Appendix 1
Supersymmetry Toolbox

This Appendix introduces the basic features of theories with global $N = 1$ supersymmetry using the language of local field theory. In the absence of gravity, $N = 1$ supersymmetry employs two collections of fields, each connected by supersymmetry transformations. The first is the chiral or Wess-Zumino supermultiplet, which consists of one left-handed Weyl spinor and one complex scalar field. The second is the gauge supermultiplet, containing the Yang-Mills gauge bosons, and the gauginos, their spin 1/2 supersymmetric partners. Since these supermultiplets already contains fields readily identified with those in the standard model, the $N = 1$ supersymmetric standard model is naturally described in terms of these multiplets and their interactions.

A1.1 The Chiral Supermultiplet

The simplest set of fields on which $N = 1$ supersymmetry is realized is the chiral or Wess-Zumino multiplet which contains the three fields,

$$\begin{aligned} &\varphi(x) \qquad , \text{ a complex scalar },\\ &\psi(x) \qquad , \text{ a Weyl spinor },\\ &F(x) \qquad , \text{ a complex auxiliary field }. \end{aligned} \tag{1.1}$$

The free Lagrangian is given by

$$\mathcal{L}_0^{WS} \equiv \partial_\mu\varphi^*\partial^\mu\varphi + \psi^\dagger\sigma^\mu\partial_\mu\psi + F^*F \ . \tag{1.2}$$

Up to a surface term, it is invariant under the transformations

$$\begin{aligned}\delta\varphi &= \widehat{\alpha}\psi \ ,\\ \delta\psi &= \alpha F - \overline{\sigma}^\mu \widehat{\alpha}^\dagger \partial_\mu \varphi \ ,\\ \delta F &= -\alpha^\dagger \sigma^\mu \partial_\mu \psi \ .\end{aligned} \tag{1.3}$$

Here α, a Weyl spinor with Grassmann components, is the global parameter of the supersymmetry transformation. Grassmann variables are just anticommuting numbers, so that for any two

$$(\widehat{\zeta}\chi)^* = \zeta^\dagger \sigma_2^* \chi^* = -\zeta^\dagger \widehat{\chi}^\dagger \ , \tag{1.4}$$

which explains the absence of an i in front of the fermion kinetic term. The fields φ and ψ, with respective canonical dimensions, -1 and $-3/2$, are physical, while F is not, with the non-canonical dimension of -2. The supersymmetry parameter has dimension $1/2$. Note that F transforms as a total divergence.

Under two supersymmetry transformations, labelled δ_1 and δ_2, with parameters α_1 and α_2, we find that

$$[\delta_1, \delta_2] \begin{pmatrix} \varphi \\ \psi \\ F \end{pmatrix} = (\alpha_1^\dagger \sigma^\mu \alpha_2 - \alpha_2^\dagger \sigma^\mu \alpha_1) \partial_\mu \begin{pmatrix} \varphi \\ \psi \\ F \end{pmatrix} \ . \tag{1.5}$$

This equation shows that the result of two supersymmetry transformations is nothing but a space-time translation by the amount

$$\delta x^\mu = (\alpha_1^\dagger \sigma^\mu \alpha_2 - \alpha_2^\dagger \sigma^\mu \alpha_1) \ , \tag{1.6}$$

recalling that $P_\mu = -i\partial_\mu$ is the generator of translations. Supersymmetry transformations act as square roots of translations, and generalize the Poincaré group. We can readily verify this algebra on one of the fields

$$\begin{aligned}\delta_1\delta_2 F &= -\alpha_2^\dagger \sigma^\mu \partial_\mu \delta_1 \psi \ ,\\ &= -\alpha_2^\dagger \sigma^\mu \alpha_1 \partial_\mu F - \alpha_2^\dagger \sigma^\mu \overline{\sigma}^\rho \widehat{\alpha}_1^\dagger \partial_\mu \partial_\rho \varphi \ .\end{aligned} \tag{1.7}$$

The symmetry of $\partial_\mu \partial_\rho \varphi$ allows us to set

$$\sigma^\mu \overline{\sigma}^\rho = \frac{1}{2}(\sigma^\mu \overline{\sigma}^\rho + \sigma^\rho \overline{\sigma}^\mu) = g^{\mu\rho} \ , \tag{1.8}$$

leading to

$$\delta_1\delta_2 F = -\alpha_2^\dagger\sigma^\mu\alpha_1\partial_\mu F - \alpha_2^\dagger\widehat{\alpha}_1^\dagger g^{\mu\nu}\partial_\mu\partial_\nu\varphi \ . \tag{1.9}$$

Now $\alpha_2^\dagger\widehat{\alpha}_1^\dagger$ is symmetric under the $(1 \leftrightarrow 2)$ interchange, and drops out from the commutator, giving the desired result

$$[\delta_1, \delta_2]F = (\alpha_1^\dagger\sigma^\mu\alpha_2 - \alpha_2^\dagger\sigma^\mu\alpha_1)\partial_\mu F \ . \tag{1.10}$$

The other two expressions for φ and ψ work out in a similar way, making use of Fierz identities applied to ψ.

All these results can be neatly summarized by introducing a two-component Weyl spinor Grassmann variable θ. We construct the *chiral superfield* $\Phi(x,\theta)$ which depends only on θ. Since θ is Grassmann, its cube vanishes, resulting, without loss of generality, in the expansion

$$\Phi(x,\theta) = \varphi(x) + \widehat{\theta}\psi(x) + \frac{1}{2}\widehat{\theta}\theta F(x) \ . \tag{1.11}$$

We then express the supersymmetry transformation as acting on the fields,

$$\delta\Phi = \delta\varphi + \widehat{\theta}\delta\psi + \frac{1}{2}\widehat{\theta}\theta\delta F \ , \tag{1.12}$$

and rewrite it as an operator acting on the coordinates

$$\delta\Phi = \left[\widehat{\alpha}\frac{\partial}{\partial\theta} + \alpha^\dagger\sigma^\mu\theta\partial_\mu\right]\Phi \ , \tag{1.13}$$

where the Grassmann derivative is defined through

$$\frac{\partial}{\partial\theta}\ \widehat{\theta} = 1 \ . \tag{1.14}$$

Expressing the supersymmetry transformations in this way enables us to directly derive the commutator formula. It also clearly shows why the change in the coefficient of $\widehat{\theta}\theta$ is a total divergence: it can only come from the term linear in θ in the supersymmetry generator, which contains the space-time derivative operator.

We can write the effect of a supersymmetry transformation on the chiral superfield in another way, namely as

$$\Phi(x^\mu,\theta) \to \Phi(x^\mu + \alpha^\dagger\sigma^\mu\theta, \theta + \alpha) \ , \tag{1.15}$$

but it is somewhat ackward since the change in x^μ is not real. If we write the change in terms of its real plus imaginary parts,

$$\alpha^\dagger\sigma^\mu\theta = \frac{1}{2}(\alpha^\dagger\sigma^\mu\theta - \theta^\dagger\sigma^\mu\alpha) + \frac{1}{2}(\alpha^\dagger\sigma^\mu\theta + \theta^\dagger\sigma^\mu\alpha) \ , \tag{1.16}$$

we find that the imaginary part of the shift can be written as the change of $\theta^\dagger\sigma_\mu\theta/2$. This suggests that we replace x^μ by

$$y^\mu = x^\mu + \frac{1}{2}\theta^\dagger\sigma^\mu\theta \ , \tag{1.17}$$

and consider the chiral superfield as a function of y_μ. It is of course no longer chiral (a function of θ alone), its expansion being given by

$$\begin{aligned}\Phi(y^\mu,\theta) &= \varphi(x) + \widehat{\theta}\psi(x) + \frac{1}{2}\widehat{\theta}\theta F(x) + \frac{1}{2}\theta^\dagger\sigma^\mu\theta\partial_\mu\varphi(x) \\ &- \frac{1}{4}\widehat{\theta}\theta\theta^\dagger\sigma^\mu\partial_\mu\psi(x) + \frac{1}{16}|\widehat{\theta}\theta|^2\partial^\mu\partial_\mu\varphi(x) \ , \end{aligned} \tag{1.18}$$

using some Fierzing and the identity

$$\theta^\dagger\sigma^\mu\theta\theta^\dagger\sigma^\nu\theta = \frac{1}{2}g^{\mu\nu}|\widehat{\theta}\theta|^2 \ . \tag{1.19}$$

The manifestly real superfield

$$V(x^\mu,\theta,\theta^*) = \Phi^*(y^\mu,\theta)\Phi(y^\mu,\theta) \ , \tag{1.20}$$

depends on both θ and θ^*,

$$V^*(x^\mu,\theta,\theta^*) = V(x^\mu,\theta,\theta^*) \ \ , \tag{1.21}$$

and transforms under supersymmetry in a more pleasing way, namely with a real change in the coordinate x^μ

$$V(x^\mu,\theta,\theta^*) \to V(x^\mu + \frac{1}{2}[\alpha^\dagger\sigma^\mu\theta - \theta^\dagger\sigma^\mu\alpha], \theta+\alpha, \theta^*+\alpha^*) \ . \tag{1.22}$$

Its expansion is given by

$$\begin{aligned}V(x,\theta,\theta^*) &= \varphi^*(x)\varphi(x) + [\widehat{\theta}\psi\varphi^* - \theta^\dagger\widehat{\psi}^\dagger\varphi] \\ &+ \frac{1}{2}[\widehat{\theta}\theta\varphi^* F - \theta^\dagger\widehat{\theta}^\dagger\varphi F^* + \theta^\dagger\sigma^\mu\theta(\varphi^*\partial_\mu\varphi - \partial_\mu\varphi^*\varphi - \psi^\dagger\sigma_\mu\psi)]\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{4}\theta^\dagger\widehat{\theta}^\dagger(2F^*\widehat{\psi} + \varphi\partial_\mu\psi^\dagger\sigma^\mu - \partial_\mu\varphi\psi^\dagger\sigma^\mu)\theta \\
&- \frac{1}{4}\widehat{\theta}\theta\theta^\dagger(2\widehat{\psi}^\dagger F + \varphi^*\sigma^\mu\partial_\mu\psi - \sigma^\mu\psi\partial_\mu\varphi^*) \\
&+ \frac{1}{8}|\widehat{\theta}\theta|^2\Big(2F^*F - \partial_\mu\varphi^*\partial^\mu\varphi + \frac{1}{2}(\varphi^*\partial^\mu\partial_\mu\varphi + \varphi\partial^\mu\partial_\mu\varphi^*) \\
&\qquad\qquad + \psi^\dagger\sigma^\mu\partial_\mu\psi - \partial_\mu\psi^\dagger\sigma^\mu\psi\Big) \ . \qquad (1.23)
\end{aligned}
$$

The alert student will recognize the last term as the Lagrange density, plus an overall divergence.

It is a bit tedious to verify the transformation laws of all the components of the real superfield. Here we illustrate it only for terms of the form $\partial_\mu\varphi\varphi^*$. On the one hand, we get from the shifts in coordinates

$$
\begin{aligned}
V(x^\mu) &+ \frac{1}{2}(\alpha^\dagger\sigma^\mu\theta - \theta^\dagger\sigma^\mu\alpha), \theta + \alpha, \theta^* + \alpha^*) \\
&= \frac{1}{2}\varphi^*(\alpha^\dagger\sigma^\mu\theta - \theta^\dagger\sigma^\mu\alpha)\partial_\mu\varphi + \frac{1}{2}(\alpha^\dagger\sigma^\mu\theta + \theta^\dagger\sigma^\mu\alpha)\varphi^*\partial_\mu\varphi + \cdots \\
&= \alpha^\dagger\sigma^\mu\theta\varphi^*\partial_\mu\varphi + \cdots \ .
\end{aligned}
$$

On the other hand, by varying the fields directly, we obtain the very same term

$$
\begin{aligned}
\widehat{\theta}\delta\psi\varphi^* &= -\widehat{\theta}\overline{\sigma}^\mu\sigma_2\alpha^*\partial_\mu\varphi\varphi^* + \cdots \ , \\
&= \alpha^\dagger\sigma^\mu\theta\partial_\mu\varphi\varphi^* + \cdots \ .
\end{aligned}
$$

We can also interpret the supersymmetric change on the real superfield in terms of the action of differential operators, namely

$$
\delta V = \Big\{\widehat{\alpha}(\frac{\partial}{\partial\theta} + \frac{1}{2}\overline{\sigma}^\mu\widehat{\theta}^\dagger\partial_\mu) - \alpha^\dagger(\widehat{\left(\frac{\partial}{\partial\theta}\right)}^\dagger - \frac{1}{2}\sigma^\mu\theta\partial_\mu)\Big\}V(x, \theta, \theta^*) \ , \qquad (1.24)
$$

where we have used the identity

$$
\theta^\dagger\sigma^\mu\alpha = -\widehat{\alpha}\overline{\sigma}^\mu\widehat{\theta}^\dagger \ .
$$

These equations suggest we introduce the generators of supersymmetry

$$
\begin{aligned}
Q &= \frac{\partial}{\partial\theta} + \frac{1}{2}\overline{\sigma}^\mu\widehat{\theta}^\dagger\partial_\mu \ , \\
\widehat{Q}^\dagger &= \widehat{\left(\frac{\partial}{\partial\theta}\right)}^\dagger - \frac{1}{2}\sigma^\mu\theta\partial_\mu \ ,
\end{aligned}
\qquad (1.25)
$$

to write the change in the real superfield

$$\delta V = (\widehat{\alpha} Q - \alpha^\dagger \widehat{Q}^\dagger) V(x, \theta, \theta^*) \ , \tag{1.26}$$

The supersymmetry generators satisfy the anticommutation relations

$$\begin{aligned} \{Q, Q\ \} &= \{Q^*, Q^*\} = 0 \ , \\ \{Q, Q^*\} &= \overline{\sigma}^\mu \frac{\partial}{\partial x_\mu} \ . \end{aligned} \tag{1.27}$$

Added to the generators of the Poincaré group, these generators form the super-Poincaré group. It follows that the particles described by supersymmetry must form irreducible representations of this supergroup.

Any representation of the super Poincaré group can be organized in terms of representations of its subgroup, the Poincaré group. It is easy to see that the supersymmetry generators commute with the translations,

$$[Q, P_\mu] = 0 \ . \tag{1.28}$$

Hence they also commute with $P_\mu P^\mu$, the mass squared Casimir operator of the Poincaré subgroup. We can therefore split our analysis in terms of the mass.

It is simplest to start with massless representations. The massless representations of the Poincaré group are labelled by the helicity λ which runs over positive and negative integer and half-integer values. In local field theory, each helicity state $|\lambda >$ is accompanied by its CPT conjugate $|-\lambda>$; for example, the left polarized photon $|\lambda = +1>$ and its CPT conjugate the right polarized photon $|\lambda = -1>$.

In the infinite momentum frame, with only $P_0 = P_3 \neq 0$, the supersymmetry algebra reduces to the Clifford algebra

$$\{Q_1, Q_1^*\} = iP_0 \ ,$$

all other anticommutators being zero. There is only one supersymmetry operator (that is why it is called $N = 1$). Together with its conjugate, they act like the creation and annihilation operators of a one-dimensional fermionic harmonic oscillator. Starting from any state $|\ \lambda >$, we can generate only one other state $Q^*|\ \lambda >$, which has helicity $\lambda + 1/2$. A second application of the raising operator yield zero since $Q_1^2 = 0$. This construction yields the only massless irreducible representation of $N = 1$ supersymmetry: two states, differing by half a unit of helicity. The lowest representations are

– The Wess-Zumino multiplet corresponds to the representation $\mid 0 > \oplus \mid 1/2 >$, together with its CPT conjugate $\mid 0 > \oplus \mid -1/2 >$. It describes one Weyl fermion and two scalar degrees of freedom.

– The gauge multiplet contains the states $\mid 1 > \oplus \mid 1/2 >$, together with their CPT conjugates $\mid -1 > \oplus \mid -1/2 >$. These describe a vector particle and a Weyl fermion.

There is an infinite of representations with higher helicities. We should note the Rarita-Schwinger representation $\mid 3/2 > \oplus \mid 1 >$ and its conjugate, as well as the graviton-gravitino combination, made up of $\mid 2 > \oplus \mid 3/2 >$, plus conjugate. The latter appears in supergravity, the local generalization of supersymmetry.

It is obvious that the number of bosonic and fermionic degrees of free dom match exactly in these representations. This can be seen in the field theory multiplets as well: using the equations of motion, the chiral multiplet has two fermionic degrees of freedom, exactly matched by the complex scalar field. If the equations of motion are not used, the number of fermions doubles, but the excess in fermions is exactly matched by adding two boson fields, the complex auxiliary field F.

The massive representations of the super Poincaré group can be obtained by assembling massless multiplets, using the group-theoretical equivalent of the Higgs mechanism. We leave to the reader the construction of the lowest lying supermultiplets.

Clearly, the real supermultiplet is highly reducible. It can be checked that the covariant derivative operator

$$\mathcal{D} \equiv \frac{\partial}{\partial \theta} - \frac{1}{2}\overline{\sigma}^{\mu}\widehat{\theta}^{\dagger}\partial_{\mu} \ , \tag{1.29}$$

and its complex conjugate anticommute with the generators of supersymmetry. By requiring that they vanish on the real superfield, we obtain the chiral superfield.

The construction of supersymmetric invariants is facilitated by the use of Grassmann variables. We have already noted that the highest component of a superfield transforms as a four-divergence, so that its integral over space-time is supersymmetric invariant. We can extract this component through the operation of Grassmann integration, defined by

$$\int d\theta \equiv 0 \ , \ \int d\theta\theta \equiv 1 \ ; \tag{1.30}$$

note that since θ has dimension 1/2, $d\theta$ has the opposite dimension, -1/2. Integration enables us to rewrite the invariant in the form

$$\int d^4x \int d^2\theta \ \Phi(x,\theta) = \int d^4xF \ . \tag{1.31}$$

It follows that the integral of a chiral superfield over superspace (x,θ), is a supersymmetric invariant.

However, any product of $\Phi(x,\theta)$ is itself a chiral superfield, since $\widehat{\theta}\theta\theta = 0$. For any number of chiral superfields $\Phi_a, \ a = 1,\ldots,N$, all the quantities

$$\int d^4x \int d^2\theta \ \Phi_{a_i} \cdots \Phi_{a_n} \quad \text{for all } a_i \text{ and } n \ , \tag{1.32}$$

are supersymmetric invariants.

For a real superfield, transforming under supersymmetry like $V(x,\theta,\theta^*)$, it is easy to show that its component along $|\widehat{\theta}\theta|^2$, called the D-term, also transforms as a four-divergence. Its space-time integral is therefore a supersymmetric invariant. By integrating over both θ and θ^*, we can extract the D-term. Indeed we have already seen that the kinetic part of the Lagrangian is a D-term

$$\mathcal{L}_{kin} = \int d^2\theta \int d^2\overline{\theta} \ |\Phi(y_\mu,\theta)|^2 \ . \tag{1.33}$$

It has the right dimension: the superfield has dimension one, and the four Grassmann integral bring dimension two.

The potential part of the Lagrangian is given by

$$V = \int d^2\theta W(\Phi) + c.c. \ , \tag{1.34}$$

where the function W is called the superpotential; it depends *only* on the chiral superfields, *not* their conjugates: it is a holomorphic function of the superfields. In renormalizable theories, it is at most cubic in the chiral superfields

$$W = m_{ij}\Phi_i\Phi_j + \lambda_{ijk}\Phi_i\Phi_j\Phi_k \ . \tag{1.35}$$

It is straightforward to see that the physical potential is simply expressed in terms of the superpotential

$$V(\varphi) = \sum_i F_i^* F_i = \sum_i |\frac{\partial W(\varphi)}{\partial \varphi_i}|^2 \; ; \tag{1.36}$$

it is obviously positive definite, a general feature of global supersymmetry. The components of the quadratic polynomial are given by

$$\int d^2\theta \Phi_1 \Phi_2 = (\varphi_1 F_2 + \varphi_2 F_1 - \widehat{\psi}_1 \psi_2) \; , \tag{1.37}$$

With this term alone in the superpotential, the equations of motions for the auxiliary fields are

$$F_1^* = -m\varphi_2 \; , \qquad F_2^* = -m\varphi_1 \; . \tag{1.38}$$

Substituting their solutions, we find

$$-m^2|\varphi_1|^2 - m^2|\varphi_2|^2 - m\widehat{\psi}_1\psi_2 \; . \tag{1.39}$$

These are the mass terms for four real scalars and one Dirac fermion of mass m, with the mass sum rule

$$\sum_{J=0} m^2 = 2 \sum_{J=1/2} m^2 \; , \tag{1.40}$$

where we count one Dirac = 2 Weyl fermions. We can rewrite this equation in the form

$$\mathcal{S}trM^2 \equiv \sum_{J=0,1/2} (2J+1)(-1)^{2J} m_J^2 = m^2 + m^2 - 2m^2 = 0 \; . \tag{1.41}$$

A cubic superpotential

$$\begin{aligned} \int d^2\theta \Phi_1 \Phi_2 \Phi_3 \quad &= \quad (\varphi_1\varphi_2 F_3 + \varphi_1 F_2 \varphi_3 + F_1 \varphi_2 \varphi_3 \\ &\qquad - \varphi_1 \widehat{\psi}_2 \psi_3 - \varphi_2 \widehat{\psi}_1 \psi_3 - \varphi_3 \widehat{\psi}_1 \psi_2) \; , \end{aligned}$$

contains the renormalizable Yukawa interactions, and, after using the equations of motion of the auxiliary fields, quartic renormalizable self-interactions. Clearly, higher order polynomials yield non-renormalizable interactions.

The kinetic term of the chiral multiplet has a special global symmetry, called R-symmetry, not found in non-supersymmetric models. It is not an internal symmetry since it does not commute with supersymmetry. R-symmetry is a global phase symmetry on the Grassmann variables

$$\theta \to e^{i\beta}\theta \ , \qquad \theta^* \to e^{-i\beta}\theta^* \ . \tag{1.42}$$

The Grassmann measures transform in the opposite way

$$d\theta \to e^{-i\beta}d\theta \ , \qquad d\theta^* \to e^{i\beta}d\theta^* \ . \tag{1.43}$$

Clearly, the Grassmann integration measure for the kinetic term is invariant, so that the most general R-type transformation that leaves the kinetic integrand invariant is

$$\Phi_i(y_\mu, \theta) \to e^{in_i\beta}\Phi_i(y_\mu, e^{i\beta}\theta) \ . \tag{1.44}$$

a In terms of components, this means that

$$\varphi_i \to e^{in_i\beta}\varphi_i \ , \qquad \psi_i \to e^{i(n_i-1)\beta}\psi_i \ . \tag{1.45}$$

This symmetry is not necessarily shared by the superpotential, unless it transforms under R as

$$W \to e^{2i\beta}W \ , \tag{1.46}$$

to match the transformation of the Grassmann measure, further restricting the form of the superpotential.

To implement internal symmetries, we simply assume that superfields transform as representation of the internal symmetry group. If the invariance is global, the kinetic part is automatically invariant, as it sums over all the internal degrees of freedom. The superpotential may or may not be invariant, depending on its form.

We close this section by working out certain functions of superfields which arise in discussing non-perturbative aspects of supersymmetric theories. As we have seen, products of chiral superfields are themselves chiral superfields, so that any special function of a chiral superfield is defined through its series expansion.

Logarithm

Given a chiral superfield

$$\Phi = \varphi(x) + \widehat{\theta}\psi(x) + \frac{1}{2}\widehat{\theta}\theta F(x) \ ,$$

we have

$$\ln \Phi = \ln \varphi + \ln[1 + \widehat{\theta}\psi'(x) + \frac{1}{2}\widehat{\theta}\theta F'(x)] \ , \tag{1.47}$$

where

$$\psi' = \frac{\psi}{\varphi} \ , \qquad F' = \frac{F}{\varphi} \ . \tag{1.48}$$

We then use the series expansion of the logarithm to obtain

$$\begin{aligned} \ln \Phi &= \ln \varphi + (\widehat{\theta}\psi' + \frac{1}{2}\widehat{\theta}\theta F') - \frac{1}{2}(\widehat{\theta}\psi' + \frac{1}{2}\widehat{\theta}\theta F')^2 \ , \\ &= \ln \varphi + \widehat{\theta}\psi' + \frac{1}{2}\widehat{\theta}\theta(F' + \frac{1}{2}\widehat{\psi}'\psi') \ , \end{aligned} \tag{1.49}$$

using the Fierz identities.

Power

The arbitrary power of a chiral superfield is given by its series expansion, since

$$\begin{aligned} \Phi^a &= \varphi^a\{1 + \widehat{\theta}\psi' + \frac{1}{2}\widehat{\theta}\theta F'\}^a \ , \\ &= \varphi^a\{1 + a\widehat{\theta}\psi' + \frac{1}{2}a\widehat{\theta}\theta F' + \frac{1}{2}a(a-1)(\widehat{\theta}\psi')^2\} \ , \end{aligned}$$

which, after a Fierz, yields the exact result

$$\Phi^a = \varphi^a[1 + a\widehat{\theta}\psi' + \frac{1}{2}\widehat{\theta}\theta(aF' - \frac{a(a-1)}{2}\widehat{\psi}'\psi')] \ . \tag{1.50}$$

A1.1.1 PROBLEMS

A. Using Fierz transformations, prove that $\theta\widehat{\theta\theta} = 0$.

B. Verify explicitly that the commutator of two supersymmetry transformations on the Weyl fermion component of a chiral superfield is indeed a translation.

C. Verify that the covariant derivative operator $\mathcal{D}$ defined in Eq. (1.29) anticommutes with the supersymmetry generators.

D. Show that a chiral superfield is a real superfield that obeys the constraint $\mathcal{D}V = 0$.

E. Show that the D-term of a real superfield transforms as a four-divergence.

A1.2 The Real Superfield

Although we have already constructed a real superfield out of a chiral superfield, we should be able to build one directly in terms of the four real Grassmann variables contained in the Weyl spinor θ. An elegant way to proceed is to rewrite the two-component Weyl into a four component Majorana spinor. In the Majorana representation, all four components of a Majorana spinor are real anticommuting degree s of freedom. The real superfield is then the most general function of the Majorana spinor

$$\Theta \equiv \begin{pmatrix} \theta \\ -\sigma_2\theta^* \end{pmatrix} , \tag{1.51}$$

shown here in the Weyl representation. Because all these components anticommute, the expansion will stop at the fourth order. We can use naive counting to determine the number of each component at each order. There are four components to the first order in Θ, $4 \cdot 3/2 = 6$ components at the second order, $4{\cdot}3{\cdot}2/(1{\cdot}2{\cdot}3) = 4$ at the third, and finally $4{\cdot}3{\cdot}2{\cdot}1/(1{\cdot}2{\cdot}3{\cdot}4) = 1$ component at the fourth. Hence a real superfield contains $(1, 4, 6, 4, 1)$ degrees of freedom, half of them commuting, half anti-commuting. We can form the six quadratic covariants

$$\overline{\Theta}\Theta, \;\; \overline{\Theta}\gamma_5\Theta, \;\; \overline{\Theta}\gamma_5\gamma_\mu\Theta , \tag{1.52}$$

where the bar denotes the usual Pauli adjoint

$$\overline{\Theta} \equiv \Theta^\dagger\gamma^0 . \tag{1.53}$$

It is easy to check the reality conditions

$$(\overline{\Theta}\Theta)^* = -\overline{\Theta}\Theta , \qquad (\overline{\Theta}\gamma_5\Theta)^* = \overline{\Theta}\gamma_5\Theta , \qquad (\overline{\Theta}\gamma_5\gamma_\mu\Theta)^* = -\overline{\Theta}\gamma_5\gamma_\mu\Theta . \tag{1.54}$$

Finally, by using the further identities

$$\overline{\Theta}\Theta\overline{\Theta} = -\overline{\Theta}\gamma_5\Theta\overline{\Theta}\gamma_5 = \frac{1}{4}\overline{\Theta}\gamma_5\gamma_\mu\Theta\overline{\Theta}\gamma_5\gamma^\mu \ , \tag{1.55}$$

$$\overline{\Theta}\gamma_5\gamma_\mu\Theta\overline{\Theta} = -\overline{\Theta}\Theta\overline{\Theta}\gamma_5\gamma_\mu \ , \qquad \overline{\Theta}\gamma_5\gamma^\mu\Theta\overline{\Theta}\gamma_5\gamma^\mu\Theta = g^{\mu\nu}(\overline{\Theta}\Theta)^2 \ , \tag{1.56}$$

we are able to write the most general Lorentz covariant expansion of a real superfield

$$\begin{aligned} V(x^\mu,\Theta) \ &= \ A(x) + i\overline{\Theta}\Psi(x) + i\overline{\Theta}\Theta M(x) + \overline{\Theta}\gamma_5\Theta N(x) \\ &\quad + i\overline{\Theta}\gamma_5\gamma^\mu\Theta A_\mu(x) + \overline{\Theta}\Theta\overline{\Theta}\Lambda(x) + (\overline{\Theta}\Theta)^2 D(x) \ . \end{aligned} \tag{1.57}$$

It can be shown that in Weyl notation the same real superfield reads

$$\begin{aligned} V(x^\mu,\theta,\theta^*) \ &= \ A(x) - i(\widehat{\theta}\psi + \theta^\dagger\widehat{\psi}^\dagger) \\ &- \ i\widehat{\theta}\theta C - i\theta^\dagger\widehat{\theta}^\dagger C^* + i\theta^\dagger\sigma^\mu\theta A_\mu \\ &+ \ \widehat{\theta}\theta\theta^\dagger\widehat{\lambda}^\dagger + \theta^\dagger\widehat{\theta}^\dagger\widehat{\theta}\lambda + |\widehat{\theta}\theta|^2 D \ , \end{aligned} \tag{1.58}$$

where

$$C(x) = M(x) - iN(x) \ , \tag{1.59}$$

and

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ -\sigma_2\psi^*(x) \end{pmatrix} \ . \tag{1.60}$$

It is evident from this equation that the real superfield contains a chiral superfield and its conjugate, made up of the non-canonical fields A, ψ, and C. We can therefore always write it in the form

$$V(x,\theta,\theta^*) = -i(\Phi(x,\theta) - \Phi^*(x,\theta)) + V'(x,\theta,\theta^*) \ , \tag{1.61}$$

where

$$\Phi(x,\theta) = \frac{1}{2}\Big(B(x) + iA(x)\Big) + \widehat{\theta}\psi(x) + \widehat{\theta}\theta C(x) \ . \tag{1.62}$$

If the real superfield is taken to be dimensionless, the vector field A_μ and the Weyl spinor λ have the right canonical dimension to represent a gauge field and a spinor field. The real superfield then describes the vector supermultiplet we have encountered in classifying the representations of the super Poincaré group, but with many extra degrees of freedom, which happen to fall neatly in chiral multiplets. This is no accident, since these extra fields in fact turn out to be gauge artifacts.

To conclude this section, let us work out the power of a real superfield, which turns out to be useful in several contexts. Consider

$$V^a = [A(1+X)]^a \ , \tag{1.63}$$

where X is a real superfield with all its components normalized by A.

$$X = i\overline{\Theta}\Psi' + i\overline{\Theta}\Theta M' + \overline{\Theta}\gamma_5\Theta N' + i\overline{\Theta}\gamma_5\gamma^\mu\Theta A'_\mu + \overline{\Theta}\Theta\overline{\Theta}\Lambda' + (\overline{\Theta}\Theta)^2 D' \ , \tag{1.64}$$

where the prime denotes division by A. Then, noting that $X^5 = 0$, a little bit of algebra gives

$$\begin{aligned} V^a &= A^a[1 + aX + a(a-1)\frac{X^2}{2!} \\ &+ a(a-1)(a-2)\frac{X^3}{3!} + a(a-1)(a-2)(a-3)\frac{X^4}{4!}] \ . \end{aligned} \tag{1.65}$$

The Fierz identity shown here for any two Dirac four component spinors

$$\Psi\overline{\Lambda} = -\frac{1}{4}\overline{\Lambda}\Psi - \frac{1}{4}\gamma_5\overline{\Psi}\gamma_5\Lambda + \frac{1}{4}\gamma_5\gamma^\rho\overline{\Psi}\gamma_5\gamma^\rho\Lambda - \frac{1}{4}\gamma^\rho\overline{\Psi}\gamma_\rho\Lambda + \frac{1}{2}\sigma_{\mu\nu}\overline{\Psi}\sigma^{\mu\nu}\Lambda \ , \tag{1.66}$$

is used repeatedly to rewrite the powers of X in terms of the standard expansion for a real superfield. We leave it as an exercise in fierce Fierzing to work out the general formula. Here we just concentrate on the D-term. The contributions to the D-term are as follows:

$$\begin{aligned} X \quad &: \quad (\overline{\Theta}\Theta)^2 D' \ ; \\ X^2 \quad &: \quad 2i\overline{\Theta}\Psi'\overline{\Theta}\Theta\overline{\Theta}\Lambda' + (i\overline{\Theta}\Theta M' + \overline{\Theta}\gamma_5\Theta N' + i\overline{\Theta}\gamma_5\gamma^\mu\Theta A'_\mu)^2 \ , \\ &= \quad (\overline{\Theta}\Theta)^2\left\{-\frac{i}{2}\hat{\overline{\Lambda}}\Psi' - M'^2 + N'^2 - A'_\mu A'^\mu\right\} \ ; \\ X^3 \quad &: \quad -3(i\overline{\Theta}\Theta M' + \overline{\Theta}\gamma_5\Theta N' + i\overline{\Theta}\gamma_5\gamma^\rho\Theta A'_\rho)(\overline{\Theta}\Psi')^2 \ , \\ &= \quad \frac{3}{4}(\overline{\Theta}\Theta)^2(iM'\overline{\Psi}'\Psi' - N'\overline{\Psi}'\gamma_5\Psi' - A'^\rho\overline{\Psi}'\gamma_5\gamma_\rho\Psi') \ ; \\ X^4 \quad &: \quad (i\Theta\Psi')^4 = \frac{1}{16}(\overline{\Theta}\Theta)^2(\overline{\Psi}'\Psi')^4[1 + 1 + g^\mu_\mu] \ . \end{aligned}$$

Putting it all together, we obtain for the D-term

$$
\begin{aligned}
(V^a)_D &= A^a[aD' + \frac{a(a-1)}{2}(-\frac{i}{2}\overline{\Lambda}'\Psi' - M'^2 + N'^2 - A'_\mu A'^\mu) \\
&+ \frac{1}{8}a(a-1)(a-2)(iM'\overline{\Psi}'\Psi' - N'\overline{\Psi}'\gamma_5\Psi' - A^{\rho'}\overline{\Psi}'\gamma_5\gamma_\rho\Psi')] \ .
\end{aligned}
\tag{1.67}
$$

One can use these formulae to show that the real superfield, expunged of its chiral components, satisfies $\hat{V}^3 = 0$.

A1.2.1 PROBLEMS

A. Verify the identities in Eqs. (1.55) and (1.56).

B. Verify the form of the expansion of a real superfield given in Eq. (1.58).

C. Show that a dimensionless real superfield that satisfies $V^3 = 0$ contains only a gauge field, a Weyl fermion and a real auxiliary field.

D. Starting from the transformation law of a real superfield, derive the transformation of the three fields, A_μ, λ, and D.

E. Derive the expression of the exponential of a real superfield.

A1.3 The Vector Supermultiplet

One massless representation of the super-Poincaré is the vector supermultiplet, containing a gauge potential and the gaugino, its associated Weyl fermion. The analysis of the previous section shows that they come accompanied by a real auxiliary field. Taking the Abelian case for simplicity, we are led to consider the three fields

$$
\begin{aligned}
&A_\mu(x): \text{ a gauge field} \\
&\lambda(x): \text{ a Weyl spinor (the gaugino),} \\
&D(x): \text{ an auxiliary field .}
\end{aligned}
\tag{1.68}
$$

The auxiliary field is here to provide the right count between bosonic and fermionic degreees of freedom. Without using the massless Dirac equation, the spinor is described by four degrees of freedom, and the gauge field is described by three degrees of freedom, leaving D to make up the balance. With the use of the equations of motion, both the Weyl field and the massless gauge field have two degrees of freedom, and the auxiliary field disappears. Sometimes the gaugino is called a Majorana fermion, but there should be no confusion between a Weyl fermion and a Majorana fermion: in two-component notation they look exactly the same. The Action

$$S = \int d^4x[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \lambda^\dagger\sigma^\mu\partial_\mu\lambda + \frac{1}{2}D^2] \ , \tag{1.69}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, is invariant under the following supersymmetry transformations

$$\begin{aligned} \delta A_\mu &= \frac{-i}{\sqrt{2}}(\lambda^\dagger\sigma_\mu\alpha + \alpha^\dagger\sigma_\mu\lambda) \ , \\ \delta\lambda &= \frac{1}{\sqrt{2}}(D + \frac{i}{2}\sigma^{\mu\nu}F_{\mu\nu})\alpha \ , \\ \delta D &= \frac{1}{\sqrt{2}}(\partial_\mu\lambda^\dagger\sigma^\mu\alpha - \alpha^\dagger\sigma^\mu\partial_\mu\lambda) \ , \end{aligned} \tag{1.70}$$

where

$$\sigma^{\mu\nu} = \frac{1}{2}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu) \ . \tag{1.71}$$

We see that D transforms as a four-divergence, making its space-time integral is a supersymmetric invariant. Let us check the commutation relations of the algebra on the fields:

$$\begin{aligned} \delta_1\delta_2 D &= \frac{1}{\sqrt{2}}(\partial_\mu\delta_1\lambda^\dagger\sigma^\mu\alpha_2 - \alpha_2^\dagger\sigma^\mu\partial_\mu\delta_1\lambda) \ , \\ &= \frac{1}{2}(\alpha_1^\dagger\sigma^\mu\alpha_2 - \alpha_2^\dagger\sigma^\mu\alpha_1)\partial_\mu D - \\ &\qquad - \frac{i}{4}\left((\sigma^{\rho\sigma}\alpha_1)^\dagger\sigma^\mu\alpha_2 - \alpha_2^\dagger\sigma^\mu\sigma^{\rho\sigma}\alpha_1\right)\partial_\mu F_{\rho\sigma} \ . \end{aligned} \tag{1.72}$$

Since we have

$$\sigma^{\rho\sigma\dagger} = -\overline{\sigma}^{\rho\sigma} = -\frac{1}{2}(\sigma^\rho\overline{\sigma}^\sigma - \sigma\overline{\sigma}^\rho) \ , \tag{1.73}$$

the identity

$$\sigma^\mu \sigma^{\rho\tau} = -i\epsilon^{\mu\rho\tau\delta}\sigma_\delta + g^{\mu\rho}\sigma^\tau - g^{\mu\tau}\sigma^\rho \ , \tag{1.74}$$

allows us to rewrite this equation as

$$[\delta_1, \delta_2]D = \alpha^\dagger_{[1}\sigma^\mu \alpha_{2]}\partial_\mu D + \frac{i}{4}\alpha^\dagger_{[1}(\overline{\sigma}^{\rho\tau}\sigma^\mu + \sigma^\mu\sigma^{\rho\tau})\alpha_{2]}\partial_\mu F_{\rho\tau} \ , \tag{1.75}$$

leading us to the simpler form

$$\begin{aligned}[\delta_1, \delta_2]D &= (\alpha^\dagger_1\sigma^\mu\alpha_2 - \alpha^\dagger_2\sigma^\mu\alpha_1)\partial_\mu D + \\ &+ \frac{1}{2}(\alpha^\dagger_1\sigma_\lambda\alpha_2 - \alpha^\dagger_2\sigma_\lambda\alpha_1)\epsilon^{\mu\rho\tau\lambda}\partial_\mu F_{\rho\tau} \ . \end{aligned} \tag{1.76}$$

The last term vanishes because of the Bianchi identity. (What if it did not? Any implications for the monopole?) Similarly, we compute

$$\begin{aligned}[\delta_1, \delta_2]A_\mu &= \frac{-i}{\sqrt{2}}(\delta_1\lambda^\dagger\sigma_\mu\alpha_2 + \alpha^\dagger_2\sigma_\mu\delta_1\lambda) - (1 \leftrightarrow 2) \ , \\ &= \frac{1}{4}\alpha^\dagger_1(\overline{\sigma}^{\rho\tau}\sigma_\mu - \sigma_\mu\sigma^{\rho\tau})\alpha_2 F_{\rho\tau} - (1 \leftrightarrow 2) \ , \\ &= (\alpha^\dagger_1\sigma^\rho\alpha_2 - \alpha^\dagger_2\sigma^\rho\alpha_1)F_{\rho\mu} \ , \end{aligned} \tag{1.77}$$

skipping over several algebraic steps. The right hand side contains the desired term, namely $\partial_\rho A_\mu$, but it also contains $-\partial_\mu A_\rho$; clearly it could not be otherwise from the transformation laws: their right-hand side is manifestly gauge invariant, whereas δA_μ certainly is not. Indeed our result can be rewritten in the suggestive form

$$[\delta_1, \delta_2]A_\mu = (\alpha^\dagger_1\sigma^\rho\alpha_2 - \alpha^\dagger_2\sigma^\rho\alpha_1)\partial_\rho A_\mu - \partial_\mu\Sigma \ , \tag{1.78}$$

where the last term is a gauge transformation, with a field dependent gauge function given by

$$\Sigma = (\alpha^\dagger_1\sigma^\rho\alpha_2 - \alpha^\dagger_2\sigma^\rho\alpha_1)A_\rho \ . \tag{1.79}$$

This equation shows clearly that a supersymmetry transformation (in this form) is accompanied by a gauge transformation. It also means that this description of the gauge multiplet is not gauge invariant, but rather in a specific gauge; this gauge is called the Wess-Zumino gauge. It is possible to eliminate the gauge transformation in the commutator of two supersymmetries by introducing extra fields which are needed for a gauge invariant description. We leave it as an exercise to derive the full gauge invariant set of fields. These fields can be neatly assembled in a real superfield, which under a gauge transformation undergoes the shift

$$V \to V + i(\Xi - \Xi^*) \ , \tag{1.80}$$

where $\Xi(x, \theta)$ is a chiral superfield. This nicely connects with the remarks of the previous section. The Wess-Zumino gauge is that for which the extraneous components of the real superfield are set to zero ($A = \psi = C = 0$). We leave it to the reader to verify the algebra on the gaugino field λ.

Generalization to the non-Abelian case is totally straightforward. The only difference is that the gaugino and auxiliary fields $\lambda^A(x)$ and $D^A(x)$ now transform covariantly as members of the *adjoint* representation of the internal symmetry group. Thus the ordinary derivative acting on $\lambda^A(x)$ has to be replaced by the covariant derivative

$$(\mathcal{D}_\mu\lambda)^A = \partial_\mu\lambda^A + ig(T^C)^A_B A^C_\mu \lambda^B \ , \tag{1.81}$$

where the representation matrices are expressed in terms of the structure functions of the algebra through

$$(T^C)^A_{\ B} = -if_B^{\ CA} \ . \tag{1.82}$$

The $N = 1$ supersymmetric non-Abelian Yang-Mills Lagrangian is then given by

$$-\frac{1}{4}G^A_{\mu\nu}G^{A\mu\nu} + \lambda^{\dagger A}\sigma^\mu(\mathcal{D}_\mu\lambda)^A + \frac{1}{2}D^A D^A \ . \tag{1.83}$$

In the Wess-Zumino gauge, there is an alternate way to represent the three fields of the vector supermultiplet, by introducing a chiral superfield which transforms as a Weyl spinor under the Lorentz group. It is given by

$$\mathcal{W}^A(x, \theta) = \lambda^A(x) + \frac{1}{2}\left[D^A(x) + \frac{i}{2}\sigma^{\mu\nu}G^A_{\mu\nu}(x)\right]\theta + \frac{1}{4}\widehat{\theta}\theta\overline{\sigma}^\mu\partial_\mu\widehat{\lambda}^{\dagger A}(x) \ , \tag{1.84}$$

suppressing the spinor index. Under a gauge transformation, this superfield transforms covariantly, as a member of the adjoint representation. One can also easily show that, under a supersymmetry transformation, $\mathcal{W}^A(x, \theta)$ does indeed transform as a chiral superfield, that is,

$$\mathcal{W}^A(x^\mu, \theta) \to \mathcal{W}^A(x^\mu + \alpha^\dagger\sigma^\mu\theta, \theta + \alpha) \ . \tag{1.85}$$

This reformulation gives us an easy way to build invariants out of products of this superfield. As for the Wess-Zumino multiplet, invariants are the F-term of the products of this superfield. This time, we must take care that Lorentz and gauge invariance be satisfied. In particular, the Yang-Mills Lagrangian is simply

$$\mathcal{L}_{SYM} = \int d^2\theta \; \widehat{\mathcal{W}}^A \mathcal{W}^A \; + \; \text{c.c.} \; . \tag{1.86}$$

The other invariant

$$\mathcal{L}_{SST} = i \int d^2\theta \; \widehat{\mathcal{W}}^A \mathcal{W}^A \; + \; \text{c.c.} \; , \tag{1.87}$$

is the usual Yang-Mills surface term

$$\mathcal{L}_{SST} = G^A_{\mu\nu} \widetilde{G}^{A\mu\nu} - i\partial_\mu(\lambda^\dagger \sigma^\mu \lambda) \; . \tag{1.88}$$

There are no other supersymmetric invariants made out of this spinor superfield that lead to renormalizable interactions. However we can easily manufacture invariant combinations of higher dimensions. For instance for $SU(N)$ with $N > 2$, we can form the gauge adjoint "anomaly" composite

$$A^B = d^{BCD} \widehat{\mathcal{W}}^C \mathcal{W}^D \; , \tag{1.89}$$

leading to the invariant

$$\int d^2\theta \; A^B A^B \; . \tag{1.90}$$

Similar constructions can be made with composites which transform as a self-dual antisymmetric second rank Lorentz tensor, and member of the adjoint representation of the gauge group, such as

$$f^{ABC} \widehat{\mathcal{W}}^B \sigma^i \mathcal{W}^C \; . \tag{1.91}$$

Some of these constructions appear in the context of non-perturbative supersymmetric models.

Finally, it is straighforward to implement R-symmetry on the gauge supermultiplet. All we need require is that

$$\mathcal{W} \to e^{i\beta} \mathcal{W} \; . \tag{1.92}$$

This means that the gaugino carries one unit of R-symmetry, while the D and gauge fields have no R-number.

A1.3.1 PROBLEMS

A. Prove the identity (1.72).

B. In the Wess-Zumino gauge, verify the commutator of the supersymmetry algebra on the gaugino field.

C. Show that the spinor superfield $\mathcal{W}$ has the correct transformation law under supersymmetry.

D. Show that the components of a real superfield in the Wess-Zumino gauge transform according to Eq. (1.67).

A1.4 Interaction of Chiral and Vector Supermultiplets

The renormalizable interactions of gauge fields with spin zero and one-half matter fields are generalized in supersymmetry to the study of the interaction of gauge supermultiplets with chiral matter supermultiplets.

Let us start with the coupling of a Wess-Zumino supermultiplet to an Abelian gauge superfield. Consider first the free action for one chiral superfield; it is clearly invariant under the global phase transformations

$$\Phi(x,\theta) \to e^{i\eta}\Phi(x,\theta) \ , \tag{1.93}$$

as long as the η is a global parameter, independent of the coordinates.

To duplicate the Yang-Mills construction, we want to modify this action to make it invariant under the most general *local* phase transformation on the chiral superfield

$$\Phi(x,\theta) \to e^{i\eta\Xi(x,\theta)}\Phi(x,\theta) \ , \tag{1.94}$$

where $\Xi(x,\theta)$ is a chiral superfield. The kinetic term loses its invariance, since

$$\Phi^*(y,\theta)\Phi(y,\theta) \to e^{i\eta(\Xi(y,\theta)-\Xi^*(y,\theta))}\Phi^*(y,\theta)\Phi(y,\theta) \ , \tag{1.95}$$

where y_μ has been previously defined. To restore invariance under the local symmetry, we generalize the kinetic term by adding the gauge supermultiplet. We have seen that it is described by a real superfield, with the suggestive gauge transformation

$$V \to V - i(\Xi - \Xi^*) \ . \tag{1.96}$$

The change of the argument translates in a redefinition of $\Lambda(x)$ and $D(x)$ in the real superfield, and does not affect the counting of the number of degrees of freedom.The Action is simply

$$\int d^4x \int d^2\theta d^2\bar{\theta} \sum_a \Phi^*(y,\theta) e^{\eta V(y,\theta,\theta^*)} \Phi(y,\theta) \ . \tag{1.97}$$

In the Wess-Zumino gauge, this expression can be shown to reduce to

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \lambda^\dagger \sigma^\mu \partial_\mu \lambda + \frac{1}{2} D^2 + \\ &+ (\mathcal{D}_\mu \varphi)^* (\mathcal{D}^\mu \varphi)^* + \psi^\dagger \sigma^\mu \mathcal{D}_\mu \psi + F^* F + \\ &+ g D \varphi^* \varphi - \sqrt{2} g \widehat{\lambda} \psi \varphi^* + \sqrt{2} g \lambda^\dagger \widehat{\psi}^\dagger \varphi \ , \end{aligned} \tag{1.98}$$

with the usual gauge covariant derivatives

$$\mathcal{D}_\mu \varphi = (\partial_\mu + igA_\mu)\varphi \ ; \qquad \mathcal{D}_\mu \psi = (\partial_\mu + igA_\mu)\psi \ . \tag{1.99}$$

The last line of this Lagrangian yields new interactions, over and above those present in the usual construction of gauge invariant theories, where derivatives are simply replaced by covariant derivatives. The reason is that the usual interaction terms created in this way, all proportional to the charge, are not supersymmetric invariants; the extra terms restore invariance under supersymmetry. However it is a bit tricky to check the invariance because we are in the Wess-Zumino gauge. This entails changes in the transformation properties of the fields of order g.

Consider the variation of the interaction of the fermion current with the gauge potential

$$\delta\left(ig\psi^\dagger \sigma^\mu \psi A_\mu\right) = ig\psi^\dagger \sigma^\mu \alpha F A_\mu + \frac{1}{\sqrt{2}} g \psi^\dagger \sigma^\mu \psi \lambda^\dagger \sigma_\mu \alpha + \text{c.c.} \ .$$

To offset the last term we need the variation

$$-\sqrt{2} g \widehat{\lambda} \psi \delta \varphi^* = -\frac{1}{\sqrt{2}} g \alpha^\dagger \sigma_\mu \lambda \psi^\dagger \sigma^\mu \psi \ .$$

By the same token, the variation

$$-\sqrt{2}g\delta\widehat{\lambda}\psi\varphi^* = -gD\widehat{\alpha}\psi\varphi^* + \cdots \ ,$$

is compensated by

$$gD\varphi^*\delta\varphi = gD\varphi^*\widehat{\alpha}\psi \ .$$

This procedure goes on *ad nauseam*. The alert student may have noticed the presence of a term proportional to F. The only way to compensate for it is to add a term in the variation of F itself. The extra variation

$$\delta_{WZ}F^* = -ig\psi^\dagger\sigma^\mu\alpha A_\mu \ ,$$

does the job. Its effect is to replace the derivative by the covariant derivative in the transformation law, which we do for all of them. Even then we are not finished: we still have one stray term proportional to F. Indeed we have

$$-\sqrt{2}g\widehat{\lambda}\delta\psi\varphi^* = -\sqrt{2}gF\widehat{\lambda}\alpha\varphi^* + \cdots \ ,$$

which can only cancelled by adding a term in the variation of F, yielding the final modification

$$\delta_{WZ}F^* = -ig\psi^\dagger\sigma^\mu\alpha A_\mu - \sqrt{2}g\alpha^\dagger\widehat{\lambda}^\dagger\varphi \ . \tag{1.100}$$

You have my word that it is the last change, but to the non-believer, I leave the full verification of the modified supersymmetric algebra in the Wess-Zumino gauge as an exercise.

This simple Lagrangian of course does not lead to a satisfactory quantum theory because of the ABJ anomaly associated with the $U(1)$, but this can be easily remedied by adding another chiral superfield with opposite charge. In this case, the extra terms beyond the covariant derivatives read

$$gD(\varphi_1^\dagger\varphi_1 - \varphi_2^\dagger\varphi_2) - \left(\sqrt{2}g\widehat{\lambda}(\psi_1\varphi_1^* - \psi_2\varphi_2^*) + \text{ c.c.}\right) \ . \tag{1.101}$$

From the equations of motion, the value of the auxiliary field is

$$D = -g(\varphi_1^\dagger\varphi_1 - \varphi_2^\dagger\varphi_2) \ ,$$

yielding the extra contribution to the potential

$$V = \frac{g^2}{2}(\varphi_1^\dagger\varphi_1 - \varphi_2^\dagger\varphi_2)^2 \ . \tag{1.102}$$

Generalization to the non-Abelian case is straightforward. We merely quote the results for a chiral matter superfield transforming as a representation $\mathbf{r}$ of the gauge group. The derivatives on the matter fields ψ_a and φ_a are replaced by the covariant derivatives

$$\mathcal{D}_\mu = \partial_\mu + ig\mathbf{T}^B A^B_\mu \ ,$$

where $\mathbf{T}^B$ represent the gauge algebra in the representation of the chiral superfield. The auxiliary fields $D^A(x)$ now couple through the term

$$gD^A \varphi^{\dagger a}(T^A)_a{}^b \varphi_b \ , \tag{1.103}$$

and the gauginos by the terms

$$-\sqrt{2}g\varphi^{\dagger a}(T^A)_a{}^b \psi^T_b \widehat{\lambda}^A + \sqrt{2}g\lambda^{A\dagger}\widehat{\psi}^{\dagger\ a}(T^A)_a{}^b \varphi_b \ , \tag{1.104}$$

where we have displayed the internal group indices (but not the spinor indices).

Lastly, we note that the gauge coupling preserves R-symmetry, irrespective of the R-value of the chiral superfield. Thus the only place R-invariance can be broken is in the superpotential.

A1.4.1 PROBLEMS

A. Evaluate the Abelian action (1.97) in the Wess-Zumino gauge.

B. Show that the modification (1.100) to the transformation of F in the Wess-Zumino gauge is sufficient to close the supersymmetry algebra.

A1.5 Supersymmetry Breaking

Exact supersymmetry implies equal masses for bosons and fermions, a feature not found in Nature. Thus any phenomenological application of supersymmetry requires an understanding of its breaking. We do not consider the so-called *hard breaking*, induced by higher-dimension (≥ 4) operators, which leaves no trace of the symmetry in the quantum field theory. Rather we discuss the more subtle breaking mechanisms which do not alter the ultraviolet properties of the theory.

Simplest is *soft breaking*, with the symmetry broken by infrared effects. This is accomplished through scale-dependent terms of dimension-two and three (relevant) operators. Intuitively, these do not affect the theory in the limit where all masses are taken to zero, relative to the scale of interest. These terms describe masses of the spin zero superpartners of the massless chiral fermions for the Wess-Zumino multiplet, and masses of the spin 1/2 gauginos for the gauge supermultiplets. The dimension-three terms describe interactions between the spin-zero partners of the chiral multiplets.

There are mass terms which do not break any symmetry other than supersymmetry by creating a mass gap between the particles within a supermultiplet (adding mass terms for the chiral fermions would break chiral symmetry, and often gauge symmetries). In the case of gauge supermultiplets, the gauge boson masses are protected by gauge symmetries, while the Majorana the gaugino mass term

$$M_i\widehat{\lambda}_i\lambda_i \ , \tag{1.105}$$

leave the gauge group invariant but break the continuous R-symmetry down to R-parity, its discrete Z_2 subgroup.

Soft breaking is not fundamental, rather a manisfestation of symmetry breaking in the effective Lagrangian language. We already encountered an example with the soft breaking of chiral symmetry in the effective chiral Lagrangian that describes the strong interactions. We understand these terms to come from the quark masses in the QCD Lagrangian, which in turn are generated by the spontaneous breaking of the electroweak symmetry. In the case of supersymmetry, the actual mechanism by which supersymmetry is broken is not known, although one can devise models where spontaneous symmetry breaking occurs naturally.

A1.5.1 Spontaneous Breaking

More fundamental is spontaneous breaking of supersymmetry. A symmetry is spontaneously broken if the field configuration which yields minimum energy no longer sustains the transformation under that symmetry. Let us remind ourselves how it works for an internal symmetry. The simplest order parameter is a complex field $\varphi(x)$ with dynamics invariant under the following transformation

$$\delta\varphi(x) = e^{i\beta}\varphi(x) \ . \tag{1.106}$$

Suppose that in the lowest energy configuration, this field has a constant value $< \varphi(x) >_0$. Expanding $\varphi(x)$ away from this vacuum configuration, and setting

$$\varphi(x) = e^{i\eta(x)}(v + \rho(x)) \ , \tag{1.107}$$

we find that under the transformation, the angle $\eta(x)$ undergoes a simple shift

$$\eta(x) \to \eta(x) + \delta \ , \tag{1.108}$$

meaning that the dynamics is invariant under that shift. Geometrically, this variable is the angle which parametrizes the closed line of minima. The dynamical variable associated with this angle is identified with the massless Nambu-Goldstone boson, $\zeta(x)$, divided by the vacuum value. It couples to the rest of the physical system universally

$$\mathcal{L}_{NG} = \frac{1}{v}\zeta(x)\partial_\mu J^\mu \ , \tag{1.109}$$

where $J_\mu(x)$ is the Noether current of the broken symmetry. Clearly, a constant shift in ζ generates a surface term and leaves the Action invariant.

Let us apply this reacquired wisdom to the supersymmetric case, starting with the chiral superfield. In analogy, we expect to see a massless fermion, since the supersymmetry parameter is fermionic, that shifts by a constant under supersymmetry. In a constant field configuration, the supersymmetry algebra reads

$$\delta\varphi_0 = \widehat{\alpha}\psi_0 \ , \qquad \delta\psi_0 = \alpha F_0 \ , \qquad \delta F_0 = 0 \ . \tag{1.110}$$

Any non-zero value of ψ_0 breaks both supersymmetry and Lorentz invariance. Since we are interested in Lorentz-invariant vacua, we set $\psi_0 = 0$, obtaining the only Lorentz invariant possibility

$$\delta\varphi_0 = 0, \quad \delta\psi_0 = \alpha F_0, \quad \delta F_0 = 0 \ ; \tag{1.111}$$

with $\varphi_0 \neq 0$ and $F_0 \neq 0$. Therefore, the only way to break the supersymmetry is through the configuration

$$F_0 \neq 0 \quad \leftrightarrow \quad \text{broken supersymmetry} \; .$$

Since F is a function of the scalar fields, it means that some $\varphi_0 \neq 0$. It must be noted that when $F_0 = 0$, and $\varphi_0 \neq 0$, any internal symmetry carried by φ_0 is broken. This fits nicely with our earlier remarks because a non-zero value for F gives the potential a positive minimum.

When $F_0 \neq 0$, the chiral fermion shifts under supersymmetry: it is the Nambu-Goldstone fermion associated with the breakdown of supersymmetry, as expected, since the broken symmetry is fermionic. It often goes under the name Goldstino.

A similar analysis carries to the vector multiplet. There, the only vacuum configuration which does not break Lorentz invariance, is that where A_μ and λ vanish in the vacuum, for which we have

$$\delta A_0^\mu = 0 \; , \; \delta\lambda_0 = \frac{1}{\sqrt{2}}\alpha D_0 \; , \; \delta D_0 = 0 \; , \tag{1.112}$$

and the only way to break supersymmetry is to give D_0 a vacuum value, and in this case, the gaugino λ is the Goldstino.

Thus, with both chiral and vector superfields, spontaneous breakdown of supersymmetry comes about when the dynamics is such that either F or D is non-zero in the vacuum. Another way of arriving at the same conclusion is to note that the potential from these theories is given by

$$V = F_i^* F_i + \frac{1}{2}D^2 \; , \tag{1.113}$$

when F_i and D take on their values obtained from the equations of motion. Since V is the sum of positive definite quantities it never becomes negative and if supersymmetry is spontaneously broken, its value at minimum is non-zero.

It is possible to formulate a general argument based on the fundamental anticommutation relations. In theories with exact supersymmetry, the vacuum state is annihilated by the generators of supersymmetry. However, the square of the same supersymmetry generators is nothing but the energy: the energy of the supersymmetric ground state is necessarily zero. Since it is also the state of lowest energy, it follows that the potential is necessarily positive definite. This is what we have just seen above.

Now suppose that supersymmetry is spontaneously broken. This requires that the action of supersymmetry on the vacuum not be zero, and therefore that the vacuum energy be positive. Comparing with the form of the potential, this can happen only if F and/or D is non-zero.

Finally, it is interesting to examine the transformation properties of composite chiral superfields which might arise in field theories as a result of strong coupling in effective infrared theories. Since products of chiral superfields are also superfields, we might consider the two simplest composites, $\Phi_{\rm matter} = \Phi\Phi$, made out of matter chiral multiplets, and $\Phi_{\rm gauge} = \widehat{\mathcal{W}}^A\mathcal{W}^A$, made out of gauge multiplets. Straightforward multiplication yields

$$\Phi_{\rm matter} = \varphi^2 + 2\widehat{\theta}\psi\varphi + \frac{1}{2}\widehat{\theta}\theta(2F\varphi - \widehat{\psi}\psi) \ , \tag{1.114}$$

so that its F-term might acquire a non-zero vacuum value if the fermion bilinear condense. Our general analysis suggests this would break supersymmetry (is this true?). In a similar way we find for the gauge field singlet composite, suppressing gauge indices,

$$\begin{aligned}\Phi_{\rm gauge} &= \widehat{\lambda}\lambda + \widehat{\theta}(\lambda D - \frac{i}{2}\sigma^{\mu\nu}G_{\mu\nu}\lambda) \\ &+ \frac{1}{2}\widehat{\theta}\theta(D^2 - 2\partial_\mu\lambda^\dagger\sigma^\mu\lambda - \frac{1}{2}G_{\mu\nu}G^{\mu\nu} + iG_{\mu\nu}\widetilde{G}^{\mu\nu}) \ .\end{aligned} \tag{1.115}$$

The gaugino condensate can get a vacuum value without breaking supersymmetry. However, $< G_{\mu\nu}G^{\mu\nu} >_0$ can break supersymmetry since it contributes to the F term, while the gaugino bilinear does not seem capable of breaking supersymmetry.

These conclusions must be examined with caution because in field theory, the algebra of products of local fields may not be the same as expected from the classical transformation laws. Indeed, Konishi (*Phys. Let.***135B**, 439(1984)) has found an anomaly in the supersymmetry transformations of the *gauge singlet* composite fermion made out of two chiral superfields. In the presence of gauge interactions, he finds that under a supersymmetry,

$$\delta(\varphi_i\psi_i) = \alpha(F_i\varphi_i - \frac{1}{2}\widehat{\psi}_i\psi_i + C\frac{g^2}{32\pi^2}\widehat{\lambda}_a\lambda_a) \ , \tag{1.116}$$

where C is the Casimir operator, and where i, a are the group indices for the matter and gauge multiplets, respectively. The appearance of two different fermion bilinears on the right hand side is intriguing. If the gauginos condense in the vacuum, the last term causes a shift in the matter composite fermion field, which must then be identified with the Goldstino, implying that supersymmetry has been broken dynamically.

A1.6 Models of Spontaneous Supersymmetry Breaking

We now present simple models where the dynamics are such that we have spontaneous supersymmetry breaking at tree level, either through the F or D terms. These are arranged in such a way that the minimum value of the potential is greater than zero.

F-breaking

The generic mechanism, invented by O'Raifeartaigh, requires at least three chiral superfields, $\Phi_j,\ j = A, B, C$, interacting through the superpotential

$$W = m\Phi_A\Phi_B + \lambda(\Phi_A^2 - M^2)\Phi_C \ . \tag{1.117}$$

This theory is invariant under one global phase symmetry

$$R' = R - \frac{2}{3}X \ , \tag{1.118}$$

where the X is an Abelian symmetry with values $x_j = (1, -2, -2)$ for the three superfields, and the R-symmetry, which does not commute with supersymmetry has the same value $(2/3)$ for all three superfields

$$\begin{aligned} X &: \quad \Phi_j(x,\theta) \ \rightarrow \ e^{ix_j\alpha}\Phi_j(x,\theta) \ , \\ R &: \quad \Phi_j(x,\theta) \ \rightarrow \ e^{i\frac{2\beta}{3}}\Phi_j(x,e^{i\beta}\theta) \ , \end{aligned} \tag{1.119}$$

Under the combined phase symmetry the components of the superfields have the following values, indicated in parentheses,

$$\begin{aligned} &\varphi_A(0) \ , \quad \psi_A(-1) \ , \quad F_A(-2) \ , \\ &\varphi_B(2) \ , \quad \psi_B(1) \ , \quad F_B(0) \ , \\ &\varphi_C(2) \ , \quad \psi_C(1) \ , \quad F_C(0) \ . \end{aligned} \tag{1.120}$$

The F equations of motion yield

$$\begin{aligned} F_A &= -m^*\varphi_B^* - 2\lambda^*\varphi_A^*\varphi_C^* \ , \\ F_B &= -m^*\varphi_A^* \ , \qquad F_C = \lambda^*(M^{2*} - \varphi_A^{*2}) \ . \end{aligned} \tag{1.121}$$

Clearly there is no field configuration for which all three F's vanish: if

$F_C = 0$, then $F_B \neq 0$, and $F_B = 0$ leads to $F_C \neq 0$. The potential is the sum of the absolute square of the F-terms

$$V = |m\varphi_B + 2\lambda\varphi_A\varphi_C|^2 + |m\varphi_A|^2 + |\lambda(\varphi_A^2 - M^2)|^2 \ , \tag{1.122}$$

from which we deduce only two independent extremum conditions

$$\begin{aligned} m\varphi_B + 2\lambda\varphi_A\varphi_C &= 0 \ , \\ |m|^2\varphi_A + 2\varphi_A^*|\lambda|^2(\varphi_A^2 - M^2) &= 0 \ , \end{aligned} \tag{1.123}$$

and their complex conjugates. From these two independent conditions, only two of the scalar fields can be determined, and the third one is left with an undetermined vacuum value. This is a general characteristic of F-type breaking: at tree level one field combination is left undetermined, so that in field space the potential has a continuous minimum along that field direction, called a *flat direction*. The vacuum manifold at tree level is a barren two dimensional plane spanned by the values of the undetermined complex field. This degeneracy can be lifted once quantum corrections are included.

For simplicity, assume the parameters m, and λ to be real (see problem). The last equation, rewritten in the form,

$$\varphi_A(|m|^2 + 2|\lambda|^2|\varphi_A|^2) - 2\lambda^2M^2\varphi_A^* = 0 \ ,$$

has two different solutions, depending on the parameters, they are

$$\begin{aligned} \text{solution I :} \qquad & M^2 - \tfrac{m^2}{2\lambda^2} < 0 \ ; \\ & \varphi_A = 0 \ ; \quad \varphi_B = 0 \ ; \quad \varphi_C \text{ undetermined} \ ; \\ & F_A = F_B = 0 \ ; \quad F_C = -\lambda M^2 \ ; \\ & V_0 = \lambda^2 M^4 \ . \end{aligned}$$

$$\begin{aligned} \text{solution II :} \qquad & M^2 - \tfrac{m^2}{2\lambda^2} > 0 \ ; \\ & \varphi_A = \sqrt{M^2 - \tfrac{m^2}{2\lambda^2}} \ ; \quad \varphi_B = -\tfrac{2\lambda}{m}\varphi_C\sqrt{M^2 - \tfrac{m^2}{2\lambda^2}} \ ; \\ & F_A = 0 \ ; \quad F_B = -m\sqrt{M^2 - \tfrac{m^2}{2\lambda^2}} \ ; \quad F_C = \tfrac{m^2}{2\lambda} \ ; \\ & V_0 = m^2(M^2 - \tfrac{m^2}{4\lambda^2}) \ , \end{aligned}$$

where V_0 is the value of the potential at minimum. In both cases, we see that supersymmetry is broken since one of the F fields does not vanish. In both cases, the R' symmetry is spontaneously broken, except at one point ($\varphi_C = 0$). Thus we expect to have two massless particles, one Goldstino and one Nambu-Goldstone boson in both cases.

• Solution I allows us to immediately identify ψ_C with the massless Goldstino, the only fermion field which gets shifted by a supersymmetry transformation, since only $F_C \neq 0$. This is consistent with the superpotential where the only fermion mass term is $-m\widehat{\psi_A\psi_B}$, which describes a Dirac fermion of mass m.

The scalar masses are given by the second derivative of the potential evaluated at the minimum. The relevant terms are

$$V = m^2|\varphi_B|^2 + m^2|\varphi_A|^2 - 2\lambda M^2(\varphi_A^2 + \varphi_A^{*2}) + \cdots \ . \tag{1.124}$$

If we let

$$\varphi_A = \frac{1}{\sqrt{2}}(a + ib) \ ,$$

we obtain the mass terms

$$m^2|\varphi_B|^2 + \frac{1}{2}(m^2 - 2\lambda^2 M^2)a^2 + \frac{1}{2}(m^2 + 2\lambda^2 M^2)b^2 \ ,$$

which allows us to extract the following mass squared for the scalars

$$0, \ \ 0, \ \ m^2, \ \ m^2, \ \ m^2 - 2\lambda^2 M^2, \ \ m^2 + 2\lambda^2 M^2 \ . \tag{1.125}$$

The effect of F-supersymmetry breaking has been to split the mass squared by equal and opposite amounts for the scalars. The scalars for which this happens belong to the superfield that couples to the superfield which gets a non-zero F-term, in this case Φ_C. This is easy to see because of the term of the form $\phi\phi F$ in the potential: when $F_0 \neq 0$, it generates a quadratic term for ϕ. Furthermore, because of the opposite split, the mass sum rule is still satisfied, namely

$$\{m^2 + m^2 + (m^2 - 2\lambda M^2) + (m^2 + 2\lambda M^2)\} - 2 \cdot 2\{m^2\} = 0 \ , \tag{1.126}$$

remembering that one Dirac = two Weyl.

• Solution II is slightly more complicated, because there are two non-zero F-terms. From the values of F_B and F_C in the vacuum, it proves convenient to introduce the angle η by

$$\tan\eta = \frac{m}{2\lambda\sqrt{M^2 - \frac{m^2}{2\lambda^2}}} \ , \tag{1.127}$$

such that in the vacuum the combination $\sin\eta F_B + \cos\eta F_C$ vanishes. From the transformation laws, we deduce that the fermion

$$\psi = \sin\eta\psi_B + \cos\eta\psi_C \tag{1.128}$$

does not shift under supersymmetry, while the orthogonal combination

$$\psi_{NG} = \cos\eta\psi_B - \sin\eta\psi_C \tag{1.129}$$

does shift; it must be identified with the massless Goldstino, which we can verify by direct computation of the tree-level fermion masses. These are given by

$$\begin{aligned} & -m\widehat{\psi}_A\psi_B - 2\lambda\sqrt{M^2 - \frac{m^2}{2\lambda^2}}\widehat{\psi}_A\psi_C \ , \\ = \quad & -2\lambda\sqrt{M^2 - \frac{m^2}{4\lambda^2}}\widehat{\psi}_A(\sin\eta\psi_B + \cos\eta\psi_C) \ , \end{aligned} \tag{1.130}$$

so that the missing orthogonal combination ψ_{NG}, is indeed the massless Goldstino as expected, while ψ becomes the Dirac partner of ψ_A; the value of the Dirac mass can also be written as

$$m_D = \frac{2\lambda}{m}(\cos\eta F_B - \sin\eta F_C) \ . \tag{1.131}$$

It should be obvious that much work can be avoided by introducing the scalar mass combinations

$$\varphi = \sin\eta\varphi_B + \cos\eta\varphi_C \ , \qquad \chi = \cos\eta\varphi_B - \sin\eta\varphi_C \ , \tag{1.132}$$

in terms of which the potential reads

$$\begin{aligned} V \ &= \ |(m\ \cos\eta - 2\lambda\varphi_A\sin\eta)\chi + (m\ \sin\eta + 2\lambda\varphi_A\cos\eta)\varphi|^2 \\ &+ \ m^2|\varphi_A|^2 + \lambda^2|\varphi_A^2 - M^2|^2 \ . \end{aligned} \tag{1.133}$$

Expand φ_A as $\langle\varphi_A\rangle_0 + \hat{\varphi}_A$, and obtain the quadratic terms

$$4\lambda^2(M^2 - \frac{m^2}{4\lambda^2})|\varphi|^2 + (4\lambda^2 M^2 - m^2)|\hat{\varphi}_A|^2 - \frac{m^2}{2}(\hat{\varphi}_A^2 + \hat{\varphi}_A^{*2}) \ ,$$

leading to the following squared masses,

$$0, \ 0, \ 4\lambda^2 M^2 - m^2, \ 4\lambda^2 M^2 - m^2, \ 4\lambda^2 M^2 - \frac{3}{2}m^2, \ 4\lambda^2 M^2 - \frac{1}{2}m^2 \ , \quad (1.134)$$

and the magic sum rule is satisfied. By now this should not appear too magical since the relevant F-term couples to $\varphi_A^2 + \varphi_A^{*2}$. The magnitude of the shift among the scalar fields again has to do with the coupling and value of the non-vanishing F-term

$$F' = \cos\eta F_B - \sin\eta F_C = -m\sqrt{M^2 - \frac{m^2}{4\lambda^2}} \ . \quad (1.135)$$

Incidentally, the presence of the massless χ fields should not be left unexplained (see problem). The main result of this example is that F-breaking leads to the tree-level fatidic sum rule

$$\mathcal{S}tr M^2 = \sum_{J=0,1/2} (-1)^{2J}(2J+1)m_J^2 = 0 \ , \quad (1.136)$$

which restricts the uses of this mechanism for phenomenology. However it can be changed by quantum corrections and by non-renormalizable couplings which appear naturally when generalizing supersymmetry to include gravity (supergravity). Note that both solutions have two massless scalars. One must be the Nambu-Goldstone associated with R'-breaking. Finally we note the ubiquitous presence of R-symmetry in this example.

D breaking

This particular mechanism was invented by Fayet and Iliopoulos. It necessarily involves an Abelian gauge supermultiplet. The simplest example that is anomaly free is a U(1) gauge theory in interaction with two chiral superfields Ψ_1 and Ψ_2 of opposite charge. The Lagrange density is

$$\begin{aligned}\mathcal{L} \ &= \ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \lambda^\dagger \sigma^\mu \partial_\mu \lambda + \frac{1}{2}D^2 \\ &\quad + \sum_{i=1}^{2}((\mathcal{D}_\mu \varphi_i)^*(\mathcal{D}^\mu \varphi_i) + \psi_i^\dagger \sigma^\mu \mathcal{D}_\mu \psi_i + F_i^* F_i)\end{aligned}$$

$$\begin{aligned} &+ \quad gD(\varphi_1^*\varphi_1 - \varphi_2^*\varphi_2) - (\sqrt{2}g\widehat{\lambda}(\psi_1\varphi_1^* - \psi_2\varphi_2^*) + \text{c.c.}) \\ &+ \quad m(\varphi_1 F_2 + \varphi_2 F_1 - \widehat{\psi}_1\psi_2 + \text{c.c.}) - \epsilon\mu^2 D \ . \end{aligned} \tag{1.137}$$

Note that we have added a D-term, the integral of which is a supersymmetric invariant; $\epsilon = \pm 1$ and μ^2 is a positive mass squared. Clearly this can only be done for a U(1) theory for which D is gauge invariant. In the above,

$$\mathcal{D}_\mu\varphi_i = (\partial_\mu + q_i g A_\mu)\varphi_i \ ; \qquad \mathcal{D}_\mu\psi_i = (\partial_\mu + iq_i g A_\mu)\psi_i \ ; \tag{1.138}$$

with $q_1 = -q_2 = 1$. This has R-symmetry, under

$$\Phi_j(x,\theta) \to e^{i2\beta}\Phi_j(x, e^{i\beta}\theta) \ , \quad j = 1,2 \ . \tag{1.139}$$

The equations of motion for the F_i and D terms yield

$$\begin{aligned} D &= \epsilon\mu^2 - g(\varphi_1^*\varphi_1 - \varphi_2^*\varphi_2) \ , \\ F_1 &= -m\varphi_2^* \ , \quad F_2 = -m\varphi_1^* \ . \end{aligned} \tag{1.140}$$

There is no field configuration for which all three vanish, and supersymmetry is necessarily broken. The potential is given by

$$V = \frac{1}{2}\{\epsilon\mu^2 - g(\varphi_1^*\varphi_1 - \varphi_2^*\varphi_2)\}^2 + m^2(\varphi_1^*\varphi_1 + \varphi_2^*\varphi_2) \ . \tag{1.141}$$

Its minimization yields

$$\varphi_1^*\{m^2 - g(\epsilon\mu^2 - g\varphi_1^*\varphi_1 + g\varphi_2^*\varphi_2)\} = 0 \ , \tag{1.142}$$

$$\varphi_2^*\{m^2 + g(\epsilon\mu^2 - g\varphi_1^*\varphi_1 + g\varphi_2^*\varphi_2)\} = 0 \ . \tag{1.143}$$

Evidently we cannot have both φ_1 and φ_2 different from zero, unless $m^2 = 0$. There are two types of solutions, one for which $\varphi_1 = \varphi_2 = 0$, and the other with $\varphi_2 = 0$, $\varphi_1 \neq 0$ (the other case, $\varphi_1 = 0$, $\varphi_2 \neq 0$ is obtained from the latter by changing the sign of g). Let us examine both cases:

a-) When $\varphi_1 = \varphi_2 = 0$ the vacuum values are

$$D = \epsilon\mu^2 \ ; \quad F_1 = F_2 = 0 \ , \quad V_0 = \frac{\mu^4}{2} \ . \tag{1.144}$$

The supersymmetry transformation laws tell us that the gaugino λ is the massles Goldstino. Only supersymmetry is broken. The gauge U(1) and the R symmetry are untouched.

The only fermion mass comes from the original Dirac mass term, $-m\widehat{\psi}_1\psi_2$. The masses of the scalars are extracted from the quadratic terms in the potential,

$$V = \frac{1}{2}\mu^4 - \epsilon\mu^2 g(\varphi_1^*\varphi_1 - \varphi_2^*\varphi_2) + m^2(\varphi_1^*\varphi_1 + \varphi_2^*\varphi_2) + \cdots \ , \tag{1.145}$$

yielding two real scalars with masses $m^2 + \epsilon\mu^2 g$, and two with masses $m^2 - \epsilon g\mu^2$. The supertrace now reads

$$\mathcal{S}tr\ M^2 = 2(m^2 + \epsilon\mu^2 g) + 2(m^2 - \epsilon\mu^2 g) - 2 \cdot 2m^2 = 0 \ . \tag{1.146}$$

Again it vanishes, but this time for a different reason. We see that the mass^2 of the first two scalars is shifted by gD_0, the second by $-gD_0$, which correspond to the value of the charge times D_0. They cancel because of our anomaly cancellation mechanism which has the sum of the charges vanishing. So we really have

$$\mathcal{S}tr\ M^2 = 2g(q_1 + q_2)D_0 \ . \tag{1.147}$$

We could have devised a different model, say with one multiplet of charge 2 and eight multiplets of charge -1. The $U(1)$ triangle anomaly diagram vanishes since

$$\sum q_i^3 = (2)^3 + 8(-1) = 0 \ , \tag{1.148}$$

however the D term is given by

$$D = \epsilon\mu^2 + 2g\varphi^*\varphi - g\sum_{i=1}^{8}\varphi_i^*\varphi_i \ , \tag{1.149}$$

so that

$$\mathcal{S}tr\ M^2 = [8(-1) + (2)]gD_0 = -6gD_0 \ . \tag{1.150}$$

We leave it to the reader to devise such a model, ugly as it may be. However this shows that D-breaking may alter the mass sum rule.

b-) The second case is described by $\varphi_2 = 0\ ,\quad \varphi_1 = v$, which we take to be real without loss of generality. This solution breaks both gauge symmetry and the R-symmetry, but leaves a linear combination invariant. Hence we expect only one Nambu-Goldstone boson, to be eaten by the gauge boson. Minimization of the potential yields

$$v^2 = \frac{1}{g^2}(g\epsilon\mu^2 - m^2) \ , \tag{1.151}$$

which taking g positive can happen only for $\epsilon = 1$ and then only if $g\mu^2 > m^2$. In the vacuum, we have

$$D = D_0 = \frac{m^2}{g} \ ; \qquad F_1 = 0 \ ; \ \ F_2 = -\frac{m}{g}\sqrt{g\mu^2 - m^2} \ , \tag{1.152}$$

In this case, both supersymmetry and the gauge symmetry are spontaneously broken, and the Goldstino is a linear combination of the gaugino λ and the chiral ψ_2. We can see this directly from the fermion mass matrix in the Lagrangian

$$(m\widehat{\psi}_2 + \sqrt{2}gv\widehat{\lambda})\psi_1 + \text{c.c.} \ . \tag{1.153}$$

This describes a Dirac fermion with mass

$$m_D = \sqrt{2g^2v^2 + m^2} \ . \tag{1.154}$$

The orthogonal combination does not have a mass term; it must be the massless Goldstino

$$\lambda_G = \frac{1}{\sqrt{2g^2v^2 + m^2}}(\sqrt{2}gv\psi_2 - m\lambda) \ . \tag{1.155}$$

One can check explicitly that it shifts by a constant under a supersymmetry transformation

$$\delta\lambda_G = -\frac{m}{\sqrt{2}g}\alpha\sqrt{2g^2v^2 + m^2} \ , \tag{1.156}$$

while the orthogonal combination experiences no shift

$$\delta\psi = \frac{\alpha}{\sqrt{2g^2v^2 + m^2}}[-m^2v + gv\frac{m^2}{g})] = 0 \ , \tag{1.157}$$

as we expected. From the spontaneous breakdown of the $U(1)$, the gauge boson gets a mass

$$m_A^2 = 2g^2v^2 \ . \tag{1.158}$$

The scalar masses are obtained by expanding the potential from its vacuum value. Letting $\varphi_1 = v + \hat{\varphi}_1, \varphi_2 = \hat{\varphi}_2$, we obtain

$$\begin{aligned} V &= V_0 + \frac{1}{2}g^2v^2(\hat{\varphi}_1 + \hat{\varphi}_1^*)^2 + (m^2 - gD_0)|\varphi_1|^2 + (m^2 + gD_0)|\hat{\varphi}_2|^2 \\ &= V_0 + \frac{1}{2}g^2v^2(\hat{\varphi}_1 + \hat{\varphi}_1^*)^2 + 2m^2|\hat{\varphi}_2|^2 \ , \end{aligned} \tag{1.159}$$

so that we have the following scalar masses squared

$$2m^2, \quad 2m^2, \quad 0, \quad 2g^2v^2 \ .$$

The zero mass scalar is of course the Nambu-Goldstone boson coming from the breaking of the gauge $U(1)$; it appears here because we are not in the unitary gauge. Now our mass sum rule, adding the gauge multiplet yields

$$\sum_{J=0,1/2,1} (-1)^{2J}(2J+1)m_J^2 = 3m_A^2 - 2\cdot 2m_D^2 + (2m^2 + 2m^2 + 2g^2v^2) = 0 \ . \tag{1.160}$$

Again, we ask the reader to convince himself or herself that the zero on the right hand side occurs really because of the particular anomaly cancellation we have chosen. The three scalar mass squared are

$$m \pm gD_0 \ , \qquad m^2 + gD_0 \ , \qquad m^2 + gD_0 + 2(g\mu^2 - m^2) \ , \tag{1.161}$$

so that

$$\mathcal{S}tr \ M^2 = g(q_1 + q_2)D_0 \ .$$

A1.7 Dynamical Supersymmetry Breaking

Over the last few years, there has been much progress in understanding supersymmetric theories which have both weak and strong coupling sectors. In these theories, supersymmetry might be dynamically broken as a result of the strong coupling. Its study therefore requires the use of non-perturbative methods, which are beyond the scope of this introductory Appendix. We will nevertheless give a glimpse of the possibilities by discussing a simple example.

In *global* supersymmetry, dynamical breaking can be studied in some generality by evaluating the order parameter of supersymmetry, the so-called Witten index

$$\Delta = Tr(-1)^F \ , \tag{1.162}$$

which counts the number of fermionic less the number of bosonic states in the vacuum, $n_F - n_B$. The case $\Delta = 0$ is ambiguous since it may mean one of two possibilities: either $n_B = N_f \neq 0$, implying the existence of states of zero energy, and unbroken supersymmetry, or both n_B and N_f are zero, in which case it is broken. However, when $\Delta \neq 0$, the unambiguous conclusion is that there exist states of zero energy, and supersymmetry is unbroken. The beauty of this index is that it is invariant under adiabatic changes of the theory. It can be computed in the perturbative regime, and then carried over to non perturbative situations. Its drawback is that it applies only to global supersymmetry, and when large field configurations become important. If Δ is computed different from zero, there is no breaking.

Asymptotically free theories come with a scale Λ, obtained by dimensional transmutation. Below that scale, the theory is strongly coupled, above it is perturbative. In order to determine its infrared behavior it is important to be able to construct the the effective low energy Lagrangian. We consider this problem, starting with the super Yang-Mills theory without matter. The Lagrangian density is

$$\mathcal{L}_{SYM} = -\frac{1}{4} G^A_{\mu\nu} G^{\mu\nu A} + \lambda^{\dagger A} \sigma^\mu (\mathcal{D}_\mu \lambda)^A + \frac{1}{2} D^A D^A \ , \tag{1.163}$$

where

$$(\mathcal{D}_\mu \lambda)^A = \partial_\mu \lambda^A + ig(T^C)^A_{\ B} A^C_\mu \lambda^B \ . \tag{1.164}$$

It is supersymmetric, and one can build a conserved supersymmetry current, S_μ, by Noether methods. In addition, the N=1 super-Yang-Mills Lagrangian density is invariant under the chiral R-symmetry which acts only on the gaugino

$$\lambda \to e^{i\beta} \lambda \ .$$

The ABJ anomaly contributes to the divergence of the R-current through the usual triangle diagram as

$$\partial^\mu J^R_\mu = 3N_c \frac{g^2}{32\pi^2} \mathrm{Tr}(G_{\mu\nu}\widetilde{G}^{\mu\nu}) \ , \tag{1.165}$$

where N_c is the number of colors (here for $SU(N_c)$). In this theory without matter, the β-function is itself proportional to the number of colors

$$\beta(g) = -3N_c \frac{g^3}{16\pi^2} \ . \tag{1.166}$$

Hence we can rewrite the anomaly as

$$\partial^\mu J^R_\mu = -\frac{\beta(g)}{g} \mathrm{Tr}(G_{\mu\nu}\widetilde{G}^{\mu\nu}) \ . \tag{1.167}$$

This anomalous symmetry is much like the PQ symmetry, except that here it is not broken spontaneously. Since the β-function has the same sign as in QCD, it leads to a strong force at large distances, which, by analogy with QCD, we expect to form fermion condensates. The prime candidate is the Lorentz-invariant condensate of two gauginos

$$\varphi_c = \widehat{\lambda}^A \lambda^A \ . \tag{1.168}$$

Does this break supersymmetry? The answer lies in the transformation of φ_c under a supersymmetric transformation. We have

$$\begin{aligned} \delta\varphi_c &= 2\widehat{\lambda}^A \delta\lambda^A \ , \\ &= \sqrt{2}(D^A \widehat{\lambda}^A \alpha + \frac{i}{2}\widehat{\lambda}^A \sigma_{\mu\nu} G^{A\mu\nu}\alpha) \ , \\ &= \sqrt{2}\widehat{\alpha}(\lambda^A D^A - \frac{i}{2}\sigma^{\mu\nu} G^A_{\mu\nu}\lambda^A) \ , \end{aligned} \tag{1.169}$$

using the identity,

$$\sigma^2 \sigma^{\mu\nu T} \sigma^2 = -\sigma^{\mu\nu} \ .$$

By comparing with the chiral multiplet transformation law, we define the composite fermion field

$$\psi_c = D^A \lambda^A - \frac{i}{2}\sigma^{\mu\nu} G^A_{\mu\nu}\lambda^A \ ; \tag{1.170}$$

its variation yields, after some tedious but straightforward algebra, the F component of the composite multiplet

$$F_c = -2\partial_\mu \lambda^{A\dagger}\sigma^\mu \lambda^A + D^A D^A - \frac{1}{2} G^A_{\mu\nu} G^{A\mu\nu} + i G^A_{\mu\nu} \widetilde{G}^{A\mu\nu} \ . \tag{1.171}$$

It is of course, up to a surface term, the Lagrange density of the super Yang-Mills theory, which is not too surprising since its integral is supersymmetric invariant. It now becomes clear that the condensate $< \widehat{\lambda}^A \lambda^A >_0$ cannot break supersymmetry since it does not tranform as an F-term. By the same token, we conclude that the "glue" condensate $< G^A_{\mu\nu} G^{A\mu\nu} >_0$ can break supersymmetry since it contributes to an F-term. Seen from another point of view, it raises the energy density of the vacuum, thus breaking supersymmetry.

This composite supermultiplet plays another role, that of the anomaly, *i.e.* it appears on the right hand side of the equations giving the divergences of the dilatation current, the axial current and the γ-trace of the supersymmetric current (the divergence of the supersymmetric current is not anomalous). In equations

$$\begin{aligned} \partial^\mu D_\mu &= \frac{\beta(g)}{g} \frac{1}{2} \mathrm{Tr}(G^A_{\mu\nu} G^{A\mu\nu}) \ , \\ r\partial^\mu J^R_\mu &= -\frac{\beta(g)}{g} \frac{1}{2} \mathrm{Tr}(G^A_{\mu\nu} \widetilde{G}^{A\mu\nu}) \ , \\ \gamma^\mu S_\mu &= \frac{\beta(g)}{2g} 2 G^A_{\mu\nu} \sigma^{\mu\nu} \lambda^A \ . \end{aligned} \tag{1.172}$$

Through supersymmetry the chiral anomaly is linked to the scale anomaly. In regular QCD, the chiral anomaly is of order $\frac{1}{N_c}$, since it is generated by fermion loops, but in the supercase, the gaugino contribution is of order $\frac{1}{N_c} \times N_c \sim 1$, and it does not go away in the large N_c limit. Hence there is no parameter which can be switched off to get rid of the anomaly. It would seem that in Super-QCD, there is no sense in which we should consider a Lagrangian invariant under both scale and chiral transformations, implemented with an "anomaly" term. Still, this approach was pursued by Veneziano and Yankielowicz (*Phys. Lett.* **B113**, 231 (1982)), in order to find out if gaugino condensates can form under the influence of the strong force.

They proceed to write the effective low energy Lagrangian for pure super Yang-Mills in terms of the gauge singlet composite chiral superfield Φ_c, given by

$$\begin{aligned}\Phi_c &= \widehat{W}^A W^A \;, \\ &= \widehat{\lambda}\lambda + \widehat{\theta}(\lambda D - \frac{i}{2}\sigma^{\mu\nu}G_{\mu\nu}\lambda) \\ &+ \frac{1}{2}\widehat{\theta}\theta(D^2 - 2\partial_\mu\lambda^\dagger\sigma^\mu\lambda - \frac{1}{2}G_{\mu\nu}G^{\mu\nu} + iG_{\mu\nu}\widetilde{G}^{\mu\nu}) \;. \end{aligned} \tag{1.173}$$

All "color" indices have been suppressed. This composite superfield has dimension three. Since we are going to use scale invariance as a guide to building the effective Lagrangian the kinetic term needs to have the correct dimension and be supersymmetric. The candidate kinetic term is the D-term of a real superfield, with dimension d such that $4 = \frac{1}{2} \times 4 + d$, since each θ has dimension one-half. Hence we need $d = 2$. Out of the chiral superfield Φ_c we can make a real superfield of dimension six, $|\Phi_c(y_\mu, \theta)|^2$. We conclude that the only candidate for a kinetic term is

$$\mathcal{L}_{\mathrm{kin}} = \int d^2\theta d^2\overline{\theta}(\Phi_c^*(y_\mu, \theta)\Phi_c(y_\mu, \theta))^{1/3} \;. \tag{1.174}$$

Under R-symmetry, the composite superfield transforms a

$$\Phi_c(y_\mu, \theta) \to e^{2i\beta}\Phi_c(y_\mu, e^{i\beta}\theta) \;, \tag{1.175}$$

and the kinetic term is manifestly invariant. Now we have to add a function of Φ_c which actually mocks up the scale and chiral (R) anomalies. We follow the procedure we have used previously in the QCD case. The relevant fact in this construction is that the right-hand side of the anomaly equations are given by the F-term of the composite superfield, Φ_c. We seek to construct a term of the form

$$\mathcal{L}_A = \int d^2\theta W(\Phi_c) + \text{c.c.} \;. \tag{1.176}$$

Under R-symmetry,

$$\begin{aligned}\mathcal{L}_A \;\longrightarrow\;& \int d^2\theta W(e^{2i\beta}\Phi_c(x_\mu, e^{i\beta}\theta)) + \text{c.c.} \;, \\ =\;& \int d^2\theta' e^{-2i\beta} W(e^{2i\beta}\Phi_c(x_\mu, \theta')) + \text{c.c.} \;, \end{aligned} \tag{1.177}$$

so that (dropping the primes)

$$\delta\mathcal{L}_A = 2i\beta \int d^2\theta(-W + \Phi_c \frac{\partial W}{\partial \Phi_c}) + \text{c.c.} \ . \tag{1.178}$$

The anomaly can be written in the form

$$-\frac{\delta\mathcal{L}_A}{\delta\alpha} = -\frac{\beta(g)}{2g} G_{\mu\nu}\widetilde{G}^{\mu\nu} \ . \tag{1.179}$$

Since

$$i\int d^2\theta\Phi_c + \text{c.c.} = -2G_{\mu\nu}\widetilde{G}^{\mu\nu} \ , \tag{1.180}$$

it suggests that

$$-W + \Phi_c \frac{\partial W}{\partial \Phi_c} = \xi\Phi_c \ , \tag{1.181}$$

where ξ is an unknown constant. Differentiating, we get

$$\Phi_c \frac{\partial^2 W}{\partial \Phi_c^2} = \xi \ ,$$

which is easily integrated to yield

$$W(\Phi_c) = \xi\Phi_c(\ln\frac{\Phi_c}{\mu^3} - 1) \ ,$$

where μ is an integration constant. The full effective low energy Lagrangian which reproduces the anomaly is then

$$\int d^2\theta d^2\overline{\theta}(\Phi_c^*\Phi_c)^{1/3} + \xi\int d^2\theta\Phi_c(\ln\frac{\Phi_c}{\mu^3} - 1) + \text{c.c.} \ . \tag{1.182}$$

It can be checked that we also get the correct anomalous equations for the dilatation and superconformal currents. We can examine the potential of this model to find its ground state. First we extract the auxiliary fields.

$$\xi\int d^2\theta\Phi_c(\ln\frac{\Phi_c}{\mu^3} - 1) = \xi(F_c \ln\frac{\varphi_c}{\mu^3} + \frac{1}{\varphi_c}\widehat{\psi}_c\psi_c) \ . \tag{1.183}$$

On the other hand, the kinetic term yields

$$\frac{1}{3}(\varphi_c^*\varphi_c)^{1/3}\frac{1}{(\varphi_c^*\varphi_c)}[F_c^*F_c + \text{ normal kinetic terms }] \ , \tag{1.184}$$

from which we deduce the equation of motion

$$F_c = 3\xi(\varphi_c^*\varphi_c)^{2/3}\ln\frac{\varphi_c^*}{\mu^3}\ , \tag{1.185}$$

leading to the potential

$$V = 9|\xi|^2(\varphi_c^*\varphi_c)^{2/3}\ln\frac{\varphi_c}{\mu^3}\ln\frac{\varphi_c^*}{\mu^3}\ . \tag{1.186}$$

One has to be careful because the kinetic term for φ_c is not canonical

$$\frac{1}{3}(\varphi_c^*\varphi_c)^{-2/3}\partial_\mu\varphi_c^*\partial^\mu\varphi_c\ , \tag{1.187}$$

a but we regain the canonical form in terms of

$$\hat{\varphi}_c = \sqrt{3}\varphi_c^{1/3}\ , \qquad \partial_\mu\hat{\varphi}_c = \frac{1}{\sqrt{3}}\varphi_c^{-2/3}\partial_\mu\varphi_c\ , \tag{1.188}$$

to arrive at the Lagrangian

$$\mathcal{L} = \partial_\mu\hat{\varphi}_c^*\partial^\mu\hat{\varphi}_c - |\xi|^2(\hat{\varphi}_c^*\hat{\varphi}_c)^2\ln\frac{\hat{\varphi}_c}{\mu}\ln\frac{\hat{\varphi}_c^*}{\mu}\ . \tag{1.189}$$

The potential is at its minimum when

$$|\hat{\varphi}_c^2|\ln\frac{\hat{\varphi}_c}{\mu} = 0\ ,$$

which has two possible solutions, $\hat{\varphi}_c = 0$ or $\hat{\varphi}_c = \mu$. The former does not make any sense because of the fermion kinetic terms. There is only one viable solution, $\hat{\varphi}_c = \mu$, showing that supersymmetry is not broken, the conclusion we wanted to reach. The non zero vacuum value of the composite superfield suggests that the gaugino condensate indeed forms, but without breaking supersymmetry. Expanding the potential away from minimum, letting

$$\hat{\varphi}_c = \mu + \varphi_c'\ ,$$

we obtain

$$\begin{aligned} V &= -|\xi|^2|\mu+\varphi'|^2\ln(1+\frac{\varphi'}{\mu})\ln(1+\frac{\varphi'^*}{\mu})\ , \\ &= -|\xi|^2\mu^2|\varphi'|^2 - \cdots\ , \end{aligned} \tag{1.190}$$

which gives the mass

$$m^2 = |\xi|^2 \mu^2 \ . \tag{1.191}$$

This mass is like that of η' in QCD, which arises because of the anomaly. Because of supersymmetry, we also have two real scalar degrees of freedom and one massive Weyl fermion.

To conclude, in pure Super-Yang-Mills without matter, a gaugino condensate may form, but without breaking supersymmetry, in accordance with the transformation properties of the condensate. It is reassuring to see how it happens, albeit through a dangerous procedure.

The lesson is that to break supersymmetry dynamically, one must devise more complicated theories. In particular, the addition of chiral matter in Super-Yang-Mills theories generates many ways to break Susy dynamically. The recent dramatic increase (see K. Intrilligator and N. Seiberg, *Lectures on Supersymmetric Gauge Theories and Electric-Magnetic Duality* in *Nucl. Phys. Proc. Suppl.* **45BC**, 1(1996)) in our understanding of non-perturbative methods in supersymmetric quantum field theories has produced many such examples.

A1.7.1 PROBLEMS

A. Show that the effective Lagrangian does reproduce the conformal anomaly and that in the trace of the supersymmetric current.

B. Derive the form of the fermion kinetic term, and show that the solution $\hat{\varphi}_c = 0$ is untenable.