9

SECOND JOURNEY: THE AXION

9.1 QCD Vacuum Energy

There is no experimental evidence that the strong interactions violate either P or CP. The absence of a measurable electric dipole moment for the neutron ($d_n \leq 10^{-26}$ esu) suggests that strong CP-violation is very small, if it exists at all. Yet QCD is capable of breaking these symmetries both spontaneously and explicitly. The former through the chiral-breaking quark condensate, the latter through interactions which violate these discrete symmetries.

The strong QCD forces spontaneously break the approximate chiral symmetry of the light quarks, through a quark-antiquark condensate

$$< \mathbf{q}_L^{\dagger i} \mathbf{q}_{Rj} >_0 \equiv \Lambda^3 e^{\frac{i\gamma}{3}} \mathbf{U}^i{}_j \ , \tag{9.1}$$

where $i, j = 1, 2, 3$ denote the three light flavors. The scale parameter Λ is real. The matrix $\mathbf{U}$ can be reduced to a unitary matrix by using a $SU(3)_L \times SU(3)_R$ chiral transformation. The phase γ is normalized to the number of flavors, by setting $\det \mathbf{U} = 1$. Under P, we have

$$\mathbf{q}_L^{\dagger i} \mathbf{q}_{Rj} \rightarrow \mathbf{q}_R^{\dagger i} \mathbf{q}_{Lj} \ , \tag{9.2}$$

and under CP

$$\mathbf{q}_L^{\dagger i} \mathbf{q}_{Rj} \rightarrow \mathbf{q}_R^{\dagger j} \mathbf{q}_{Li} \ , \tag{9.3}$$

so that P-invariance of the condensate requires that

$$\mathbf{U}^\dagger \mathbf{U}^* = e^{\frac{2i\gamma}{3}} \mathbf{1} \ , \tag{9.4}$$

while CP-invariance yields

$$\mathbf{U}^\dagger \mathbf{U}^\dagger = e^{\frac{2i\gamma}{3}} \mathbf{1} \ . \tag{9.5}$$

Taking the determinant of these equations, and using det $\mathbf{U} = 1$, we see that CP and P invariances require that $\gamma = 0$ or π, but we have (at present) no means of calculating the phase γ, since it is determined by strong QCD.

Explicit breaking of P and CP comes from the term

$$\frac{\theta}{32\pi^2} \mathrm{Tr}(\mathbf{G}_{\mu\nu} \tilde{\mathbf{G}}_{\mu\nu}) \ . \tag{9.6}$$

The dimensionless parameter θ can be absorbed in the quark masses, but the combination $\overline{\theta} = \theta -$ arg det $\mathbf{M}_q$, where $\mathbf{M}_q$ is the quark mass matrix, is physical. This term and/or the complex quark masses break the discrete symmetries explicitly. The experimental limit on the neutron's electric dipole moment requires the fundamental parameter of the standard model, $\overline{\theta}$, to be smaller than 10^{-9}.

In the chiral Lagrangian, the condensates and the quark masses produce the chirality-breaking interactions

$$\mathcal{L}_{\mathrm{break}} = \Lambda^3 \sum_j m_j {\mathbf{U}^j}_j e^{i\frac{\gamma + \overline{\theta}}{3}} + \mathrm{c.c.} \ , \tag{9.7}$$

showing that the low energy physics depends on the sum of the condensate and explicit phase $\gamma + \overline{\theta}$. This expresses the lowest order shift in the vacuum energy, and it has an absolute minimum at

$$\frac{\gamma + \overline{\theta}}{3} = \pi \ , \tag{9.8}$$

which is necessary for generating successful mass relations among the pseudoscalars.

The QCD vacuum energy $E = E(\overline{\theta})$, depends on $\overline{\theta}$. In addition, the dependence must be cyclic, since the angle $\overline{\theta}$ is defined modulo 2π

$$E(\overline{\theta} + 2\pi) = E(\overline{\theta}) \ . \tag{9.9}$$

We have already seen, in the context of the chiral Lagrangian, that $E(\overline{\theta})$ acquires its minimum value at $\overline{\theta} = 0$ (mod 2π). This can also be shown in a very elegant and direct proof due to C. Vafa *Vafa-Witten theorem* and E. Witten (*Phys. Rev. Lett.* **53**, 535(1984)), starting from the fundamental QCD Lagrangian. Their proof is part of a more general theorem which states that QCD does not spontaneously break any vector-like symmetry. Of course the theorem does not apply to the chiral symmetries, which are evidently broken. These authors start with the Euclidean path integral for QCD (neglecting any electroweak effects). In a volume V, it is

$$e^{-VE(\theta)} = \int \mathcal{D}\mathbf{A}\mathcal{D}\mathbf{q}\mathcal{D}\overline{\mathbf{q}}\ \exp\left(-\int d^4x \mathcal{L}_{QCD}(\theta)\right) \ , \tag{9.10}$$

where

$$\mathcal{L}_{QCD}(\theta) = -\frac{1}{4g^2}\mathrm{Tr}\,(\mathbf{G}_{\mu\nu}\mathbf{G}_{\mu\nu}) + \sum_i \overline{\mathbf{q}}_i(\not{D} + m_i)\mathbf{q}_i + \frac{i\theta}{32\pi^2}\mathrm{Tr}\left(\mathbf{G}_{\mu\nu}\widetilde{\mathbf{G}}_{\mu\nu}\right) \ . \tag{9.11}$$

We take all the quark masses to be real, so that in the above we have chosen $\theta = \overline{\theta}$. Integrating out the fermions we obtain

$$e^{-VE(\theta)} = \int \mathcal{D}\mathbf{A}\ \det(\not{D} + \mathbf{M}) e^{\int d^4x [\frac{1}{4g^2}\mathrm{Tr}(\mathbf{G}_{\mu\nu}\mathbf{G}_{\mu\nu}) - \frac{i\theta}{32\pi^2}\mathrm{Tr}(\mathbf{G}_{\mu\nu}\widetilde{\mathbf{G}}_{\mu\nu})]} \ . \tag{9.12}$$

It is crucial to the analysis that the imaginary unit i appears only in the last term. In the continuation from Minkowski to Euclidean space, the i in the Minkowski exponent is absorbed by the Wick rotation of dt, but an extra i appears in the Wick rotation of the Levi-Cività symbol (allowing the volume, $\epsilon_{\mu\nu\rho\sigma}dx^\mu dx^\nu dx^\rho dx^\sigma$, to remain real).

In QCD, $\det(\not{D} + \mathbf{M})$ is positive and real; this is a consequence of the vector-like couplings of the quarks: for each eigenvalue λ of $\mathcal{D}$, there is another one with the opposite sign. Thus

$$\begin{aligned} \det(\not{D} + \mathbf{M}) &= \prod_\lambda (i\lambda + \mathbf{M}) = \prod_{\lambda>0}(i\lambda + \mathbf{M})(-i\lambda + \mathbf{M}) \ , \\ &= \prod_{\lambda>0}(\lambda^2 + \mathbf{M}^2) > 0 \ . \end{aligned} \tag{9.13}$$

Note that if $\lambda = 0$, $\mathbf{M}$ appears by itself but we can always take $\mathbf{M} > 0$ (this is the unphysical $\theta \to \overline{\theta}$ ambiguity).

Thus if θ were zero, the integrand would be made up of purely real and positive quantities. Now, inclusion of the θ term with its i can only *reduce* the value of the path integral, which is the same as *increasing* the value of $E(\theta)$. It follows that $E(\overline{\theta})$ is minimized at $\overline{\theta} = 0$ (mod 2π).

Keeping in mind the slight *caveat* that with Yukawa couplings, the fermion determinant may no longer be positive nor real, it should now be clear that the vacuum energy, expressed either in the context of QCD or of its associated chiral model, is minimized when $\overline{\theta}$ is zero. We emphasize that this does not mean that $\overline{\theta}$ is actually zero, since in the standard model, $\overline{\theta}$ is just a parameter, not a dynamical variable, to be determined from experiment.

Thus CP-violation by the strong interactions is a *prediction* of QCD. Yet there is no experimental whiff of CP-violation by the Strong Interactions, setting a limit, $\overline{\theta} < 10^{-9}$. Why should this number be so small? This is the one of the remaining outstanding puzzle of the standard model. Theorists have come up with several possible explanations:

• The ultraviolet up-quark mass is zero. This would seem to be in conflict with chiral QCD, which measures a non-zero up-quark mass. However this is an infrared mass, not the fundamental ultraviolet quark mass: the two are known to differ by *additive* corrections generated by strong coupling QCD effects. Although the size of these corrections are presently impossible to calculate, and there is no good theoretical rationale for the u quark mass to vanish, this simple scheme would explain the lack of CP-violation by the strong interactions.

• The tree-level Lagrangian is CP-invariant: the $\overline{\theta}$ parameter is naturally zero at tree-level. This requires the tree-level determinant of the quark mass matrix to be real. The potential is arranged to break CP spontaneously, and in one class of models, the ensuing CP-violation turns out to be very small. These models simply augment the standard model chiral quarks with vector-like weak doublet with standard model hypercharge, and two vector-like weak singlets of charges $2/3$ and $-1/3$:

$$\mathbf{V} = \begin{pmatrix} \mathbf{U} \\ \mathbf{D} \end{pmatrix}_{1/3} \ , \quad \overline{\mathbf{V}} = \begin{pmatrix} \mathbf{U} \\ \mathbf{D} \end{pmatrix}_{-1/3} \ , \quad \mathbf{U}'_{4/3} + \overline{\mathbf{U}}'_{-4/3} \ , \quad \mathbf{D}'_{-2/3} + \overline{\mathbf{D}}'_{2/3} \ .$$

The standard model Higgs is required to couple *only* to the chiral quarks $\mathbf{d}\overline{\mathbf{d}}$, but *not* to $\mathbf{d}\overline{\mathbf{D}}'$ $\mathbf{D}\overline{\mathbf{D}}'$ $\mathbf{D}\overline{\mathbf{d}}$, $\mathbf{D}'\overline{\mathbf{D}}$. $\Delta I_w = 0$ masses in the $\mathbf{D}\overline{\mathbf{D}}$, $\mathbf{D}'\overline{\mathbf{D}}'$ entries are allowed. CP violation comes about only through the phase carried by the $\Delta I_w = 0$ Higgs that couples to $\mathbf{d}\overline{\mathbf{D}}$ and $\mathbf{D}'\overline{\mathbf{d}}$. These rules yield the mass matrix

$$(\mathbf{d}, \mathbf{D}, \mathbf{D}') \begin{pmatrix} < H_d > & S & 0 \\ 0 & M & 0 \\ S' & 0 & M' \end{pmatrix} \begin{pmatrix} \overline{\mathbf{d}} \\ \overline{\mathbf{D}} \\ \overline{\mathbf{D}}' \end{pmatrix} \ , \tag{9.14}$$

shown here in block-diagonal form, and where $< H_d >, M, M'$ are real, and S, S' are complex. Its determinant is real. A similar matrix obtains in the charge 2/3 sector. There will be loop corrections to the phase of the determinant which will induce a non-zero value of $\overline{\theta}$, but these can be shown to be generically small.

• The lack of CP-violation in the strong interactions is traced to a new particle. The crucial observation, due to Peccei and Quinn (*Phys. Rev.* **D16**, 1791(1977)), is that if $\overline{\theta}$ can be made into a dynamical variable, it will naturally be driven to the value that minimizes the energy, which we have just seen is zero, thereby explaining the lack of P and CP violation by the strong interactions! This extra dynamical variable, identified by Weinberg (*Phys. Rev. Lett.* **40**, 223(1978)) and F. Wilczek (*Phys. Rev. Lett.* **40**, 279(1978)), is called it the *axion*, the (pseudo) Nambu-Goldstone boson of the global PQ symmetry.

Of these three possibilities, we will only study the last one in some detail because of its rich spectrum of experimental consequences and its possible implications for cosmology.

9.1.1 PROBLEMS

A. Show that, in the quark-antiquark condensate, the matrix $\mathbf{U}$ can always be chosen to be unitary.

B. Show that with N_f flavors of light quarks, the absolute minimum of the first order shift occurs when $\gamma + \overline{\theta} = N_f\pi$.

C. Show, using the Vafa-Witten theorem, that for QCD with three flavors, we have $\gamma = \pi$ and $\overline{\theta} = 0$ (see P. Sikivie and C.B. Thorn *Phys. Lett.* **B234**, 132(1990)).

D. Suppose the up-quark mass in the standard model Lagrangian is zero. Estimate the size of the effective m_u that appears in the chiral Lagrangian, and comment on the possible origin of the effect. For further details, see T. Banks, N. Seiberg and Y. Nir in *Yukawa Couplings and the Origin of Masses*, (International Press, 1994), and references therein.

E. Show that the determinant of the quark mass matrix Eq. (nelson?) is real. Estimate the size of $\overline{\theta}$ generated at the one-loop level. Estimate the size of the resulting CP-violation in the weak interactions. For references, see A. Nelson *Phys. Lett.* **136B**, 387(1984), and S. Barr *Phys. Rev. Lett.* **53**, 329(1984).

9.2 Axion Properties

The classical QCD action is invariant under a constant shift of the parameter $\overline{\theta}$, since it appears multiplied by the four-divergence $\mathrm{Tr}(\mathbf{G}_{\mu\nu}\widetilde{\mathbf{G}}_{\mu\nu})$. Hence if it were to be replaced by a field, the physics should be be invariant under a constant shift of that field. This describes the dynamics of the Nambu-Goldstone boson $a(x)$, which appears in the Lagrangian through the term

$$\mathcal{L}_{NG} = \frac{a(x)}{V_{PQ}}\partial_\mu J_\mu \ , \tag{9.15}$$

where V_{PQ} is the mass scale at which the global symmetry generated by the current J_μ is spontaneously broken. We use two hints to identify this symmetry. First, comparison with the QCD Lagrangian suggests that the divergence of this $U(1)$ current must be proportional to $\mathrm{Tr}(\mathbf{G}_{\mu\nu}\tilde{\mathbf{G}}_{\mu\nu})$, implying that this symmetry is anomalous, and therefore with computable strength through the usual triangle graph. Secondly, since a massless quark makes $\overline{\theta}$ unobservable through its redefinition by the extra chiral symmetry, this new symmetry must both be chiral and act on the quarks. The same conclusion follows from the *anomaly* diagram which involves the quark fields circulating in the triangle.

The standard model has no such global chiral U(1) symmetry; all of its global symmetries are accounted for, and none are chiral on the quarks.

To accomodate the Peccei-Quinn (PQ) symmetry, the standard model must be extended to include new degrees of freedom, assumed to be represented by local field. There must be (at least) one new complex field, which changes by a phase under the PQ chiral $U(1)$. Its phase is the axion which shifts under PQ as the quarks transform chirally, and it couples to the anomalous divergence of the current which is proportional to $\mathrm{Tr}(\mathbf{G}_{\mu\nu}\widetilde{\mathbf{G}}_{\mu\nu})$. The PQ chiral symmetry is *explicitly broken* in two ways, by QCD effects (for instance *instantons*) and by the quark masses, since it is chiral. It follows that the axion, like the pion, is not massless, but a pseudo Nambu-Goldstone boson. Its mass must be proportional to both the strength of the strong interactions which violate the symmetry, and also to the light quark masses. This suggests that

$$M_a \sim \frac{M_\pi F_\pi}{V_{PQ}} \ ; \tag{9.16}$$

the first factor M_π is expected because as M_π tends to zero, so should the axion mass, F_π labels the strength of the strong interactions, and V_{PQ} is the scale at which the PQ symmetry is broken. When we analyze this relation in more detail, we will find it to be a little more subtle.

There are two types of axion models. In the first, the scalar field whose vacuum expectation value breaks the PQ symmetry, transforms under $SU_2 \times U_1$; in the second it is an electroweak singlet.

Assume first that the PQ order parameter belongs to scalar fields which carry electroweak quantum numbers. In order not to break electric charge, it must have one electrically-neutral component, and thus transform under the weak SU_2. The simplest choice is an SU_2 doublet, just like the Higgs doublet already present in the standard model, although one can readily envisage more complicated models, with weak triplets, etc... .

With no new quarks, the $U(1)_{PQ}$ symmetry can be simply implemented in the standard model by inventing a new Higgs field which couples to the quarks. Of several logical possibilities, simplest is to introduce two electroweak doublets, each doublet coupling to one quark charge sector only. This avoids tree-level flavor changing neutral current effects. Calling these Higgs doublets H_u and H_d, we have

$$\mathcal{L}_Y = i\widehat{\mathbf{Q}}_i\overline{\mathbf{u}}_j Y_{ij}^{(2/3)} H_u + i\widehat{\mathbf{Q}}_i\overline{\mathbf{d}}_j Y_{ij}^{(-1/3)} H_d + i\widehat{L}_i\overline{e}_j Y_{ij}^{(-1)} H_d \ . \tag{9.17}$$

The leptons couple to the same Higgs doublet as the down quarks.

This Yukawa Lagrangian has the required global PQ symmetry. The Higgs doublets, H_u and H_d, have opposite hypercharge, but the same PQ charge. The hypercharge and PQ values are summarized in the following table

	H_u	H_d	$\mathbf{Q}$	$\overline{\mathbf{u}}$	$\overline{\mathbf{d}}$	L	$\overline{e}$
Y	1	-1	$\frac{1}{3}$	$-\frac{4}{3}$	$\frac{2}{3}$	-1	2
PQ	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

The PQ charges of the quarks and leptons are chosen to be chiral and flavor symmetric to ensure no possible admixture of the vector-like baryon number and the lepton numbers. With two electroweak doublets, the Higgs potential is slightly more complicated. The full potential that leaves invariant the PQ symmetry is given by

$$V(H_u, H_d) = \sum_{a=u,d} (-\mu_a^2 H_a^\dagger H_a + \lambda_{aa}(H_a^\dagger H_a)^2) + \lambda_{ud} H_u^\dagger H_u H_d^\dagger H_d + \lambda'_{ud} H_u^\dagger H_d H_d^\dagger H_u \ , \tag{9.18}$$

so that with the modest addition of another Higgs doublet, the number of parameters in the potential increases dramatically from two (μ, λ) to six $(\mu_u,\ \mu_d,\ \lambda_{uu},\ \lambda_{dd},\ \lambda_{ud},\ \lambda'_{ud})$.

In this model, when both H_u and H_d acquire vacuum values, the PQ symmetry is broken at the same scale as the electroweak symmetry. This possibility makes the axion properties almost totally predictable: since V_{PQ} breaks $SU_2 \times U_1$ it cannot be larger than $G_F^{-1/2}$, the scale of weak interactions set by the Fermi coupling constant. This sets stringent bounds on the mass and coupling strength of the axion. Our empirical formula yields an axion mass in the keV range, with predicted coupling strength. This "visible" axion model has been ruled out by experiments: if the axion exists at all, it must be that $V_{PQ} \gg G_F^{-1/2}$.

Hence the PQ symmetry can only be broken by a field with no electroweak quantum numbers. The first model of this kind was proposed by J. E. Kim (*Phys. Rev. Lett.* **43**, 103(1979)), who introduced an extra quark coupled to an electroweak-singlet PQ order parameter. Since the axion couples to matter with strength inversely proportional to V_{PQ}, this type of model leads to an "invisible axion".

Another way to build such a model is to add a new electroweak singlet Higgs field $\Phi(x)$ to the previous model with two Higgs doublets. Its role is to break the PQ symmetry at a scale much larger than the electroweak scale. This field does not couple directly to quarks and leptons, but acquires a PQ quantum number by coupling to the Higgs doublets. One can envisage two couplings

$$H_u^t \tau_2 H_d \Phi \ , \qquad H_u^t \tau_2 H_d \Phi^2 \ , \tag{9.19}$$

together with their conjugates.

A model with the cubic interaction was first proposed by Zhitnitskii (*Sov. J. Nucl. Phys.* **31**, 260(1980)). The cubic interaction can be set to zero naturally, by imposing the discrete symmetry $\Phi \to -\Phi$. A model with only the quartic term is due to M. Dine, W. Fischler and M. Srednicki (*Phys. Lett.* **104B**, 199(1981)). In that case, the field Φ has PQ charge of -1, and the total potential is simply

$$V(H_u, H_d, \Phi) = V(H_u, H_d) + V_0(\Phi) + \Delta H_u^T \tau_2 H_d \Phi^2 + \text{c.c.} \ , \tag{9.20}$$

where $V_0(\Phi)$ is given by

$$V_0(\Phi) = -M^2 \Phi^* \Phi + \Lambda(\Phi^* \Phi)^2 \ .$$

This axion model has now ten parameters in its Higgs potential!

When the three neutral scalar fields in the potential acquire vacuum values, two phase symmetries will be broken; one is the usual linear combination of hypercharge and third component of weak isospin which breaks the electroweak symmetry, the other is the PQ symmetry.

In addition to being broken spontaneously by the electroweak singlet Φ, the PQ symmetry is explicitly broken by QCD instanton-like effects, with strength determined by the anomaly diagram

$$N\delta_{AB} = \sum_a \text{Tr}(Q_{PQ} Q_A^a Q_B^a) \ , \tag{9.21}$$

where Q_A^a is the color charge of quark a, and Q_{PQ} its PQ charge. The effect of this anomalous divergence can be accounted for by adding to the Lagrangian an effective interaction proportional to the determinant of the quark mass matrix, $\det(\mathbf{q}_L^\dagger \mathbf{q}_R)$ and its conjugate. Indeed, under the PQ transformation

$$\mathbf{q} \to e^{i\alpha\gamma_5} \mathbf{q} \ , \tag{9.22}$$

the determinant certainly breaks the overall continuous chiral U(1), since for N_f quark flavors

$$\det(\mathbf{q}_L^\dagger \mathbf{q}_R) \to e^{-2iN_f\alpha} \det(\mathbf{q}_L^\dagger \mathbf{q}_R) \ . \tag{9.23}$$

Sikivie (*Phys. Rev. Lett.* **48**, 1156(1982)) remarked that for the special values

$$\alpha = \frac{\pi k}{N_f} \ , \qquad k = 1, 2, \ldots \ , \tag{9.24}$$

the interaction term stays invariant. Consequently the Z_{2N_f} discrete subgroup of $U(1)_{PQ}$ survives strong coupling effects. Still, this discrete subgroup is spontaneously broken by the Higgs vacuum value $< \Phi >$. For instance, in the DFS model, Φ transforms under $U(1)_{PQ}$, as

$$\Phi \to e^{i2\alpha}\Phi \ , \tag{9.25}$$

leaving, for α a multiple of π, the discrete subgroup Z_2 invariant. The spontaneous breaking of $U(1)_{PQ}$ breaks the true discrete symmetry, namely Z_{2N_f} down to Z_2. This produces N_f degenerate regions, separated by *domain walls*, with possibly nefarious consequences in cosmology.

At any case, the field configuration of the vacuum has a two-fold continuous degeneracy, parametrized by two angles, $\theta_0(x)$, $\theta_1(x)$, which are related to the vacuum fields by

$$\begin{aligned} < H_u >_0 &= \frac{v_u}{\sqrt{2}} e^{i[\theta_0(x)+\theta_1(x)]} \ , \\ < H_d >_0 &= \frac{v_d}{\sqrt{2}} e^{i[\theta_0(x)-\theta_1(x)]} \ , \\ < \Phi >_0 &= \frac{V_{PQ}}{\sqrt{2}} e^{-i\theta_0(x)} \ , \end{aligned} \tag{9.26}$$

where all three vacuum values are taken to be real without loss of generality. The vacuum values of the doublets are limited by the Fermi constant

$$G_F^{-1} \equiv v^2 = v_u^2 + v_d^2 \ . \tag{9.27}$$

There is no such restriction on V_{PQ}. Since Φ is a gauge singlet, quantum corrections to this potential do not detune the hierarchy $V_{PQ} \gg G_F^{-1/2}$. The kinetic terms for these fields are easily seen to be

$$\mathcal{L}_{\rm Kin} = \frac{1}{2}(\partial_\mu\theta_0, \partial_\mu\theta_1) \begin{pmatrix} V_{PQ}^2 + v_u^2 + v_d^2 & v_u^2 - v_d^2 \\ v_u^2 - v_d^2 & v_u^2 + v_d^2 \end{pmatrix} \begin{pmatrix} \partial_\mu\theta_0 \\ \partial_\mu\theta_1 \end{pmatrix} \ . \tag{9.28}$$

Upon diagonalization to canonical form, we find

$$\mathcal{L}_{\rm Kin} = \frac{1}{2}\partial_\mu a(x)\partial_\mu a(x) + \frac{1}{2}\partial_\mu\theta(x)\partial_\mu\theta(x) \ , \tag{9.29}$$

where the canonical fields are the axion field

$$a(x) = \sqrt{V_{PQ}^2 + v^2 \sin^2 2\beta}\ \theta_0(x) \equiv V_{PQ}\theta_0(x) \ , \tag{9.30}$$

and

$$\theta(x) = v\cos 2\beta\theta_0(x) + v\theta_1(x) \ , \tag{9.31}$$

is the field that will be eaten in the unitary gauge to become the longitudinal Z-boson. We have introduced the angle β through $\tan\beta \equiv v_u/v_d$. It is easy to read off the couplings of the axion to the quarks, by rewriting the Yukawa couplings in the form

$$i\mathbf{u}^\dagger_{Ri}\mathbf{u}_{Li}m_{ui}e^{i\frac{a(x)}{V_{PQ}}2\sin^2\beta} + i\mathbf{d}^\dagger_{Ri}\mathbf{d}_{Li}m_{di}e^{i\frac{a(x)}{V_{PQ}}2\cos^2\beta} + \text{ c.c. .} \tag{9.32}$$

Because of the relative i between the mass term and the axion term, we see that the axion couples to $\overline{\mathbf{u}}\gamma_5\mathbf{u}$ and $\overline{\mathbf{d}}\gamma_5\mathbf{d}$, the divergences of the axial currents $\overline{\mathbf{u}}\gamma_5\gamma_\mu\mathbf{u}$ and $\overline{\mathbf{d}}\gamma_5\gamma_\mu\mathbf{d}$. The coupling to the lepton is the same as that to the down quarks.

An equivalent way to arrive at this coupling is to observe that, as a Nambu-Goldstone boson, the axion couples to the divergence of the PQ current

$$J^{PQ}_\mu = V_{PQ}\partial_\mu a(x) + 2\sin^2\beta\overline{\mathbf{u}}_i\gamma_\mu\gamma_5\mathbf{u}_i + 2\cos^2\beta(\overline{\mathbf{d}}_i\gamma_\mu\gamma_5\mathbf{d}_i + \overline{e}_i\gamma_\mu\gamma_5 e_i)\ , \tag{9.33}$$

where i is the summed-over family index. So far the relevant part of the Lagrangian, not showing the leptons and the potential, reads

$$\begin{aligned}\mathcal{L} \;&=\; im_{ui}\mathbf{u}^\dagger_{Ri}\mathbf{u}_{Li}e^{i\frac{a(x)}{V_{PQ}}2\sin^2\beta} + im_{di}\mathbf{d}^\dagger_{Ri}\mathbf{d}_{Li}e^{i\frac{a(x)}{V_{PQ}}2\cos^2\beta} + \text{ c.c.}\\ &+\; \frac{1}{2}\partial_\mu a(x)\partial_\mu a(x) + \theta\frac{g_3^2}{32\pi^2}\mathrm{Tr}(\mathbf{G}_{\mu\nu}\widetilde{\mathbf{G}}_{\mu\nu}) + \cdots\ ,\end{aligned} \tag{9.34}$$

Under the chiral rotation

$$\mathbf{u}_{L,Ri} \to e^{\mp i\sin^2\beta\frac{a(x)}{V_{PQ}}}\mathbf{u}_{L,Ri}\ , \qquad \mathbf{d}_{L,Ri} \to e^{\mp i\cos^2\beta\frac{a(x)}{V_{PQ}}}\mathbf{d}_{L,Ri}\ , \tag{9.35}$$

the quark kinetic term picks up the extra pieces

$$\frac{-i}{V_{PQ}}\left(\sin^2\beta\overline{\mathbf{u}}_i\gamma_\mu\gamma_5\mathbf{u}_i + \cos^2\beta\overline{\mathbf{d}}_i\gamma_\mu\gamma_5\mathbf{d}_i\right)\partial_\mu a(x)\ . \tag{9.36}$$

This transformation is anomalous so that the fermion measure in the path integral picks up a phase proportional to $\mathrm{Tr}(\mathbf{G}_{\mu\nu}\widetilde{\mathbf{G}}_{\mu\nu})$, leading to a shift in the θ parameter

$$\theta \to \theta' \;=\; \theta + \frac{N_f}{2}(2\sin^2\beta + 2\cos^2\beta)\frac{a(x)}{V_{PQ}}\ ,$$

$$= \quad \theta + N_f \frac{a(x)}{V_{PQ}} \; . \tag{9.37}$$

We set the parameter θ to zero by redefining the phases in the quark masses, yielding the final Lagrangian

$$\begin{aligned} \mathcal{L} \;&=\; \overline{\mathbf{u}}_i(\not\partial + i m_{ui})\mathbf{u}_i + \overline{\mathbf{d}}_i(\not\partial + i m_{di})\mathbf{d}_i + \frac{1}{2}\partial_\mu a(x)\partial_\mu a(x) \\ &\quad - \frac{i}{V_{PQ}}\partial_\mu a(x)\left[\sin^2\beta\,\overline{\mathbf{u}}_i\gamma_\mu\gamma_5\mathbf{u}_i + \cos^2\beta\,\overline{\mathbf{d}}_i\gamma_\mu\gamma_5\mathbf{d}_i\right] \\ &\quad + N_f\frac{a(x)}{V_{PQ}}\frac{g_3^2}{32\pi^2}\mathrm{Tr}(\mathbf{G}_{\mu\nu}\tilde{\mathbf{G}}_{\mu\nu}) + \cdots \; . \end{aligned} \tag{9.38}$$

We are pleased to recognize, after integration by parts, that the axion couples manifestly to the divergence of J_μ^{PQ}, augmented by its anomalous part. We have achieved our purpose in replacing the θ parameter by the dynamical axion field $a(x)$. This is the fundamental Lagrangian of QCD, to which we have added the axion couplings.

9.2.1 PROBLEMS

A. Devise an invisible axion model which contains a new quark and only one Higgs doublet.

B.1-) Show that the potential of Eq. (pot?) is the most general potential with PQ symmetry.

2-) Find the necessary conditions among the couplings to insure the the tree-level inequality $V \gg v_u, v_d$.

3-) Show using diagrammatic techniques that this inequality is stable against radiative corrections (technically natural).

9.3 The Axion Chiral Lagrangian

We now proceed to incorporate the axion into the QCD chiral Lagrangian. To that effect, we first integrate over the heavy quarks, and then make a PQ transformation on the light quarks to transfer the axion coupling into their masses. There are three light quarks for which $m \leq \Lambda_{QCD}$. In the following we assume only two light quarks, leaving it as an exercise to the reader to construct the three-light flavor axion effective low energy chiral Lagrangian. We start with

$$\mathbf{u} \to = e^{i\sin^2\beta\frac{a(x)}{V_{PQ}}\gamma_5}\mathbf{u}\ , \qquad \mathbf{d} \to e^{i\cos^2\beta\frac{a(x)}{V_{PQ}}\gamma_5}\mathbf{d}\ , \tag{9.39}$$

together with a change of the anomaly coefficient

$$N_f\frac{a(x)}{V_{PQ}} \to (N_f - 2\sin^2\beta - 2\cos^2\beta)\frac{a(x)}{V_{PQ}} = (N_f - 2)\frac{a(x)}{V_{PQ}}\ . \tag{9.40}$$

After neglecting (integrating over) the heavy quark fields, we arrive at the relevant part of the Lagrangian

$$\begin{aligned} \mathcal{L} &= im_u\overline{\mathbf{u}}e^{i2\sin^2\beta\frac{a(x)}{V_{PQ}}\gamma_5}\mathbf{u} + im_d\overline{\mathbf{d}}e^{i2\cos^2\beta\frac{a(x)}{V_{PQ}}\gamma_5}\mathbf{d} \\ &+ (N_f - 2)\frac{a(x)}{V_{PQ}}\frac{g_3^2}{32\pi^2}\mathrm{Tr}(\mathbf{G}_{\mu\nu}\widetilde{\mathbf{G}}_{\mu\nu}) + \cdots\ . \end{aligned} \tag{9.41}$$

Note that the presence of the heavy quarks is only reflected in the anomaly term, which is to be expected since anomalies span scales with impunity. This expression forms the basis for the formulation of the low energy chiral Lagrangian.

However, in the real world effective chiral Lagrangian with axion, there is one further complication because the axion is not the only neutral (pseudo) Nambu-Goldstone boson in the problem. With three light quark flavors, there are the pion and the η. With two light quarks, there is only the pion. Recall that the η' is not light (it is in fact heavier than the proton) because its corresponding symmetry has been explicitly broken by strong coupling effects (e.g. instantons). In the absence of the axion, the two-flavor chiral Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu\eta'\partial_\mu\eta' + \frac{F_\pi^2}{16}\mathrm{Tr}(\partial^\mu\mathbf{\Sigma}\partial_\mu\mathbf{\Sigma}^{-1}) - \frac{1}{2}m_{\eta'}^2\eta'\eta' \\ &+ \frac{F_\pi^2}{8}m_0\mathrm{Tr}(e^{\frac{-2i}{F_\pi}\boldsymbol{\tau}^0\eta'}\mathbf{\Sigma}^{-1}\mathbf{M} + \mathrm{h.c.})\ , \end{aligned} \tag{9.42}$$

where

$$\mathbf{\Sigma}(x) = e^{\frac{2i}{F_\pi}\boldsymbol{\tau}^A\pi^A}\ , \tag{9.43}$$

$\mathbf{M}$ is the light quark mass matrix, and $m_{\eta'}$ is the mass of the heavy η' meson.

What is the effect of the anomaly on this system? In the case of two flavors, the θ-term can be transmitted to the quarks by the chiral transformation

$$\mathbf{q} \to e^{i\frac{\theta}{4}\gamma_5}\mathbf{q}\ , \qquad \mathbf{q} = \mathbf{u}, \mathbf{d}, \mathbf{s}\ , \tag{9.44}$$

which results in the shift in the phase of the condensate, or in the effective Lagrangian language,

$$\frac{\eta'}{F_\pi} \to \frac{\eta'}{F_\pi} - \frac{\theta}{4}\ . \tag{9.45}$$

To introduce the axion field in the chiral Lagrangian, we simply add its kinetic term, and rewrite the phase in terms of the axion field

$$\theta \to (N_f - 2)\frac{a(x)}{V_{PQ}} \equiv n_{lf}\frac{a(x)}{V_{PQ}}\ , \tag{9.46}$$

with the result

$$\begin{aligned}\mathcal{L}_{eff} = \frac{1}{2}\partial_\mu a \partial_\mu a + \frac{1}{2}\partial_\mu \eta' \partial_\mu \eta' + \frac{F_\pi^2}{16}\mathrm{Tr}(\partial_\mu \mathbf{\Sigma}^\dagger \partial_\mu \mathbf{\Sigma}) \\ -m_{\eta'}^2(\eta'(x) + n_{lf}\frac{F_\pi}{4}\frac{a(x)}{V_{PQ}})^2 \\ +m_0\frac{F_\pi^2}{8}\mathrm{Tr}\left\{e^{\frac{-2i}{F_\pi}\boldsymbol{\tau}^0\eta'}\mathbf{\Sigma}^{-1}\left(m_u e^{i2\sin^2\beta\frac{a(x)}{V_{PQ}}} \quad 00 \quad m_d e^{i2\cos^2\beta\frac{a(x)}{V_{PQ}}}\right) + \text{ h.c.}\right\}\ .\end{aligned} \tag{9.47}$$

It follows that the field

$$\hat{\eta}'(x) = \frac{1}{\sqrt{1+\xi^2}}\left[\eta'(x) + \xi a(x)\right]\ , \tag{9.48}$$

where

$$\xi = \frac{n_{lf}}{4}\frac{F_\pi}{V_{PQ}}\ , \tag{9.49}$$

picks up a mass generated by instanton-like effects

$$m_{\hat{\eta}'} = m_{\eta'}\sqrt{1+\xi^2}\ , \tag{9.50}$$

while the orthogonal combination

$$\hat{a}(x) = \frac{1}{\sqrt{1+\xi^2}}\left[a(x) - \xi\eta'(x)\right]\ , \tag{9.51}$$

only picks up a "light" mass generated by the quark masses. We rewrite the quark mass terms in the hatted, yielding in the neutral sector

$$m_0 \frac{F_\pi^2}{8} \mathrm{Tr}\mathbf{\Sigma}^{-1} \begin{pmatrix} m_u e^{i(\frac{\hat{a}}{V_{PQ}} g_u + \frac{\hat{\eta}'}{F_\pi} h_u)} & 0 \\ 0 & m_d e^{i(\frac{\hat{a}}{V_{PQ}} g_d + i\frac{\hat{\eta}'}{F_\pi} h_d)} \end{pmatrix} + \text{ h.c.} \ , \tag{9.52}$$

where

$$g_u = \frac{1}{\sqrt{1+\xi^2}}[2\sin^2\beta + \frac{n_{lf}}{2}] \ , \qquad g_d = \frac{1}{\sqrt{1+\xi^2}}[2\cos^2\beta + \frac{n_{lf}}{2}] \ , \tag{9.53}$$

and

$$h_u = \frac{1}{\sqrt{1+\xi^2}}(-2+\sin^2\beta \frac{n_{lf}}{2}\left(\frac{F_\pi}{V_{PQ}}\right)^2) \, , h_d = \frac{1}{\sqrt{1+\xi^2}}(-2+\cos^2\beta \frac{n_{lf}}{2}\left(\frac{F_\pi}{V_{PQ}}\right)^2) \, , \tag{9.54}$$

Computations are made simpler by concentrating solely on the neutral sector. It is easy to see that the potential is given by

$$V = -m_0 \frac{F_\pi^2}{4}(m_u \cos\theta_u + m_d \cos\theta_d) \ , \tag{9.55}$$

with

$$\theta_u = \frac{\hat{a}(x)}{V_{PQ}} g_u + \hat{\eta}'(x)\frac{h_u}{F_\pi} + \frac{2}{F_\pi}\pi_3(x) \ , \qquad \theta_d = \frac{\hat{a}(x)}{V_{PQ}} g_d + \hat{\eta}'(x)\frac{h_d}{F_\pi} - \frac{2}{F_\pi}\pi_3(x) \ ,$$

In the above, $m_{u,d}$ are positive and real, so that the minimum of the potential occurs when

$$\theta_u = 2n_u\pi \ , \qquad \theta_d = 2n_d\pi \ , \tag{9.56}$$

where $n_{u,d} = 0, 1, 2, \ldots$. We have already seen that there is a mass term for the $\hat{\eta}'$ field, which fixes it to zero at minimum. Hence, neglecting ξ^2, the minimum conditions reduce to

$$\frac{\hat{a}(x)}{V_{PQ}} N_f = 2(n_u + n_d)\pi \ , \tag{9.57}$$

which shows that the PQ solution does indeed solve the problem, by allowing

$$\overline{\theta} = \frac{N_f \hat{a}}{V_{PQ}} \ , \tag{9.58}$$

to relax to zero. This analysis also shows that the vacuum is degenerate *mod* 2π, reproducing our earlier observation.

The masses of the light bosons (π and $\hat{a}$) obtained by expanding the potential, and setting $\hat{\eta}' = 0$,

$$V \approx m_0 \frac{F_\pi^2}{4}[m_u(\frac{\hat{a}(x)}{V_{PQ}}g_u + \frac{2}{F_\pi}\pi_3(x))^2 + m_d(\frac{\hat{a}(x)}{V_{PQ}}g_d - \frac{2}{F_\pi}\pi_3(x))^2] + \cdots \ , \tag{9.59}$$

leading to the mass matrix

$$m_0(\hat{a} \quad \pi^3)\begin{pmatrix} \frac{F_\pi^2}{4V_{PQ}^2}(m_u g_u^2 + m_d g_d^2) & \frac{F_\pi}{2V_{PQ}}(m_u g_u - m_d g_d) \\ \frac{F_\pi}{2V_{PQ}}(m_u g_u - m_d g_d) & (m_u + m_d) \end{pmatrix}\begin{pmatrix} \hat{a} \\ \pi^3 \end{pmatrix} \ . \tag{9.60}$$

Since we are dealing with invisible axion models, we can simplify life and expand in $\frac{F_\pi}{V_{PQ}}$. The analysis yields the neutral pion mass,

$$m_{\pi^0}^2 = m_0(m_u + m_d) + \mathcal{O}((\frac{F_\pi}{V_{PQ}})^2) \ ,$$

and the axion mass

$$m_a^2 = m_{\pi^0}^2 N_f^2 (\frac{F_\pi}{2V_{PQ}})^2 \frac{m_u m_d}{(m_u + m_d)^2} + \mathcal{O}(\frac{F_\pi^4}{V_a^4}) \ . \tag{9.61}$$

The eigenstates are

$$\begin{aligned} \pi^0(x) &= \pi_3(x) + \frac{F_\pi}{2V_{PQ}}\left\{\frac{m_u g_u - m_d g_d}{m_u + m_d}\right\}\hat{a}(x) + \mathcal{O}(\frac{F_\pi^2}{V_{PQ}^2}) \ , \\ a_{phys}(x) &= \hat{a}(x) - \frac{F_\pi}{2V_{PQ}}\left\{\frac{m_u g_u - m_d g_d}{m_u + m_d}\right\}\pi_3(x) + \mathcal{O}(\frac{F_\pi^2}{V_{PQ}^2}) \ . \end{aligned} \tag{9.62}$$

In the invisible axion model, it is clear that the mixing is minimal since $F_\pi \ll V_{PQ}$. It is important to realize that the properties of the invisible axion are determined by very few parameters, namely V_{PQ} which fixes both the overall strength of its couplings to matter and its mass, as well as model-dependent coupling constants to specific quarks and leptons. In the next section, we address phenomenological constraints on V_{PQ} and possible modes of detection. These rely on the electromagnetic interactions of the axion, which we have hitherto neglected.

9.3.1 Axion detection

In the original axion model where the breaking of the PQ symmetry is at electroweak scale, all of the axion properties are determined in terms of one parameter, the angle β. Its mass is given by

$$m_a = \frac{N_f}{\sin 2\beta}\frac{\sqrt{m_u m_d}}{m_u + m_d}\frac{F_\pi m_{\pi^0}}{v} \approx \frac{147.2}{\sin 2\beta}\ \text{keV}\ . \tag{9.63}$$

Its couplings to matter are through the term

$$\mathcal{L}_{int} = \frac{1}{2v}a(x)\partial_\mu J_\mu^{PQ}\ , \tag{9.64}$$

where, neglecting the lepton contributions,

$$\begin{aligned} J_\mu^{PQ} &= v\partial_\mu a - \frac{1}{2}\sum_{i=1}^{3}(\cot\beta\overline{\mathbf{u}}_i\gamma_\mu\gamma_5\mathbf{u}_i - \tan\beta\overline{\mathbf{d}}_i\gamma_\mu\gamma_5\mathbf{d}_i) \\ &+ \frac{3}{(m_u + m_d)\sin 2\beta}(m_d\overline{\mathbf{u}}_1\gamma_\mu\gamma_5\mathbf{u}_1 + m_u\overline{\mathbf{d}}_1\gamma_\mu\gamma_5\mathbf{d}_1)\ . \end{aligned} \tag{9.65}$$

This current is the canonical PQ current minus a contribution from the two light quarks to make it anomaly-free, and orthogonal to the neutral pion current. This original axion model is in contradiction with experiment.

In invisible axion models, V_{PQ} is no longer constrained by the electroweak scale. This has two consequences; it reduces its couplings to matter and makes it extremely light. Thus, the possibility of its detection through laboratory experiments seems remote, hence the name invisible.

But there are serious restrictions on V_{PQ} : a lower bound set by astrophysical considerations, and an upper bound based on the axion's contribution to the universe's energy density. Remarkably the two still leave a "small" (we are conforming to the standard accuracy of pre-nucleosynthesis cosmology by displaying the error in the exponent) window

$$10^{9\pm 1}\ \text{GeV} \le f_a = \frac{V_{PQ}}{N} \le 10^{12\pm 1}\ \text{GeV}\ . \tag{9.66}$$

Any value outside this range leads to severe contradictions with our present understanding of stellar and cosmological evolutions.

In the following we do not detail these bounds but refer the reader to the many excellent texts on the subject, notably *The Early Universe*, *po. cit.*.

The lower bound is set by noting that the axion provides a pathway for energy loss in stars. Once created inside a star, the wee-weakly interacting axion can escape the star, causing greater energy loss and faster aging. The lower bound is derived from the existence of old stars, such as red giants.

The tracing of the cosmological evolution of the axion provides the upper bound . The axion stays in thermal equilibrium with the primal heat bath until it decouples at a temperature $T \approx V_{PQ}$. Relativistic axions are simply red-shifted away. As the universe cools to a temperature of the order of Λ_{QCD}, the axion develops a mass, and therefore a parabolic potential. At later times, the axion is at the bottom of this parabola, but earlier it might be anywhere. Thus it starts oscillating about the minimum, with a classical field of the form

$$< a >= A(t) \cos m_a t \ , \tag{9.67}$$

where $A(t)$ is of order of V_{PQ}. The axion hardly decays and so the oscillations are not damped. The result of this coherent oscillation is to produce an energy density in the axion field which evolves like matter. Thus it grows faster than radiation. Since it is proportional to V_{PQ}, it can close the universe for too large a value of V_{PQ}. The complicated detailed calculations, performed by many groups, all lead to the same upper bound.

Although this range of allowed V_{PQ} leads to a very tiny coupling, Sikivie (*Phys. Rev. Lett.* **51**, 1415(1983)) pointed out that the coupling to electromagnetism can still lead to an effect observable in the laboratory: the conversion of an axion into a photon inside an electromagnetic cavity.

At the level of the fundamental quark-lepton Lagrangian, electromagnetic interactions are introduced by replacing the derivative by the Maxwell covariant derivative. However when quarks and leptons undergo chiral rotations, the anomalous divergence induces a term along $F_{\mu\nu}\widetilde{F}_{\mu\nu}$ term as well, leading to an induced interaction for the axion of the form

$$\mathcal{L}_{int} = N_e \frac{1}{32\pi^2} \frac{a(x)}{V_{PQ}} F_{\mu\nu}\widetilde{F}_{\mu\nu} \ , \tag{9.68}$$

where in the DFS model, with three chiral families, $N_e = 4$.

To obtain the full coupling in the effective low energy chiral Lagrangian, we have to shift to account for the chiral transformation of the light quarks, and also for the diagonalization of the axion-η'- pion system. Subtracting out the anomaly of the **u** and **d** quarks means that the anomaly coefficient is now

$$N'_e = N_e - \frac{8}{3}\sin^2\beta - \frac{2}{3}\cos^2\beta \ . \tag{9.69}$$

This equation forms the basis for the possible detection of the *invisible* axion. The more axions in the cavity, the more conversions will be observed. If axions are the main constituents of dark matter, it is conceivable their presence could be detected in such cavities. An ongoing experiment has so far been unable to "tune" on the axion (C. Hagmann *et al*, *Phys. Rev. Lett.* **80**, 2043(1998)).

9.3.2 PROBLEMS

A. Show, using symmetry arguments, that the *axion mass* vanishes if any quark mass is zero.

B. Show that in the original "visible" axion model (see W. Bardeen and H. Tye, *Phys. Lett.* **B74**, 229(1978)), the axion mass is given by

$$m_a = \frac{N_f}{\sin 2\beta}\frac{\sqrt{m_u m_d}}{m_u + m_d}\frac{F_\pi m_{\pi^0}}{v} \tag{9.70}$$

C. Start from the results of chapter 5 to derive the axion mass for three flavors. Verify that in the limit of $m_{u,d}/m_s \to 0$ it reduces to the expression derived in the text.