Journeys Beyond the Standard Model

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To My Girls

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Introduction

1.1 INTRODUCTION

As the study of the standard model requires many tools and techniques, it is useful to gather in one place salient facts about relativistic invariance, the description of fields of different spin, and the construction of gauge theories, spontaneous breaking, and group theory. This introductory chapter offers a short review of these basic ingredients. It also serves as a brief description of the notations and conventions we will follow throughout the book. The reader unfamiliar with these concepts is encouraged to consult standard texts on Advanced Quantum Mechanics and Field Theory.

1.2 Lorentz Invariance

We set $\hbar = c = 1$. Our metric is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The space-time coordinates are given by

$$x^\mu = (x^0, x^i) = (t, \vec{x}) \ ; \qquad x_\mu = g_{\mu\nu} x^\nu = (t, -\vec{x}) \ , \tag{1.1}$$

$$\partial_\mu = (\frac{\partial}{\partial t}, \vec{\nabla}) \ ; \qquad \partial^\mu = (\frac{\partial}{\partial t}, -\vec{\nabla}) \ . \tag{1.2}$$

The Lorentz-invariant inner product of two vectors is

$$x^2 = g_{\mu\nu} x^\mu x^\nu = x_\mu x^\mu = t^2 - \vec{x} \cdot \vec{x}. \tag{1.3}$$

We also have

$$\partial_\mu x^\rho = \delta_\mu^\rho \ , \tag{1.4}$$

where δ_μ^ρ is the diagonal Kronecker delta.

In four space-time dimensions, the algebra of the *Lorentz group* is isomorphic to that of $SU_2 \times SU_2$ (up to factors of i), the first generated by $\vec{J} + i\vec{K}$, the second by $\vec{J} - i\vec{K}$; $\vec{J}$ are the generators of angular momentum and $\vec{K}$ are the boosts. These two SU_2 are thus connected by complex conjugation ($i \to -i$) and/or parity ($\vec{K} \to -\vec{K}, \vec{J} \to \vec{J}$), and are therefore left invariant by the combined operation of CP. In accordance with this algebraic structure, spinor fields appear in two varieties, left-handed spinors, transforming under the first SU_2 as spin $\frac{1}{2}$ representations, and right-handed spinors transforming only under the second SU_2. They are represented by two-component complex spinor fields, called *Weyl spinors*,

$$\psi_L(x) \sim (\mathbf{2}, \mathbf{1}) \ , \qquad \psi_R(x) \sim (\mathbf{1}, \mathbf{2}). \tag{1.5}$$

The spinor fields must be taken to be anticommuting Grassman variables, in accordance with the Pauli exclusion principle. Their Lorentz transformation properties can be written in terms of the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.6}$$

which satisfy

$$\sigma_i \sigma_j = \delta_{ij} + i \ \epsilon_{ijk} \sigma_\kappa. \tag{1.7}$$

It is useful to express *Fierz transformations* in Weyl language. Let ζ and η be complex two-component Weyl spinors, each transforming as ($\mathbf{2}$,$\mathbf{1}$) of the Lorentz group. The first Fierz decomposition is

$$\zeta \eta^t \sigma_2 = -\frac{1}{2} \sigma_2 \eta^t \sigma_2 \zeta - \frac{1}{2} \sigma^i \eta^t \sigma_2 \sigma^i \zeta, \tag{1.8}$$

where t means transpose, corresponding to

$$(\mathbf{2}, \mathbf{1}) \otimes (\mathbf{2}, \mathbf{1}) = (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{1}). \tag{1.9}$$

We will find it useful to use the notation

$$\widehat{\eta} \equiv \eta^t \sigma_2. \tag{1.10}$$

Since the combinations $\sigma_2 \zeta^*$ and $\sigma_2 \eta^*$ transform according to the ($\mathbf{1}$,$\mathbf{2}$) representation, we have another Fierz relation

$$\zeta\eta^\dagger = -\frac{1}{2}\eta^\dagger\zeta - \frac{1}{2}\sigma^i\eta^\dagger\sigma^i\zeta, \tag{1.11}$$

associated with

$$(\mathbf{2},\mathbf{1}) \otimes (\mathbf{1},\mathbf{2}) = (\mathbf{2},\mathbf{2}). \tag{1.12}$$

The right hand side of this equation does indeed correspond to the Lorentz vector representation, since it can be written succintly as

$$\zeta\eta^\dagger = -\frac{1}{2}\overline{\sigma}^\mu\eta^\dagger\sigma_\mu\zeta, \tag{1.13}$$

in terms of the matrices

$$\sigma^\mu = (\sigma^0 = 1, \sigma^i) \ ; \qquad \overline{\sigma}^\mu = (\sigma^0 = 1, -\sigma^i). \tag{1.14}$$

The Pauli matrices property

$$\sigma_2\sigma^i\sigma_2 = -(\sigma^i)^t = -\sigma^{i*},$$

translates in the covariant language as

$$\sigma_2\overline{\sigma}^\mu\sigma_2 = (\sigma^\mu)^t. \tag{1.15}$$

The Lorentz group acts on the spinor fields as

$$\psi_{L,R} \to \mathbf{\Lambda}_{L,R}\psi_{L,R} \equiv e^{\frac{i}{2}\vec{\sigma}\cdot(\vec{\omega}\mp i\vec{\nu})}\psi_{L,R}, \tag{1.16}$$

where $\vec{\omega}$ and $\vec{\nu}$ are the real rotation and boost angles, respectively. This corresponds to the representation where

$$\vec{J} = \frac{\vec{\sigma}}{2} \ ; \qquad \vec{K} = -i\frac{\vec{\sigma}}{2}. \tag{1.17}$$

Note that the two-component Weyl spinors transform non-unitarily, as expected: unitary representations of non-compact groups, such as the Lorentz group, are infinite-dimensional.

This allows us to construct by simple conjugation left-handed spinors out of right-handed antispinors, and vice-versa. One checks that

$$\overline{\psi}_L \equiv \sigma_2\psi_R^* \sim (\mathbf{2},\mathbf{1}) \ , \qquad \overline{\psi}_R \equiv \sigma_2\psi_L^* \sim (\mathbf{1},\mathbf{2}). \tag{1.18}$$

Under charge conjugation the fields behave as

$$C\ :\ \psi_L \to \sigma_2 \psi_R^* \ , \qquad \psi_R \to -\sigma_2 \psi_L^*, \tag{1.19}$$

and under parity

$$P\ :\ \psi_L \to \psi_R \ , \qquad \psi_R \to \psi_L. \tag{1.20}$$

This purely left-handed notation for the fermions which we use in most of this book is especially suited to describe the parity violating weak interactions. For instance the neutrinos appear only as left-handed fields, while the antineutrinos are purely right-handed. On the other hand, fermions which interact in a parity invariant way as in QED and QCD, have both left- and right-handed parts. In that case, it is far more convenient to use the Dirac four-component notation. The fields ψ_L and ψ_R are then assembled in a four-component Dirac spinor

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{1.21}$$

on which the operation of parity is well-defined. This is the Weyl representation, in which the anticommuting Dirac matrices are (in 2×2 block form)

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \ , \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since one can generate right-handed fields starting from left-handed ones, it is enough to consider only polynomials made out of left-handed fields. For instance, the well-known group-theoretic product rule of the SU_2 representation

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}, \tag{1.22}$$

tells us that we can build a Lorentz scalar bilinear in the left-handed fields

$$\text{spin } 0 : \quad \widehat{\eta}_L^a \eta_L^b \ , \quad \text{symmetric under } a \leftrightarrow b, \tag{1.23}$$

where a and b label the fields, or else a space-time tensor bilinear

$$\text{tensor} : \quad \widehat{\eta}_L^a \sigma^i \eta_L^b \ , \quad \text{antisymmetric under } a \leftrightarrow b. \tag{1.24}$$

On the other hand, one needs both left and right handed fields to generate the currents

$$i(\eta_L^\dagger \eta_L,\ \eta_L^\dagger \vec{\sigma} \eta_L) \equiv i\ \eta_L^\dagger \sigma^\mu \eta_L. \tag{1.25}$$

One can check that their four components transform as $(\mathbf{2}, \mathbf{2})$, *i.e.* like a four-vector. The invariant actions built out of η_L and η_R are given by

$$\mathcal{L}_L = \frac{1}{2} \eta_L^\dagger \sigma^\mu \overleftrightarrow{\partial}_\mu \eta_L\ , \qquad \mathcal{L}_R = \frac{1}{2} \eta_R^\dagger \overline{\sigma}^\mu \overleftrightarrow{\partial}_\mu \eta_R, \tag{1.26}$$

representing the kinetic terms of the fermion fields. For charged fermions which have both left- and right-handed components, the kinetic term is just the parity-invariant sum of the two, leading to the massless Dirac equation. Since the action has units of $\hbar$, it is dimensionless in our natural system. It follows that the Lagrangian has units of (length)$^{-4}$, which is the same as (mass)^4. Since the derivative operator has the dimension of inverse length, we deduce that the fermion fields have engineering dimension of (length)$^{-3/2}$. Two Lorentz invariant spinor bilinears can appear in the Lagrangian. Using only one left-handed field, we can form only one complex spin-zero combination, $\widehat{\eta}_L \eta_L$. In order to appear in the Lagrangian, it must be added to its complex conjugate. It has engineering dimension of (length)$^{-3}$, and can enter the Lagrangian density with a prefactor with the dimension of mass. This type of mass is called the *Majorana mass*. It is only relevant for neutral fermions.

For charged fermions, which have independent left- and right-handed parts, the diagonal Majorana mass term is not allowed because of charge conservation. However, the off-diagonal Majorana mass is nothing but the Dirac mass term. To see this, consider the combination

$$\widehat{\eta}_L^1 \eta_L^2 = \psi_R^\dagger \psi_L, \tag{1.27}$$

where we have made the identifications $\eta_L^1 = \sigma_2 \psi_R^*$, $\eta_L^2 = \psi_L$. We can now form two real combinations

$$i\overline{\Psi}\Psi = i(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R)\ , \qquad \overline{\Psi}\gamma_5 \Psi = \psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R,$$

corresponding to scalar (Dirac mass) and pseudoscalar combinations, respectively.

Our notation is simply related to that introduced long ago by Van der Waerden. A left-handed spinor, transforming as the $(\mathbf{2}, \mathbf{1})$, is assigned an two-valued lower undotted index α, while the right-handed $(\mathbf{1},\mathbf{2})$ spinor is assigned a two-valued dotted index $\dot{\alpha}$. Each index can be raised by the antisymmetric $\epsilon^{\alpha\beta}$, or $\epsilon^{\dot{\alpha}\dot{\beta}}$. Thus the two-spinor singlet

$$\text{singlet} \ : \qquad \psi_\alpha \zeta_\beta \epsilon^{\alpha\beta} = \psi_\alpha \zeta^\alpha . \tag{1.28}$$

In this notation, the Lorentz vectors have both dotted and undotted indices.

1.2.1 PROBLEMS

A. Show that the combination $\overline{\psi}_L \equiv \sigma_2 \psi_L^*$ transforms as a right handed field, given that ψ_L is left-handed.

B. Given the transformation properties on the fields, show that the current combinations $i\eta_L^\dagger \sigma_\mu \eta_L$ does indeed transform as a four-vector.

C. Show that the kinetic term for a neutrino field is CP invariant. What happens to the Majorana mass under CP?

1.3 Gauge Fields

Gauge fields and their interactions are introduced through the covariant derivatives

$$\mathcal{D}_\mu = \partial_\mu + i\mathbf{A}_\mu \ , \tag{1.29}$$

where $\mathbf{A}_\mu$ is a matrix in the Lie algebra $\mathcal{G}$,

$$\mathbf{A}_\mu(x) = \sum_{A=1}^{N} A_\mu^A(x) \mathbf{T}^A \ , \tag{1.30}$$

and $\mathbf{T}^A$ are the matrices which generate $\mathcal{G}$; they are hermitian and satisfy the commutation relations

$$[\mathbf{T}^A, \mathbf{T}^B] = i f^{ABC} \mathbf{T}^C \ . \tag{1.31}$$

In the fundamental representation of the algebra, they are normalized as

$$\mathrm{Tr}(\mathbf{T}^A \mathbf{T}^B) = \frac{1}{2} \delta^{AB} \ . \tag{1.32}$$

Under the gauge transformations generated by the unitary matrices

$$\mathbf{U}(x) = e^{i\omega^A(x)\mathbf{T}^A} \ , \tag{1.33}$$

the covariant derivatives transform (as their name indicates) covariantly

$$\mathcal{D}_\mu \to \mathcal{D}'_\mu = \mathbf{U}\mathcal{D}_\mu\mathbf{U}^\dagger \ , \tag{1.34}$$

leading to the transformations on the gauge fields

$$\mathbf{A}_\mu \to \mathbf{A}'_\mu = \mathbf{U}\mathbf{A}_\mu\mathbf{U}^\dagger - i\mathbf{U}\partial_\mu\mathbf{U}^\dagger \ . \tag{1.35}$$

For infinitesimal gauge transformations, these correspond to

$$\delta A^A_\mu = -\partial_\mu\omega^A - f^{ABC}\omega^B A^C_\mu \ . \tag{1.36}$$

The covariant Yang-Mills field strengths are built from the covariant derivatives

$$\mathbf{F}_{\mu\nu} = -i[\mathcal{D}_\mu, \mathcal{D}_\nu] \ , \tag{1.37}$$

$$= \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + i[\mathbf{A}_\mu, \mathbf{A}_\nu] \ , \tag{1.38}$$

or in component form,

$$\mathbf{F}_{\mu\nu} = F^A_{\mu\nu}\mathbf{T}^A \ , \tag{1.39}$$

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - f^{ABC}A^B_\mu A^C_\nu \ . \tag{1.40}$$

The field strengths transform covariantly,

$$\mathbf{F}_{\mu\nu} \to \mathbf{U}\mathbf{F}_{\mu\nu}\mathbf{U}^\dagger \ , \tag{1.41}$$

and satisfy the integrability conditions (Bianchi identities)

$$\epsilon^{\alpha\beta\gamma\delta}\mathcal{D}_\beta\mathbf{F}_{\gamma\delta} = 0 \ . \tag{1.42}$$

Finally, the Yang-Mills action is given by

$$S_{YM} = -\frac{1}{4g^2}\int d^4x \mathrm{Tr}(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}) \ ; \tag{1.43}$$

it is invariant under both Lorentz and gauge transformations. The dimensionless Yang-Mills coupling constant g appears in the denominator, but it can be absorbed by redefining the potentials $\mathbf{A}_\mu \to g\mathbf{A}_\mu$, in which case it reappears in the interaction terms. One may build another quadratic invariant,

$$\int d^4x \mathrm{Tr}(\mathbf{F}_{\mu\nu}\mathbf{F}_{\rho\sigma})\epsilon^{\mu\nu\rho\sigma} \ . \tag{1.44}$$

It is easy to show that it does not affect the equations of motion because the integrand is a total derivative. It also differs from the Yang-Mills action in its properties under discrete symmetries. In quantum theory, where there is much more than the classical equations of motion, terms of this type do play an important role.

The gauge fields couple to matter (fermions, scalars) only through the covariant derivative, thus insuring gauge invariance. Consider a complex boson field $\Phi(x)$, with components $\varphi_a(x)$, $a = 1, 2, \cdots, n$, which transforms according to the n-dimensional representation of a gauge group G. Its coupling to the gauge fields is achieved by simply replacing the normal derivative by the covariant derivative

$$\partial_\mu\varphi_a \to (\mathcal{D}_\mu\varphi)_a = \partial_\mu\varphi_a + iA_\mu^A\mathbf{T}_{ab}^A\varphi_b \ . \tag{1.45}$$

Here the $\mathbf{T}^A$ are the $n \times n$ matrices which represent the Lie algebra in the representation of the scalar field. The invariant Lagrangian is just the absolute square

$$\mathcal{L} = (\mathcal{D}_\mu\Phi)^\dagger\mathcal{D}^\mu\Phi \ . \tag{1.46}$$

The coupling to fermions proceeds in the same way, by changing the normal derivative into the covariant derivative. This can be done equally well for Weyl and Dirac fermions. This results in the Weyl-Dirac Lagrangian

$$\mathcal{L}_{WD} = \mathbf{f}_L^\dagger\sigma^\mu\mathcal{D}_\mu\mathbf{f}_L \ , \tag{1.47}$$

where the left-handed fermion fields $\mathbf{f}$ transform as some representation of the gauge algebra,

$$\mathbf{f}_L \to \mathbf{f}'_L = \mathbf{U}\mathbf{f}_L \ . \tag{1.48}$$

One can form a similar Lagrangian for right-handed fields,

$$\mathcal{L}_{WD} = \mathbf{f}_R^\dagger\bar{\sigma}^\mu\mathcal{D}_\mu\mathbf{f}_R \ , \tag{1.49}$$

where the right-handed fields need not transform in the same way as the left-handed fields. When both helicities transform the same, the theory is invariant under parity, and it is said to be *vector-like*; otherwise they are said to be *chiral*. Chiral gauge theories run the risk of developing perturbative anomalies which spoil their renormalizability. Most groups have no such anomalies, except $U(1)$, $SU(n)$, for $n > 2$. The standard model is a chiral theory, which contains a $U(1)$ gauge group. Although it could in principle be anomalous, it escapes through a remarkable cancellation between quarks and leptons.

1.3.1 PROBLEMS

A. Show that with the gauge transformation Eq. (1.35), the covariant derivative does transform as advertised in the text.

B. Show that the quadratic invariant given by Eq. (1.44) is a divergence of a four-vector. Find its expression in terms of the gauge potentials.

1.4 Spontaneous Symmetry Breaking

First invented by Heisenberg to describe ferromagnetism, spontaneous symmetry breaking (SSB) is a ubiquitous phenomenon. It appears in superconductivity, superfluidity, is the mechanism by which the strong interactions break chiral invariance, and also describes the breaking of electroweak symmetry, to name but a few of its applications.

In quantum mechanics, symmetries play a central role. The dynamics of a quantum system is determined by the Hamiltonian $\mathcal{H}$ whose eigenvalues describe its allowed energies; to each energy eigenvalue correspond one or more states which contain the detailed information about the system at that energy, and satisfy

$$\mathcal{H}|\Psi_a\rangle = E_a|\Psi_a\rangle \ . \tag{1.50}$$

If the Hamiltonian is invariant under some symmetry operation represented by the operator $\mathcal{S}$, then

$$\mathcal{H} \to \mathcal{H}' = \mathcal{S}\mathcal{H}\mathcal{S}^{-1} = \mathcal{H} \ ; \tag{1.51}$$

if the symmetry is continuous and analytic, we have

$$[\mathcal{H}, T] = 0 \ , \tag{1.52}$$

where T is any of the generators of the symmetry. In that case, the state functions $\mathcal{S}|\Psi_a\rangle$ belong to the same eigenvalue as $|\Psi_a\rangle$. How does this analysis apply to the ground state of the system?

The ground state of a physical system is special in the sense that it is unique. Suppose it were not, and a student finds that the Hamiltonian he is considering has two states of lowest energy. She may envisage two possibilities. In the first, she finds that he made a mistake, and closer scrutiny indicates that there is a non-vanishing transition amplitude between the two “vacua”. This means that the starting Hamiltonian was not properly analyzed, and the effect of this transition must be included. This effect can be subtle, as in tunnelling. Rediagonalization breaks the degeneracy: the linear combination with a lower energy is the unique ground state. It could be that no mistake has been made, and there is no physical transition from one “vacuum” state to the other. Starting with each state, she may build a tower of excited states, with no physical transitions between the two towers. These two sets of states span different Hilbert spaces, and both describe a consistent physical system. In the absence of any criterion to distinguish them, they must yield the same physics. A trivial example is the Bohr atom with electron spin included in its kets, but not in the Hamiltonian, resulting in two non-communicating equivalent Hilbert spaces. Of course, when immersed in a magnetic field, the two Hilbert spaces mesh into one, and one unique ground state emerges after diagonalization of the new Hamiltonian.

Since the ground state $|\Omega\rangle$ is unique, what then is $S|\Omega\rangle$ when S is a symmetry of the Hamiltonian? The first possibility is that the ground state is invariant under all symmetries, in which case $S|\Omega\rangle = |\Omega\rangle$, up to a phase, that is $T|\Omega\rangle = 0$ for infinitesimal generators. This is true for many physical systems on which the symmetry is then said to be *linearly realized.* The symmetry is manifest in the degeneracy of the excited states, and can be even be deduced from the multiplicity of the degenerate states.

A prototypical example is the degeneracy structure of the states of the Bohr Hydrogen atom. It has a unique ground state, on which the symmetry operations are trivial. The first excited state contains four states; one s-wave, and three p-wave states. The rotation symmetry of the Hamiltonian is identified by recognizing the degenerate states as representations of the rotation group. Since there are two different representations of angular momentum at the first excited level, it hints of a symmetry more general than the rotation symmetry. There must be three extra symmetry operations, each rotating the s-wave singlet into one component of the p-wave triplet. This symmetry with six generators (three rotations and three "others") is the hidden, but well-known, Runge-Lenz symmetry of the Hydrogen atom. It is identified with the group $SO(4)$, and the four states of the first excited level act as the components of its four dimensional representation. The symmetry of the Hamiltonian has been linearly realized on its eigenstates.

However, another possibility may arise. It may happen that under the action of the symmetry operation on the ground state, a different state is produced, so that the symmetry generators do not vanish on the ground state, even though the Hamiltonian is symmetric. If the symmetry is indeed present in the Hamiltonian, but not on its ground state, the symmetry is said to be *spontaneously* broken. The vacuum happens not to be an eigenstate of the symmetry operations, whose action on the vacuum generates other states of zero energy. If the symmetry is discrete, there is a finite number of such vacuum states; when the symmetry operation is continuous and compact, the action of the symmetry yields a closed family of equivalent ground states parametrized by the parameters of the symmetry. All of these vacuum states are equivalent, each belonging to a different Hilbert space. Since there can be only one state of lowest energy, the resulting physics of this system must not depend on the angles that parametrize these states.

Since to each possible "ground state" there corresponds an equivalent physical theory, *any* one of them can serve as *the* ground state. This chosen vacuum state is no longer invariant under the action of the symmetry group; the symmetry is spontaneously broken, an unfortunate description since the symmetry is not really broken, just expressed differently. The symmetry is *non-lineraly realized* on the states; it is still encoded in the system, but in a more subtle way. In fact, there is a limit in which one can recover the symmetry in its linear realization. The degeneracy of highly excited states, with energies much larger than the energy scale associated with the breaking of the symmetry, will reflect the original symmetry of the system.

The celebrated example of spontaneous symmetry breaking is the Heisenberg ferromagnet, described by a lattice of spins in interaction with one another. These spins can only take two values; they can be either "up" or "down". The Hamiltonian is invariant under the "up-down" symmetry. In the ground state, the spins are either all aligned or anti-aligned, depending on the sign of the coupling between the spins. If the anti-aligned configuration is favored, there is no preferred "up" or "down" direction, and the vacuum preserves the "up-down" symmetry. On the other hand, there are two possible ground state configurations with all the spins aligned, "up-up" and "down-down". For either choice, the "up-down" symmetry has been broken, since there is a preferred direction in both configurations. The Heisenberg Hamiltonian is given by

$$\mathcal{H} = -J \sum_{<ij>} \sigma_i \cdot \sigma_j \ , \tag{1.53}$$

where $\sigma_i = \pm$, and the sum is over nearest neighbors. It is invariant under a change of sign of the two spins. For $J > 0$, the Hamiltonian is at minimum when all the spins are aligned, breaking the "up-down" invariance. In this example, the broken symmetry is discrete, but it is not hard to generalize the model to continuous symmetry by replacing the elementary spins by normalized three-dimensional spin vectors at each site, $\vec{S}_i$.

We are primarily interested in describing spontaneous breaking in the context of field theory. To do so, we start with a simple system with an Abelian symmetry, a complex scalar field $\phi(x)$, with Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi^* \phi) \ , \tag{1.54}$$

where the potential is chosen to be of the form

$$V(\phi^* \phi) = \lambda(\phi^* \phi - \frac{v^2}{2})^2 \ ; \tag{1.55}$$

This Lagrangian is clearly invariant under the $U(1)$ transformation

$$\phi \to e^{i\alpha} \phi \ , \tag{1.56}$$

where α is a constant phase. Using Noether methods, we obtain a conserved current

$$j_\mu = i \phi^* \overleftrightarrow{\partial}_\mu \phi \ , \qquad \partial_\mu j^\mu = 0 \ . \tag{1.57}$$

The field configurations which yields a minimum energy density are constant in space time; otherwise the kinetic energy term would give a positive contribution. Its actual values are determined by minimizing the potential,

$$\phi^* \phi = \frac{v^2}{2} \ . \tag{1.58}$$

There are therefore an infinite number of possible vacuum field configurations, all with zero energy,

$$\phi_0 = \frac{1}{\sqrt{2}} v e^{i\theta} \ , \tag{1.59}$$

each labelled by an angle θ. Since the symmetry is compact, they form a closed circle of states. We note that while the continuous phase symmetry is broken by these configurations, the vacuum condition leaves one discrete symmetry,

$$\Phi_0 \to -\Phi_0 \ , \tag{1.60}$$

so that the angle θ is defined *modulo* π. We expand the field away from this minimum configuration

$$\phi(x) = \frac{1}{\sqrt{2}} e^{i\frac{\xi(x)}{v}} (v + \rho(x)) \ , \tag{1.61}$$

a parametrization introduced by *Kibble*. The phase θ has been absorbed in the definition of the field $\xi(x)$. Rewriting the Lagrangian in terms of these new fields, we obtain

$$\mathcal{L} = \frac{1}{2}\partial_\mu \rho \partial^\mu \rho + \frac{1}{2}\partial_\mu \xi \partial^\mu \xi (1 + \frac{\rho}{v})^2 - \frac{1}{2} m^2 \rho^2 - \frac{\lambda}{4}\rho^4 - v\lambda\rho^3 \ . \tag{1.62}$$

We recognize kinetic terms for two fields, $\rho(x)$ and $\xi(x)$. The first has a mass $m = v\sqrt{2\lambda}$, the second is massless. The minimum of the energy density now corresponds to zero values for both fields. The Lagrangian is invariant under the constant shift

$$\xi(x) \to \xi(x) + \theta \ , \tag{1.63}$$

with the ρ field unchanged. This shift is in one-to-one correspondance with the original $U(1)$ phase symmetry. Its linear realization on ϕ is replaced by a non-linear realization (by a shift) on the massless ξ field, which represents the *Nambu-Goldstone boson.* The invariance of the Lagrangian under a constant shift indicates that all vacua differing by their value of θ are indeed physically equivalent.

The Nambu-Goldstone boson couples to itself and to the other field in a very special way, restrited by the shift invariance. To see it, we rewrite (after integration by parts) the coupling of the Nambu-Goldstone boson in the more transparent form

$$\mathcal{L}_{int} = -\frac{1}{v}\xi(x)\partial_\mu j^\mu(x) \ , \tag{1.64}$$

since the current is just

$$j_\mu = -v(1 + \frac{\rho(x)}{v})^2 \partial_\mu \xi(x) \ . \tag{1.65}$$

Under a constant shift of the Nambu-Goldstone field, the Lagrangian density picks up a total divergence, which does not affect the equations of motion. This is a general feature: Nambu-Goldstone bosons couple to the divergence of the current that was broken, with strength proportional to the inverse of the scale of breaking, v. It is the dynamical variable associated with the angle that parametrizes the continuous circle of minima.

It is instructive to verify that these features generalize when we add fermions. This is done through the Yukawa coupling

$$\mathcal{L}_{Yu} = y\chi_L^\dagger(x)\chi_R(x)\phi(x) + \text{c.c} \ ; \tag{1.66}$$

invariance is retained as long as the fermions transform as

$$\chi_L \to e^{-i\alpha}\chi_L \ ; \qquad \chi_R \to \chi_R \ . \tag{1.67}$$

It is a bit more involved to show that the NG boson still couples to the divergence of the current, and we leave it as a problem.

In generalizing this discussion to non-Abelian symmetries, we find the mathematics to be slightly more involved, but the conclusions to be the same as in the Abelian case. To each broken continuous symmetry corresponds a massless Nambu-Goldstone boson, which couples to the divergence of the broken current. One new feature is that generally not all the continuous symmetry is spontaneously broken, and it is sometimes tricky to identify the unbroken symmetry.

To study this case, we consider the example of a real field $\Phi(x)$, with components $\varphi_a(x)$, $a = 1, 2, \ldots, N$, with interactions described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi^a - V(\varphi_a\varphi^a) \ , \tag{1.68}$$

where the sum over a is implicit. It is clearly invariant under the transformations

$$\Phi \to \Phi' = \mathbf{R}\Phi(x) \ , \tag{1.69}$$

where $\mathbf{R}$ is a constant rotation matrix

$$\mathbf{R} = e^{i\vec{\theta}\cdot\vec{\mathbf{T}}} \ , \qquad \mathbf{R}\mathbf{R}^t = 1 \ , \tag{1.70}$$

written in terms of the $N(N-1)/2$ antisymmetric real matrices $\mathbf{T}^A$ that generate the Lie algebra of $SO(N)$. To each of these generators corresponds a conserved current

$$J^A_\mu = i\Phi^t\mathbf{T}^A\overleftrightarrow{\partial}_\mu\Phi \ , \qquad \partial^\mu J^A_\mu = 0 \ . \tag{1.71}$$

The potential is chosen to be

$$V = \lambda(\Phi^t\Phi - \frac{v^2}{2})^2 \ , \tag{1.72}$$

so as to produce an infinite number of minimum field configurations written in the form

$$\Phi_0 = \frac{1}{\sqrt{2}}e^{i\vec{\theta}\cdot\vec{\mathbf{K}}}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix} \ . \tag{1.73}$$

The vacuum expectation value v has been aligned along the Nth component of the vector; this entails no loss of generality, since it can always be aligned this way by a suitable rotation. The vacuum value of Φ clearly singles out one direction in the N dimensional internal space. It breaks the $SO(N)$ symmetry, but all rotations in the $(N-1)$-dimensional plane perpendicular to that direction are left invariant. The generators of the $(N-1)$ broken symmetries are denoted by $\vec{\mathbf{K}}$. The vacuum configuration depends on the $(N-1)$ angles which parametrize these broken rotations. The generators of $SO(N)$ can be decomposed in two classes

$$\vec{\mathbf{T}} = \vec{\mathbf{K}} \oplus \vec{\mathbf{H}} \ , \tag{1.74}$$

where the generators $\vec{\mathbf{H}}$ generate the Lie algebra of the unbroken subalgebra, $SO(N-1)$,

$$[H_a, H_b] = if_{abc}H_c \ , \quad a, b, c, = 1\cdots, \frac{(N-1)(N-2)}{2} \ , \tag{1.75}$$

and the $\mathbf{K}_i$, $i = 1, \cdots, N-1$ no longer form a Lie algebra, and transform as a $(N-1)$ vector under the unbroken symmetry, $SO(N-1)$. They span a space called the *coset* space. Their commutators yield generators of the unbroken subalgebra. Symbolically we may write

$$[H, H] \subset H \ ; \qquad [H, K] \subset K \ ; \qquad [K, K] \subset H \ . \tag{1.76}$$

Note that the generators of the unbroken subalgebra vanish on the vacuum. The coset generators do not vanish on the vacuum, although their commutators do.

We expand $\Phi(x)$ away from its vacuum configuration *à la Kibble*,

$$\Phi(x) = \frac{1}{\sqrt{2}} e^{i\frac{\vec{\xi}}{v}\cdot\vec{\mathbf{K}}} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v+\rho(x) \end{pmatrix} , \tag{1.77}$$

$$\equiv \frac{1}{\sqrt{2}} \mathbf{U}(x)(v+\rho(x))\mathbf{n}_0 \ , \tag{1.78}$$

where $\mathbf{n}_0$ is a unit vector pointing along the Nth direction. As before the potential is simply written in terms of the ρ field alone,

$$V = \frac{1}{2} m^2 \rho^2 + \frac{\lambda}{4} \rho^4 + v\lambda\rho^3 \ . \tag{1.79}$$

The kinetic part of the Lagrangian yields

$$\frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} \partial_\mu \xi^i \partial^\mu \xi^j (1 + \frac{\rho}{v})^2 \mathbf{n}_0^t \frac{\partial}{\partial \xi^i} \mathbf{U}^\dagger \frac{\partial}{\partial \xi^j} \mathbf{U} \mathbf{n}_0 \ , \tag{1.80}$$

which again exhibits the canonical kinetic term for the ρ field. The indices i, j run over the $(N-1)$ dimensions of the coset. The kinetic piece for the ξ^i fields can be simplified considerably. For a general group element

$$\mathbf{U} = e^{i\vec{\theta}\cdot\vec{\mathbf{T}}} \ , \tag{1.81}$$

it can be shown that

$$\frac{\partial}{\partial \theta^A} \mathbf{U} = i\mathbf{U} X_{AB} \mathbf{T}^B \ , \tag{1.82}$$

where the indices A, B run over the whole algebra, where the matrix $\mathbf{X}$ is given by

$$X_{AB} = \delta_{AB} + \frac{1}{2!}(\vec{\theta} \cdot \vec{F})_{AB} + \frac{1}{3!}((\vec{\theta} \cdot \vec{F})^2)_{AB} + \cdots \ , \tag{1.83}$$

where the $\mathbf{F}^A$ are the antisymmetric matrices that represent the algebra in its adjoint representation; they are simply related to the structure functions

$$(F^A)_{BC} = -if^A{}_{BC} \ . \tag{1.84}$$

It follows that

$$\frac{\partial}{\partial\theta^A}\mathbf{U}^\dagger = -i\mathbf{T}^B X^{-1}_{BA}\mathbf{U}^\dagger \ , \tag{1.85}$$

Applying these to the case in hand, we obtain

$$\begin{aligned} \mathbf{n}_0^t \frac{\partial}{\partial\xi^i}\mathbf{U}^\dagger \frac{\partial}{\partial\xi^j}\mathbf{U}\mathbf{n}_0 = &\quad X^{-1}_{li} X_{jk} \mathbf{n}_0^t \mathbf{K}^l \mathbf{K}^k \mathbf{n}_0 \ , \\ = &\quad X^{-1}_{Ci} X_{jD} \delta^{CD} = \delta^{ij} \ , \end{aligned}$$

since the unbroken group generators vanish on the unit vector $\mathbf{n}_0$. Thus the full Lagrangian looks much the same as in the Abelian case,

$$\frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}(1+\frac{\rho}{v})^2\partial_\mu\xi^i\partial^\mu\xi_i - \frac{1}{2}m^2\rho^2 - \frac{\lambda}{4}\rho^4 - v\lambda\rho^3 \ . \tag{1.86}$$

The only difference is that there are now as many *Nambu-Goldstone bosons* as there are broken symmetries, and we have invariance under $(N-1)$ constant shifts

$$\xi_i(x) \to \xi_i(x) + \theta_i \ . \tag{1.87}$$

These invariances are made obvious made manifest by writing the couplings of the Nambu-Goldstone bosons in terms of the divergence of the broken currents,

$$\mathcal{L}_{int} = \frac{1}{v}\xi^i(x)\partial^\mu J^i_\mu(x) = -\frac{1}{v}J^i_\mu(x)\partial^\mu\xi^i(x) \ , \tag{1.88}$$

where we have integrated by parts, and dropped the surface terms. In this case, we see that the unbroken symmetry $SO(N-1)$ is realized linearly on the Nambu-Goldstone bosons which transform as a $(N-1)$-dimensional vector,

$$\Xi(x) \to e^{i\vec{\theta}\cdot\vec{\mathbf{K}}}\Xi(x) \ , \tag{1.89}$$

where Ξ is the vector with components $\xi(x)$. The number of degrees of freedom has not changed; we started with N fields, and we end up with $(N-1)$ massless Nambu-Goldstone bosons and one massive Higgs boson.

This form of the interaction between the Nambu-Goldstone bosons and the curents is quite suggestive, when compared to the interaction of a gauge field and the same current.

Suppose we gauge the starting symmetry, in this case $SO(N)$. We expect as many gauge potentials as there are generators of $SO(N)$. They interact with the currents J^A_μ via the interaction term

$$\mathcal{L}_{int} = gA^A_\mu J^{A\mu} \ . \tag{1.90}$$

By comparing Eqs. (1.88) and (1.90) for the broken directions, we note that the interaction term of the Nambu-Goldstone bosons can be cancelled exactly by performing the gauge transformation

$$\delta A^i_\mu = -\frac{1}{g}\partial_\mu \xi^i + f^{ijk}A^j_\mu \xi^k \ . \tag{1.91}$$

Hence the Nambu-Goldstone bosons are to be viewed as gauge artifacts. They no longer correspond to physical particles, since they can be absorbed by redefining the gauge fields. Each gauge field that corresponds to a broken symmetry acquires an extra degree of freedom, and becomes massive, to keep the description relativistic.

In order to see exactly what happens, we start from the Lagrangian density (1.68), with the normal derivative replaced by the covariant derivative

$$\partial_\mu \Phi \to (\partial_\mu + ig\mathbf{A}_\mu)\Phi \ , \tag{1.92}$$

with

$$\mathbf{A}_\mu = \mathbf{T}^A A^A_\mu \ ,$$

where the $N \times N$ matrices $\mathbf{T}^A$ generate the Lie algebra of $SO(N)$. We now simply retrace the steps starting with the vacuum configuration (1.73).

The only difference is that the covariant derivative on the vacuum configuration leaves behind a term linear in those gauge potentials associated with the broken generators

$$\mathcal{D}_\mu \Phi = \left(\partial_\mu + igA^i_\mu(x)\mathbf{T}^i\right) \frac{v + \rho(x)}{\sqrt{2}} \hat{n}_0 \ . \tag{1.93}$$

When squared in the Lagrangian, these yield a mass term for these gauge fields which correspond to the broken directions. This is the Higgs mechanism: the gauge potential associated to each broken generator becomes massive, after "eating" the Nambu-Goldstone boson, which then serves as its third degree of freedom. The gauge potentials along unbroken directions stay massless. The advantage of giving a mass to the gauge bosons in this particular way is that it does not affect the ultraviolet properties of the theory. This is equivalent to the solid state case where spontaneous symmetry breaking, being a property of the vacuum is more apparent on states near the vacuum and become less relevant for the highly excited states. Thus a spontaneously broken gauge theory has the same renormalizability properties as an unbroken one.

1.4.1 PROBLEMS

A. When fermions are added in the Abelian model with the Yukawa coupling of Eq. (1.66) find the new expression for the current and show that the Nambu-Goldstone boson still couples to the divergence of the broken current.

B. Show that the vacuum configuration Φ_0 is left invariant by $SO(N-1)$ rotations.

C. Verify the form of the $\mathbf{X}$ matrix given by Eq. (1.83), and show that the kinetic term for the Nambu-Goldstone bosons is indeed invariant under constant shifts.

D. 1-) Show that, in the Non-Abelian case, the couplings of the Nambu-Goldstone bosons can be expressed solely in terms of the divergences of the currents associated with the broken generators.

2-)Add to the theory a set of N left- and right-handed fermions which transform as the vector representation of $SO(N)$. Write an $SO(N)$ invariant Yukawa coupling of these fermions to the Higgs. Show that the Nambu-Goldstone bosons still couple to the divergences of the broken currents.

1.5 Group Theory

This section summarizes the essentials group-theoretic facts the reader will need to follow some of the material in this book. Much of it follows the presentation that can be found in R. Slansky, *Phys. Reports* **79**, 1(1981). Continuous groups of physical interest are generated by Lie algebras. The semisimple Lie algebras, classified by Cartan a century ago, come in four infinite families, $A_n,\ B_n,\ C_n,\ D_n$, $n=1,2...$, and in five exceptional algebras, $G_2,\ F_4,\ E_6,\ E_7,\ E_8$. The subscript denotes the the *rank* of the algebra, the number of its commuting generators. Here follows a brief description:

• The unitary series, A_n , $n=1,2,\cdots$, is called SU_{n+1} by physicists. This Lie algebra is generated by the n^2-1 independent $n\times n$ traceless hermitian matrices. They generate transformations that leaves the real quadratic form

$$z_1^* z_1 + z_2^* z_2 + \cdots + z_n^* z_n \ , \tag{1.94}$$

invariant, where the z_i are complex numbers. Acting on the complex n-dimensional vector, these transformations are represented by unitary matrices. They are expressed as exponentials of i times hermitian matrices

$$U(\theta_A) = e^{i\theta_A \mathbf{T}^A} \ , \tag{1.95}$$

where θ_A are real parameters, $A=1,2,\ldots,n^2-1$, and

$$U^\dagger = U^{-1} \quad \to \quad \mathbf{T}^{A\dagger} = \mathbf{T}^A \ . \tag{1.96}$$

• The infinite orthogonal series which generates transformations that leave the quadratic form

$$x_1^2 + x_2^2 + \cdots + x_n^2 \ , \tag{1.97}$$

invariant, where the x_i are real numbers. These transformations are represented by orthogonal *real* matrices which are exponentials of antisymmetric hermitian $n\times n$ matrices. There are $n(n-1)/2$ such matrices.

$$U(\omega_A) = e^{i\omega_A \mathbf{T}^A} \ , \tag{1.98}$$

where the ω_A are the $n(n-1)/2$ real rotation angles, and

$$U^T = U^{-1} \ \to \ \mathbf{T}^{A\dagger} = \mathbf{T}^A \ . \tag{1.99}$$

These transformations are called SO_n by physicists. However they have very different structure depending on whether n is even or odd. Accordingly, mathematicians split them up in two infinite series, B_n and D_n, corresponding to SO_{2n+1} and SO_{2n}, with $n(2n+1)$ and $n(2n-1)$ generators respectively. While identical in their tensor representations, their spinor representations are very different. Each B_n has one real fundamental spinor irreducible representation (irrep). D_n have two fundamental spinor irreps; for n even they are real, and for n odd they are conjugate of one another.

• The symplectic series C_n, which we denote by Sp_{2n} generate transformations that leave invariant the *real antisymmetric* quadratic form

$$\sum_{i,j}^{n} x_i J_{ij} y_j \ , \tag{1.100}$$

with the antisymmetric matrix $\mathbf{J}$ given by

$$\mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \ , \tag{1.101}$$

where I is the unit matrix in n dimensions. Symplectic transformations are represented by symmetric $2n \times 2n$ real matrices with $n(2n+1)$ parameters.

• There are five exceptional Lie algebras, G_2, F_4, E_6, E_7, E_8, with dimensions 14, 26, 78, 133, and 248, respectively. It is not as easy to characterize the groups generated by these algebras as they leave invariant not only quadratic but also higher-order polynomials.

All these Lie algebras have common features as they consist of $d_\mathbf{a}$ hermitian matrices $\mathbf{T}^A$, $A = 1, 2..., d_\mathbf{a}$, which satisfy an algebra of the form

$$[\mathbf{T}^A, \mathbf{T}^B] = i f^{ABC} \mathbf{T}^C \ , \tag{1.102}$$

implying that these matrices must be traceless. The real antisymmetric f^{ABC} are called the *structure functions* of the algebra. Since matrices naturally satisfy the Jacobi identity

$$[\mathbf{T}^A, [\mathbf{T}^B, \mathbf{T}^C]] + [\mathbf{T}^B, [\mathbf{T}^C, \mathbf{T}^A]] + [\mathbf{T}^C, [\mathbf{T}^A, \mathbf{T}^B]] = 0 \ , \tag{1.103}$$

it follows that the f^{ABC} obey the relations

$$f^{ADE} f^{BCD} + f^{BDE} f^{CAD} + f^{CDE} f^{ABD} = 0 \ . \tag{1.104}$$

We can therefore define a $d_{\mathbf{a}} \times d_{\mathbf{a}}$ matrix

$$(T^C)_{AB} \equiv -if_{ABC} \ , \tag{1.105}$$

which satisfies the same Lie algebra. These matrices form a real representation of the Lie Algebra in a $d_{\mathbf{a}}$-dimensional vector space called the *adjoint* representation; its dimension is equal to the number of generators. Each Lie algebra has its own unique adjoint representation.

Lie algebras can also be represented in different finite-dimensional Hilbert space, in infinitely many ways. The properties of these representations are the starting point for physical applications. Each representation is labelled by a number of invariant combinations of the generators, called *Casimir operators*; a Lie algebra of rank l contains l Casimir operators. The $\mathbf{T}_{\mathbf{r}}^A$ matrices which represent the algebra in the $\mathbf{r}$ representation satisfy a normalization condition

$$\mathrm{Tr}({\mathbf{T}_{\mathbf{r}}}^A {\mathbf{T}_{\mathbf{r}}}^B) = C_{\mathbf{r}} \delta^{AB} \ , \tag{1.106}$$

where the coefficient $C_{\mathbf{r}}$ is called the *Dynkin index* of the representation. By multiplying by δ^{AB} and summing over A, B, we find

$$d_{\mathbf{r}} C^{[2]}(\mathbf{r}) = C_{\mathbf{r}} d_{\mathbf{a}} \ , \tag{1.107}$$

where $d_{\mathbf{r}}$ is the dimension of the representation, $d_{\mathbf{a}}$ is the number of generators of the Lie algebra, and $C^{[2]}(\mathbf{r})$ is the quadratic Casimir operator of the representation $\mathbf{r}$, defined through

$$\sum_A^{d_{\mathbf{a}}} \mathbf{T}^A \mathbf{T}^A = C^{[2]}(\mathbf{r}) \mathbf{I} \ , \tag{1.108}$$

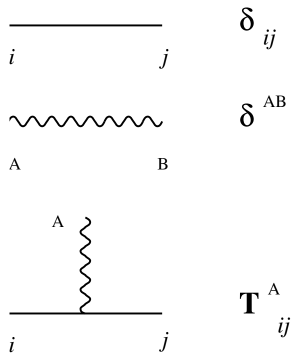
where $\mathbf{I}$ is the $d_{\mathbf{r}} \times d_{\mathbf{r}}$ unit matrix. The Dynkin index is the first of the group-theoretic numbers associated with each representation. We single it out because of its importance in physical applications.

Representations can be multiplied together to yield other representations. An elementary example is that of vector multiplication. A real vector in three dimensions transforms as the $\mathbf{3}$ representation of the orthogonal group $SO(3)$. We learn in high school that, given two vectors, we can either form another vector by taking their antisymmetric (cross) product, or form a rotation invariant by taking their symmetric dot product, or else form a five-component symmetric traceless second rank tensor (quadrupole). In group-theoretic terms, this would simply read

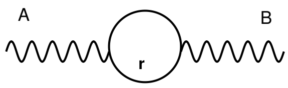
$$\mathbf{3} \otimes \mathbf{3} = \mathbf{3}_A \oplus (\mathbf{5} \oplus \mathbf{1})_S \ . \tag{1.109}$$

Each algebra has a fundamental representation out of which one can generate all of its representations by repeatedly taking their (Kronecker) product.

There is a useful graphical way to represent group-theoretic numbers associated with any representation. We represent the $\mathbf{r}$ representation by a solid line, and the adjoint representation by a wavy line. The matrices in the representation $\mathbf{r}$ can then be written as a vertex between two solid lines and a wavy line:



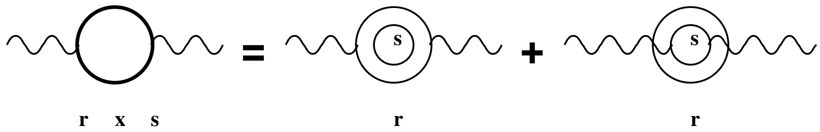
The Dynkin index is represented by the vacuum polarization



This graphical representation provides an easy way to deduce the Dynkin index of other representations. The Dynkin indices of two representation $\mathbf{r}$ and $\mathbf{s}$ of dimensions $d_{\mathbf{r}}$ and $d_{\mathbf{s}}$ are related to that of the compound $\mathbf{r} \otimes \mathbf{s}$ reducible representation through

$$C_{\mathbf{r}\otimes\mathbf{s}} = d_{\mathbf{r}} C_{\mathbf{s}} + d_{\mathbf{s}} C_{\mathbf{r}} \ , \tag{1.110}$$

which can be understood graphically



To derive this, we have used two facts: a closed line without any external line is equal to the dimension of the representation, and a closed loop with one adjoint external line is zero, since the matrices are traceless. We can go further and compute the Dynkin index of the symmetric or antisymmetric products of representations, using our graphical representation.

Consider the group SU_n. It is represented in a complex n-dimensional vector space by one-half times the Gell-Mann matrices, $\boldsymbol{\lambda}^A$. Thus

$$\mathrm{Tr}\left(\frac{\boldsymbol{\lambda}^A}{2}\frac{\boldsymbol{\lambda}^B}{2}\right) = \frac{1}{2}\delta^{AB} \ , \tag{1.111}$$

so that the Dynkin index of the fundamental representation is $C_{\mathbf{n}} = \frac{1}{2}$, independent of n. One can check that the matrices $-(\boldsymbol{\lambda}^A)^t/2$ represent the algebra in the conjugate representation, $\overline{\mathbf{n}}$, so that $C_{\mathbf{n}} = C_{\overline{\mathbf{n}}}$. It is now easy to deduce the Dynkin index of the adjoint representation $\mathbf{n^2} - \mathbf{1}$ which appears in the product

$$\mathbf{n} \otimes \overline{\mathbf{n}} = (\mathbf{n^2} - \mathbf{1}) \oplus \mathbf{1} \ . \tag{1.112}$$

The composition law yields

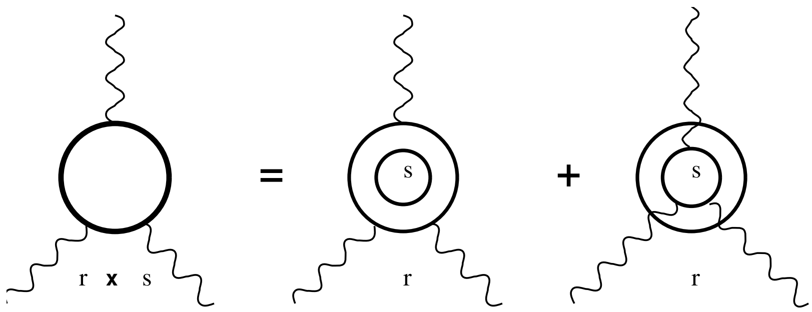
$$C_{\mathbf{n^2}-\mathbf{1}} = C_{\mathbf{n}\otimes\overline{\mathbf{n}}} = n\frac{1}{2} + n\frac{1}{2} = n \ . \tag{1.113}$$

One can proceed to build the indices of other representations using this technique. This graphical construction can be extended to diagrams with more than two wavy lines. Consider the case of three wavy lines. This corresponds to the symmetric product of three adjoint representations; algebraically

$$\mathrm{Tr}(\mathbf{T}^A_{\mathbf{r}}\mathbf{T}^B_{\mathbf{r}}\mathbf{T}^C_{\mathbf{r}}) = \mathcal{A}_{\mathbf{r}} d^{ABC} \ . \tag{1.114}$$

For most Lie algebras, it is zero, because the symmetric product of three adjoint representations does not contain an invariant. It is only for the unitary series $SU(n)$ with $n \geq 3$ that it does not vanish (as well as for the

Abelian algebra $U(1)$). It is a number, called the *anomaly* number associated with every representation. Our graphical rules give us an easy way to derive its composition law



One uses the fact that all three wavy lines must connect to the same internal line, to find the composition law for the symmetric and antisymmetric products of representations (see problem). Applying this to $SU(3)$,

$$\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}}_A \oplus \mathbf{6}_S \ , \tag{1.115}$$

it is easy to show that the anomaly of the triplet is minus that of the antitriplet. We can proceed to deduce that the anomaly of the real adjoint representation is zero. In fact these results generalize: the anomaly of any representation is equal to minus that of its conjugate, and thus it is zero for real representations.

One may apply this construction to diagrams with many external adjoint lines. We leave this as an exercise. The nice feature is that one gets composition rules for higher indices that are independent of the algebra.

While we have proceeded by giving salient examples, it is clear that a more systematic approach to the study of the representations of Lie algebras is warranted. While many techniques can be found in the literature, they are often patterned after a particular algebra,. Below we present the method of Dynkin, as the most general and simplest way to study the representations of Lie algebras.

1.5.1 Dynkinese

The essence of this approach is to summarize the relevant properties of all Lie Algebras, using only two-dimensional diagrams, called *Dynkin diagrams*.

We present only the necessary mathematical facts, leaving proofs to more learned texts. Any Lie algebra of rank l contains l commuting generators, $H_i,\ i = 1, \ldots, l$,

$$[H_i, H_j] = 0 \ , \qquad H_i^\dagger = H_i \ . \tag{1.116}$$

The remaining $(d_{\mathbf{a}} - l)$ generators of the algebra split in two conjugate groups. These are the $E_{\boldsymbol{\alpha}}$, labelled by a $l-$dimensional *root* vector,

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_l) \ , \tag{1.117}$$

which satisfy

$$[H_i, E_{\boldsymbol{\alpha}}] = \alpha_i E_{\boldsymbol{\alpha}} \ . \tag{1.118}$$

The remaining are their conjugates

$$E_{\boldsymbol{\alpha}}^\dagger = E_{-\boldsymbol{\alpha}} \ , \tag{1.119}$$

so that $-\boldsymbol{\alpha}$ is also a root. We have the further commutation relations

$$[E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}] = \alpha^i H_i \ , \tag{1.120}$$

where the α^i are related to the components of the root vector by the metric g_{ij},

$$\alpha^i \equiv g^{ij} \alpha_j \ . \qquad g^{ij} g_{jk} = \delta^i_k \tag{1.121}$$

Finally, we have

$$[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}] = \begin{cases} 0 & \text{if } \boldsymbol{\alpha} + \boldsymbol{\beta} \text{ is not a root} \ , \\ N_{\boldsymbol{\alpha}\boldsymbol{\beta}} E_{\boldsymbol{\alpha}+\boldsymbol{\beta}} & \text{if } \boldsymbol{\alpha} + \boldsymbol{\beta} \text{ is a root} \ . \end{cases} \tag{1.122}$$

Thus a Lie algebra is characterized by its root vectors, the metric function, and the coefficients $N_{\boldsymbol{\alpha}\boldsymbol{\beta}}$.

To see that this abstract notation is actually very physical, consider $SU(2) = A_1$, with its generators represented by the Pauli spin matrices. The rank is one, and we identify

$$H_1 = \mathbf{T}^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ ,$$

as well as

$$E_\alpha = \mathbf{T}^1 + i\mathbf{T}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad E_{-\alpha} = \mathbf{T}^1 - i\mathbf{T}^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

The commutation relations are

$$[H_1, E_{\pm\alpha}] = \pm E_{\pm\alpha} , \tag{1.123}$$

so that there are two one-dimensional roots, one is positive, $\alpha = 1$, and the other negative, $\alpha = -1$. Physicists recognize E_α as the ladder operator which raises the magnetic quantum number by one unit. Since

$$[E_\alpha, E_{-\alpha}] = 2H_1 , \tag{1.124}$$

the metric is g_{ij} is just $\delta_{ij}/2$. This easily generalizes to our next example $SU(3) = A_2$. Here we have two commuting generators, H_1 and H_2. In the **3** representation, they are given by the diagonal Gell-Mann matrices

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad H_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} .$$

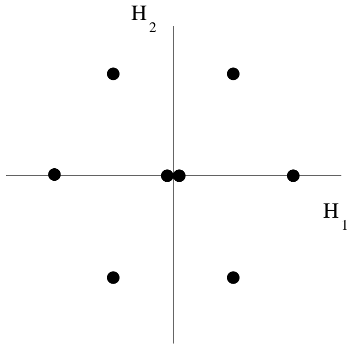
The other generators are given by matrices which have only one non-zero entry in the off-diagonal position; there are six such generators, three represented by upper diagonal (nilpotent) matrices,

$$E_{\boldsymbol{\beta}_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad E_{\boldsymbol{\beta}_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad E_{\boldsymbol{\beta}_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

The other three, represented by lower diagonal matrices, correspond to the negative roots $E_{-\boldsymbol{\beta}_i}$. The commutation relations yield the three root vectors

$$\boldsymbol{\beta}_1 = (1, 0) , \qquad \boldsymbol{\beta}_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}) , \qquad \boldsymbol{\beta}_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}) , \tag{1.125}$$

with the other three roots given by their negatives. A simple computation shows that the metric is also one-half times the unit matrix. It is instructive to plot these roots in the $H_1 - H_2$ plane



The choice of a convenient basis to express these roots is not obvious. Since half of the roots are the negative of the others, let us first split them in positive roots and negative roots. We define *positive roots* as those for which the first non-zero component is positive. Clearly the three positive roots $\boldsymbol{\beta}_1$, βb_2, and $-\boldsymbol{\beta}_3$ are not all independent, and we can express

$$\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 + [-\boldsymbol{\beta}_3] \ , \tag{1.126}$$

in such a way that the expansion coefficients are positive (and in this case equal) integers. The two roots $\boldsymbol{\beta}_2 \equiv \boldsymbol{\alpha}_1$, and $-\boldsymbol{\beta}_3 \equiv \boldsymbol{\alpha}_2$, are called the *simple* roots. There are as many simple roots as the rank of the algebra; they form a basis, called the α-*basis*, for the root space. Their properties characterize the algebra. In $SU(3)$, for example, the two simple roots have the same length, and from

$$g^{ij}(\alpha_1)_i(\alpha_2)_j \equiv \boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2 = -1 \ , \tag{1.127}$$

the angle between them is $120°$, as can easily be seen from the root diagram above. All this information about the simple roots is contained in the 2×2 *Cartan matrix*

$$A_{ij} \equiv 2\frac{\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_j}{\boldsymbol{\alpha}_j \cdot \boldsymbol{\alpha}_j} \ , \tag{1.128}$$

which for $SU(3)$ is just

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \ .$$

This matrix is clearly independent of the basis in which we have expressed the simple roots. We can introduce another convenient basis, called the ω-*basis*, defined by the relations

$$2\frac{\boldsymbol{\omega}_i \cdot \boldsymbol{\alpha}_j}{\boldsymbol{\alpha}_j \cdot \boldsymbol{\alpha}_j} = \delta_{ij} \ . \tag{1.129}$$

In the case of $SU(3)$, the $\boldsymbol{\omega}$ vectors are given by

$$\boldsymbol{\omega}_1 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}) \ , \qquad \boldsymbol{\omega}_2 = (\frac{1}{2}, -\frac{1}{2\sqrt{3}}) \ . \tag{1.130}$$

The relation between the two bases can be written in the form

$$\boldsymbol{\alpha}_1 = 2\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2 \ , \qquad \boldsymbol{\alpha}_2 = -\boldsymbol{\omega}_1 + 2\boldsymbol{\omega}_2 \ , \tag{1.131}$$

so that the expansion coefficients are the rows of the Cartan matrix.

Finally we mention a third basis, called the *orthonormal basis*, in which the simple roots are conveniently expressed. Introduce three orthonormal vectors in a 3-dimensional space, $\mathbf{e}_i$, $i = 1, 2, 3$, such that $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$. Then we have

$$\boldsymbol{\alpha}_1 = \mathbf{e}_1 - \mathbf{e}_2 \ , \qquad \boldsymbol{\alpha}_2 = \mathbf{e}_2 - \mathbf{e}_3 \ . \tag{1.132}$$

Let us apply this notation to the case of $SU(4) = A_3$. This rank-three algebra has three commuting generators, represented in its fundamental complex 4-dimensional representation by the traceless diagonal matrices,

$$H_1 = \frac{\mathrm{diag}}{2}(1, -1, 0, 0) \ , \quad H_2 = \frac{\mathrm{diag}}{2\sqrt{3}}(1, 1, -2, 0) \ , \quad H_3 = \frac{\mathrm{diag}}{2\sqrt{6}}(1, 1, 1, -4) \ .$$

There are twelve ladder operators associated with the six root vectors and their negatives. The first three are the same as in $SU(3)$,

$$\boldsymbol{\beta}_1 = (1, 0, 0) \ , \qquad \boldsymbol{\beta}_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0) \ , \qquad \boldsymbol{\beta}_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0) \ ,$$

with the rest given by

$$\boldsymbol{\beta}_4 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \sqrt{\frac{2}{3}}) \ , \ \boldsymbol{\beta}_5 = (-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \sqrt{\frac{2}{3}}) \ , \ \boldsymbol{\beta}_6 = (0, -\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}) \ .$$

The six positive roots are $\boldsymbol{\beta}_1, \ \boldsymbol{\beta}_2, \ -\boldsymbol{\beta}_3, \ \boldsymbol{\beta}_4, \ -\boldsymbol{\beta}_5, \ -\boldsymbol{\beta}_6$. As for $SU(3)$, some positive roots are linear combinations of three with positive (and in this case also equal) integer coefficients:

$$\boldsymbol{\beta}_1 = [-\beta_3] + \beta_4 + [-\boldsymbol{\beta}_6] \ , \ \boldsymbol{\beta}_2 = \boldsymbol{\beta}_4 + [-\boldsymbol{\beta}_6] \ , \ [-\boldsymbol{\beta}_5] = [-\boldsymbol{\beta}_3] + [-\boldsymbol{\beta}_6] \ .$$

So we identify three three simple roots

$$\boldsymbol{\alpha}_1 \equiv -\boldsymbol{\beta}_3 \ , \ \boldsymbol{\alpha}_2 \equiv -\boldsymbol{\beta}_6 \ , \ \boldsymbol{\alpha}_3 \equiv \boldsymbol{\beta}_4 \ . \tag{1.133}$$

Other properties are also similar: they have the same length, and the angle between $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_2$ and $\boldsymbol{\alpha}_3$ is 120°; however, the angle between $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_3$ is 90°. For $SU(4)$, the Cartan matrix is then

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \ .$$

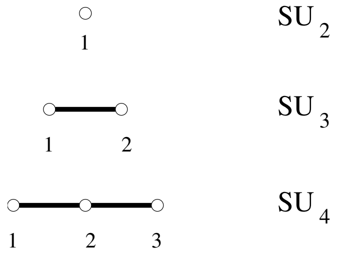
The ω-basis vectors are related to the α-basis by

$$\begin{aligned} \boldsymbol{\alpha}_1 &= 2\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2 \ , \\ \boldsymbol{\alpha}_2 &= -\boldsymbol{\omega}_1 + 2\boldsymbol{\omega}_2 - \boldsymbol{\omega}_3 \ , \\ \boldsymbol{\alpha}_3 &= -\boldsymbol{\omega}_2 + 2\boldsymbol{\omega}_3 \ . \end{aligned} \tag{1.134}$$

The orthogonal basis spans a four-dimensional space with

$$\boldsymbol{\alpha}_1 = \mathbf{e}_1 - \mathbf{e}_2 \ , \qquad \boldsymbol{\alpha}_2 = \mathbf{e}_2 - \mathbf{e}_3 \ , \qquad \boldsymbol{\alpha}_3 = \mathbf{e}_3 - \mathbf{e}_4 \ . \tag{1.135}$$

By now the pattern should be obvious. The Cartan matrix contains a lot of information, and Dynkin devised a convenient way to describe its properties by means of a two-dimensional diagram. There, a simple root is entered as a dot. Two simple roots at 120° are joined by a single line. Dots not directly connected turn out to be at at 90° to one another. These rules produce the *Dynkin diagram* for the $SU(n+1)$ algebras, as a linear chain of dots connected by a single line.



The symmetric $n \times n$ Cartan matrix associated with the Lie algebra $SU(n+1)$ has integer or zero coefficients, given by

$$\begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ -1 & 2 & -1 & \cdot & \cdot & \cdot & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & -1 & 2 \end{pmatrix} .$$

Notice that most simple roots are perpendicular to one another.

Dynkin found that the same procedure could be generalized, allowing him to draw a specific Dynkin diagram for each Lie algebra.

This is made possible by the following facts about simple roots which we quote without proof:

– The angle between two simple roots can only be 90°, 120°, 135°, or 150°.

– Two simple roots at 120° have the same length.

– Two simple roots at 135° have a $1 : \sqrt{2}$ length ratio.

– Two simple roots at 150° have a $1 : \sqrt{3}$ their length ratio.

– All positive roots can be expressed as linear combinations of the simple roots with positive integer coefficients.

These facts about Lie algebras motivated Dynkin to associate to any Lie algebra, two-dimensional diagrams, according to the following rules:

– Associate with a short simple root a filled dot: $\bullet$.

– Associate with a long simple root an empty dot: $\circ$.

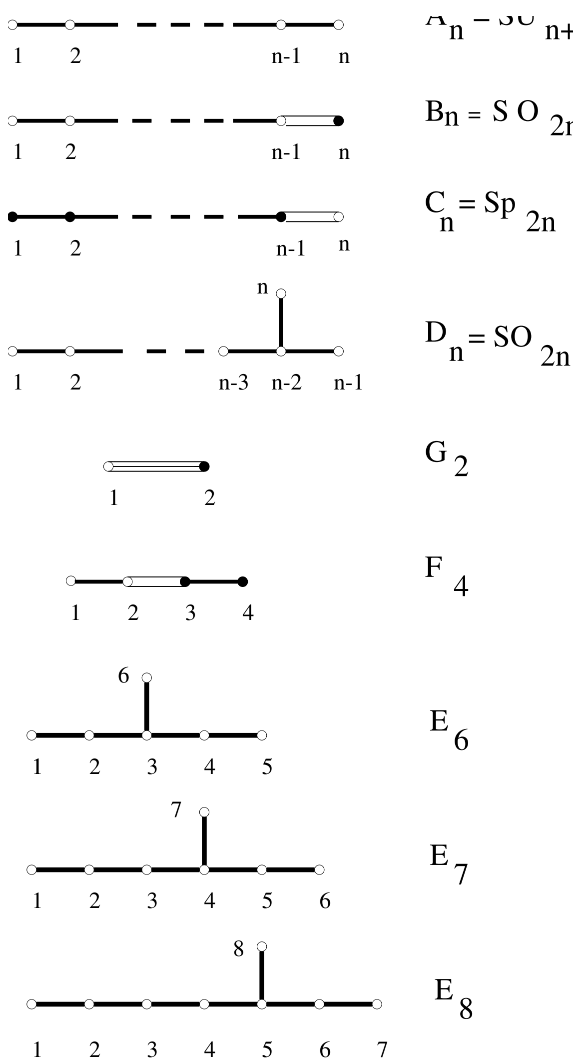
– Connect the dots of two simple roots by a single line if they are at 120° .

– Connect the dots of two simple roots by two lines if they are at 135°.

– Connect the dots of two simple roots by three lines if they are at 150°.

–Two simple roots not directly connected to one another are necessarily perpendicular.

This produces the following *Dynkin diagrams* for all the semi-simple Lie Algebras:



It is straightforward to simply read-off the Cartan matrix from the Dynkin diagram. Another immediate consequence is that there are many redundant diagrams for low rank algebras. In particular we see that there is only one algebra of rank one, since its Dynkin diagram is just one dot. This yields the following equivalences between Lie algebras

$$A_1 = B_1 = C_1 \ . \tag{1.136}$$

Although these Lie algebras are strictly equal, the groups they generate are not. For instance, the spinor representation of the rotation group is double valued, so that we say $SO_3 = SU_2/Z_2$. Finally for algebras of rank two and three, we have the equivalences

$$B_2 = C_2 \ ; \qquad D_2 = A_1 + A_1 \ ; \qquad A_3 = D_3 \ . \tag{1.137}$$

We also note that many diagrams display discrete symmetries; their significance will become clear later.

However, the information displayed by these diagrams does not allow us to reconstruct the full root diagram of the algebra. For instance, the Dynkin of $SU(3)$ tells us, from the number of dots that there are two roots of zero length, in addition to two simple roots of equal length at 120° from one another. The determine the other roots, we only have two facts: 1-) the negative of every root is itself a root, and 2-) the remaining positive roots are linear combinations of the two simple roots with integer coefficients of the same sign, but the Dynkin diagram does not tell us the values of these coefficients, nor the number of positive roots. Additional information must be provided to develop an algorithm to determine these coefficients.

1.5.2 Representations

Consider any state within a representation of a Lie algebra of rank l. It can be labelled by a set of l numbers, the eigenvalues of the commuting generators, which can be thought of as the coordinates of an l-dimensional vector, Λ, called the *weight* of the state. In the adjoint representation, the weight vectors are nothing but the roots of the algebra.

The action of a ladder operator associated with a root α on any state of weight Λ simply produces a different state labelled by a new weight vector

$$\boldsymbol{\Lambda}' = \boldsymbol{\Lambda} + \boldsymbol{\alpha} \ . \tag{1.138}$$

As a result, we can think of any representation as a collection of weights, differing from one another by the roots of the algebra. We can express the weights in any basis we choose. In terms of the basis of simple roots, we have

$$\mathbf{\Lambda} = 2\sum_{i=1}^{l} \overline{\lambda}_i \frac{\boldsymbol{\alpha}_i}{\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_i} \ , \tag{1.139}$$

where the $\overline{\lambda}_i$ describe the weight in the *dual* basis. The factor of 2 is traditional. Dynkin introduced a different basis with components

$$a_i = 2\frac{\mathbf{\Lambda} \cdot \boldsymbol{\alpha}_i}{\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_i} \ , \tag{1.140}$$

from which we see that

$$\mathbf{\Lambda} = \sum_{i=1}^{l} a_i \boldsymbol{\omega}_i \ , \tag{1.141}$$

which are the expansion coefficients of the weight in the ω basis. We also have

$$a_i = 2\sum_{i=1}^{l} \bar{\lambda}_j \frac{A_{ji}}{\boldsymbol{\alpha}_j \cdot \boldsymbol{\alpha}_j} \ . \tag{1.142}$$

For the jth simple root, $\overline{\lambda}_i = \delta_{ij}$: the components of the simple roots in the Dynkin basis are integers since they are just the rows of the Cartan matrix. The other roots, being combinations of these with integer coefficients, have themselves integer components in this basis. In the Dynkin basis, all weights in the adjoint representation have integer coordinates.

It can be shown that this is true for all representations, that is *all* weights have integer coefficients in the Dynkin basis. It follows that any state within a unitary irreducible representation can be labelled by a set of l integers, $(a_1, a_2, \ldots, a_k, \ldots, a_l)$: any weight can be represented by integers associated with the dots of the Dynkin diagram. We need only agree on a numbering convention to order these integers according to the sequences of the Dynkin diagrams.

This labelling does not tell us *which* representation the state belongs to; for example, the Dynkin label of states for spin $SU(2)$ is just twice the magnetic quantum number, and we know that states with Dynkin label (1), meaning $j_3 = 1/2$, occur in all half-integer spin representations. This example gives us the clue to identify the representation, as each representation of $SU(2)$ has only one state which has the maximum allowed value of the magnetic quantum number. It is the state $|j, j_3 = j\rangle$ that is annihilated by half the ladder operators. This feature generalizes to all unitary representations: each has exactly one state where the commuting generators have their maximum allowed eigenvalues, which are zero or positive . This state is called the *highest weight state*; it *uniquely* labels unitary irreducible representations with l *positive or zero* integers.

Conversely, any set of positive (or zero) integers uniquely labels a representation of the algebra. In the weight space, these weights are confined to being inside (or at the boundary if some highest weight components are zero) a region bounded by the l vectors $\boldsymbol{\omega}_i$. The inside of this region is called the *fundamental Weyl chamber* of the weight space. For $SU(3)$, it is the sliver bounded by the two vectors $(1/2, 1/2\sqrt{3})$ and $(0, 1/\sqrt{3})$. This region of weight space can be mapped into the whole weight space by the action of a discrete group of transformations, called the *Weyl* group. Each Lie algebra has its own characteristic Weyl group. For $SU(3)$, it is S_3, the permutation group on three objects. Its action can be described in terms of reflections about the ω-basis vectors. Any weight inside the fundamental Weyl chamber is called a *dominant* weight. Under the action of the Weyl group, it is taken into a finite number of weights all outside the fundamental chamber (since they are reflections about its boundaries). This set of weights is called the *Weyl orbit*, which is a closed in weight space. Thus we may view the whole weight space as an infinite number of non-intersecting Weyl orbits, each with one dominant weight in the fundamental chamber. For example the adjoint (octet) irrep of $SU(3)$ has highest weight state with Dynkin label $(1, 1)$. Under S_3, this weight goes around its Weyl orbit of six weights. The other two states of the adjoint have Dynkin label $(0, 0)$; they are at the origin of the Weyl cone, and are their own orbit. We can infer from this example the general anatomy of any irrep: it contains its highest weight state *once*, together with its Weyl orbit; it also contains a set of dominant weights, with their Weyl orbits; there can be several copies of the same dominant weight within one irrep. Thus in order to compute the content of the irrep, we need only know its dominant weights and their multiplicities. This method for describing irreps is favored in the mathematical literature, and we mention it to add to the (not inconsiderable) erudition of the reader.

We now proceed with our description of irreps along more conventional lines, essentially by acting the lowering operators on the highest weight state. This is done by subtracting from it the positive simple roots until one reaches either the negative of the highest weight for a real representation, or the negative of its conjugate for a complex representation. A weight that is obtained by k subtractions from the highest weight, is said to be at the kth level of the representation. The number of levels is called the *height* of the representation. The height of any representation can be computed, using the *level vector* $\mathbf{R} = (R_1, R_2, \ldots, R_l)$ which depends on the algebra, not the representation. It is given by

$$T = \sum_{i=1}^{l} R_i a_i \ , \tag{1.143}$$

where $(a_1\ a_2 \cdots\ a_l)$ is the highest weight, used to label a representation of a Lie algebra. The level vectors for the different Lie algebras are given by

$$\begin{aligned}
A_n : &\quad [n, 2(n-1), \ldots, (n-2)3, (n-1)2, n] \ , \\
B_n : &\quad [2n, 2(2n-1), \ldots, (n-1)(n+2), n(n+1)/2] \ , \\
D_n : &\quad [2n-2, 2(2n-3), .., (n-2)(n+1), n(n-1)/2, n(n-1)/2] \ , \\
C_n : &\quad [2n-1, 2(2n-2), \ldots, (n-1)(n+1), n^2] \ , \\
G_2 : &\quad [10, 6] \ , \qquad F_4 : \ [22, 42, 30, 16] \ , \\
E_6 : &\quad [16, 30, 42, 30, 16, 22] \ , \qquad E_7 : \ [34, 66, 96, 75, 52, 27, 49] \ , \\
E_8 : &\quad [92, 182, 270, 220, 168, 114, 58, 136] \ .
\end{aligned} \tag{1.144}$$

Note that for $SO(2n)$ $(SO(2n+1))$, the last two (one) components break the pattern.

To make the construction of representations even simpler, Dynkin proved that the distribution of weights in any representation is spindle-shaped so that the number of weights at the kth level is the same as that of the $(T-k)$th level.

By definition, adding any positive simple root to the highest weight state does not produce another weight, while substracting a simple root does yield another state. For example the action of J_- on $|j, j\rangle$ yields, up to normalization, the state $|j, j-1\rangle$. In the Dynkin language, this translates as: the Cartan matrix, 2, gives the value of the one dimensional simple root, thus starting with the representation with highest weight (1), we obtain another weight within the representation by substracting the simple root, yielding (-1), the negative of the simple root; it is of course the state with $j_3 = -\frac{1}{2}$, and this yields the two-state spinor representation.

Starting with the representation $(2j)$, we generate the state $(2j-2k)$, by substracting the simple root k times until we reach the negative $(-2j)$. The Dynkin label of the adjoint of $SU(2)$ is clearly (2).

To reconstruct the root system of any algebra, we need to know, besides the Dynkin diagram, the Dynkin label (the highest weight state) of its adjoint representation. The other states are then generated by subtracting the positive simple roots. The Dynkin labels (highest weight state) of the adjoint of each Lie algebras are:

$$
\begin{array}{rcl}
\mathrm{A}_l : & (1\;0\;0\;\ldots\;0\;0\;1) & l \neq 1\;, \\
\mathrm{B}_l : & (0\;1\;0\;\ldots\;0\;0\;0) & \\
\mathrm{C}_l : & (2\;0\;0\;\ldots\;0\;0\;0) & \\
\mathrm{D}_l : & (0\;1\;0\;\ldots\;0\;0\;0) & \\
\mathrm{G}_2 : & (1\;0) & \\
\mathrm{F}_4 : & (1\;0\;0\;0) & \\
\mathrm{E}_6 : & (0\;0\;\;0\;0\;0\;1) & \\
\mathrm{E}_7 : & (1\;0\;0\;0\;0\;0\;0) & \\
\mathrm{E}_8 : & (0\;0\;0\;0\;0\;0\;1\;0) &
\end{array}
$$

We are now in a position to construct the states within a given representation. The crucial element is that any state with a weight with positive Dynkin coordinate in the ith entry, a_i, has a_i quanta of the ith root, which can then be subtracted that many times.

Consider the algebra $SU(4)$. Its simple roots are the rows of the Cartan matrix, $\boldsymbol{\alpha}_1 = (2, -1, 0)$, and $\boldsymbol{\alpha}_2 = (-1, 2, -1)$, $\boldsymbol{\alpha}_3 = (0, -1, 2)$. Its level vector is $\mathbf{R} = (3, 6, 3)$. Let us start with its simplest representation with Dynkin label (100). Its level is therefore $T = 3$. The rule is that for any state of weight $(a_1 a_2 \ldots a_k \ldots a_l)$, the simple root $\boldsymbol{\alpha}_k$ can be subtracted k times, corresponding to the k-fold application of an annihilation operator, as long as a_k is positive. Hence we can subtract $\boldsymbol{\alpha}_1$ once from (100), obtaining $(-1\ 1\ 0)$, from which we can subtract $\boldsymbol{\alpha}_2$ once, obtaining $(0 - 1\ 1)$, from which we can subtract $\boldsymbol{\alpha}_3$ once, yielding $(0\ 0 - 1)$. Since there are no more positive Dynkin we are done. This representation has four states; it is the **4** of $SU(4)$. A similar procedure applied to the representation (001) yields another four-dimensional representation. It is the conjugate $\overline{\mathbf{4}}$.

Now for (101), the adjoint representation of $SU(4)$. It has six levels. We can subtract $\boldsymbol{\alpha}_{1,3}$ once, obtaining two states at the first level: $(-1\ 1\ 1)$ and $(1\ 1 - 1)$. Similar subtractions yield at the next level the states $(0 - 1\ 2)$, $(-1\ 2 - 1)$, $(-1\ 2 - 1)$, $(2 - 1\ 0)$. These are the simple roots, but one of them appears twice. Clearly there is only one state associated with that simple root, and we only have three states at this level. The next level yields the three roots of zero length. Further subtraction reproduces the states but with negative entries. Hence we have $1+2+3+3+3+2+1=15$ states, corresponding to the generators of $SU(4)$.

The beauty of the Dynkin notation is that it applies equally well to a familiar algebra like $SU(4)$ as for an unfamiliar one like E_7.

There are many other advantages in describing representations by their Dynkin labels. Consider again the case of $SU_4 \approx SO_6$. We have seen that its lowest representation is the four dimensional complex vector, $\mathbf{4}$, represented in Dynkinese by (100), while its complex conjugate $\overline{\mathbf{4}}$ is represented by (001). We note that complex conjugation corresponds to a symmetry of the Dynkin diagram: reflection about the vertical. The smallest real (self-conjugate) representation is therefore the (010). It is nothing but the six-dimensional vector representation of SO_6! In $SU(4)$ language, it corresponds to a second rank antisymmetric tensor, with the quantum numbers of the Kronecker product $\mathbf{4}\otimes\mathbf{4}_A=\mathbf{6}$.

Representations can be multiplied together to yield new representations. In general the product of two representations will contain the sum of several irreducible representations. However it will always contain a state of highest eight equal to the sum of the highest weights of the product representations. For the product of low-lying representations, that sum can be determined by the fact that the sum of their dimensions is equal to the product of the dimensions of the factor representations, and that the sum of their Dynkin indices is equal to the dimension of one factor representation times the Dynkin of the other, plus the other way around. In some cases, the product representation will be irreducible: the p-times antisymmetric product of the fundamental of SU_{n+1} always yields an irreducible representation, which can be written as a totally antisymmetric tensor of rank p. In Dynkinese, it is labelled with zeros except for a one at the pth position.

The product of two representations always contains one representation which is labeled by the sum of their Dynkin labels. For instance, in SU_4, the adjoint representation is just (101); note that it is symmetric under inversion and therefore real, as required.

There are two more types of Dynkin diagrams with interesting symmetries. The D_n Dynkin is symmetric under the interchange of the two dots at the n and $n-1$ positions. Representations with a one at one of these positions and zeros elsewhere are the spinor representations, and they are indeed complex; under conjugation, they flip into one another. For instance, in $D_5=SO_{10}$, the complex spinor representation, $\mathbf{16}$ is represented by (00010), while its conjugate, $\overline{\mathbf{16}}$ is just (00001). On the other hand, the representation (10000) is manifestly real; it corresponds to the ten-dimensional vector. The representation (01000) represents the antisymmetric second rank tensor, and (00100) is the three-times antisymmetric tensor.

The upscale reader may have noticed that the Dynkin diagram for $SO_8 = D_4$ is just the Mercedes-Benz symbol which has a three-fold symmetry. There the (100) labels the eight-dimensional vector representation, while (010) and (001) label two eight-dimensional spinors. There is a special triality symmetry between these three representations, which has dramatic consequences in the formulation of superstring theories.

There is one more diagram which has a special symmetry. E_6 does indeed have complex representations. For instance, the complex **27** is written in Dynkinese as (100000), while its conjugate, $\overline{\mathbf{27}}$ is (000010). Groups with complex representations play a special role because they are the only ones that can describe the complex Weyl spinors that appear in the theory of weak interactions.

1.5.3 PROBLEMS

A. Using graphical methods, show that the Dynkin index of the symmetric product of two representations is equal to $(d_{\mathbf{r}} + 2)c_{\mathbf{r}}$.

B. Using graphical methods, show that the anomaly of the symmetric product of two representations is equal to $(d_{\mathbf{r}} + 4)A_{\mathbf{r}}$, where $d_{\mathbf{r}}$ and $A_{\mathbf{r}}$ are the dimension and anomaly of the representation. Deduce that

1-) the anomaly of the $\overline{\mathbf{3}}$ of $SU(3)$ is opposite that of the **3**.

2-) the anomaly of the sum of the two $SU(5)$ representations $\overline{\mathbf{5}} \oplus \mathbf{10}$ is free of anomalies, where **10** is the antisymmetric product of two **5**'s.

C. 1-) Using graphical methods, find the composition law for the trace of four representation matrices over a product of two representations.

2-) Explicitly construct the trace over four matrices in the fundamental and adjoint representation of $SU(2)$, and verify the composition law you have just derived.

D. Repeat the last part of problem C when the algebra is $SU(3)$.

E. Starting from its Dynkin diagram, find the Cartan matrix of G_2. Then work out and plot the roots of the algebra G_2, in the adjoint representation. The highest weight for the adjoint representation (10) (in the Dynkin basis), and the level vector has components $(10, 6)$.

F. There are many different ways to choose the two simple roots of $SU(3)$. Show that they are all related by the action of a discrete set of Weyl transformations, S_3, acting on the root diagram. Proceed to analyze the $(2,0)$ irrep in terms of its Weyl orbits.