}$, the two corresponding vierbeins are given by

$$e^m_\mu(x) = \frac{\partial\xi^m}{\partial x^\mu} \quad , \quad \bar{e}^m_\mu(\bar{x}) = \frac{\partial\xi^m}{\partial\bar{x}^\mu} \ , \tag{6.289}$$

so that they are easily related to one another through the chain rule.

Now suppose that the two ways of labeling P are equivalent for this particular gravitational field in the sense that the vierbeins are the same in both systems, *i.e.* that their *functional* dependence is the same. This happens whenever there is no physical difference between the two labeling schemes. Alternatively, we can say that an observer who labels P with x will see the same physics as one who labels P by $\bar{x}$. Whenever this is true there is a conserved symmetry operation in going from x to $\bar{x}$. Mathematically, this form invariance means that for an arbitrary label z,

$$e^m_\mu(z) = \bar{e}^m_\mu(z) \ , \tag{6.290}$$

whenever the difference between the two systems is a symmetry operation. In order to examine the consequences, let us specialize to an infinitesimal transformation

$$\bar{x}^\mu = x^\mu + \epsilon\zeta^\mu \ , \tag{6.291}$$

where ϵ is infinitesimal and $\zeta^\mu(x)$ is an arbitrary vector. The requirement of symmetry means that

$$e^m_\mu(x) = e^m_\alpha(x + \epsilon\zeta)(\delta^\alpha_\mu + \epsilon\partial_\mu\zeta^\alpha) \ . \tag{6.292}$$

Use of the Taylor expansion yields to lowest order the *Killing* equation

$$\zeta^\alpha\partial_\alpha e^m_\mu(x) + e^m_\alpha\partial_\mu\zeta^\alpha = 0 \ . \tag{6.293}$$

This equation represents the necessary condition on the gravitational field for the change ζ to be a symmetry operation. The vector ζ^α is called the *Killing* vector. If such a vector can be found for a given gravitational field, there is a corresponding symmetry operation. We leave it to the motivated reader to show that this equation is indeed the usual Killing equation (see problem).

6.4.1 PROBLEMS

A. Verify Eqs (6.4.31) and (6.4.55)

B. Show that in the Palatini formalism, variation of the Einstein action with respect to the connection implies vanishing torsion.

C. In four space-time dimensions, build all possible scalar invariants of dimension -4 which are built from the Riemann and torsion tensors only. Then repeat the procedure, this time including scalar fields, Dirac fields, gauge fields, and Rarita-Schwinger fields. In the last two cases be sure to respect the extra gauge invariances.

*D.Consider the following theory defined by the action

$$S = \int d^3x E R_{mn}{}^{pq} S_{pq}{}^{r} \epsilon^{mn}{}_{r} \ ,$$

in three space-time dimensions. R and S are the curvature and torsion tensors respectively, and E is the vierbein determinant. Derive the equations of motion and discuss their properties as far as you can.

**E. Consider a scalar field $\phi(x)$, coupled to a gravitational field. The coupling is minimal except for the extra term

$$\mathcal{L}_{extra} = \frac{f}{2} R \phi^2 \ ,$$

where R is the scalar curvature. Derive the equations of motion and show that there exist solutions where ϕ is constant.

When the potential density is given by

$$V = v(1 - 8\pi G f \phi^2)^2 \ ,$$

show that there exist an infinite set of such solutions. Discuss their meaning. Compute the one-loop radiative correction to this potential, keeping the gravitational field at the classical level. Show that it removes the degeneracy. Finally compute the vacuum value of the scalar field to O($\hbar$).