Group Theory: A Physicist's Survey

Pierre Ramond

Institute for Fundamental Theory, Physics Department

University of Florida

To Richard Slansky
Who Would Have Written This Book

Contents

1

Preface: The Pursuit of Symmetries

Symmetric objects are so singular in the Natural World that our ancestors must have noticed them very early. Indeed, symmetrical structures were given special magical status. The Greeks' obsession with geometrical shapes led them to the enumeration of platonic solids, and to adorn their edifices with various symmetrical patterns. In the ancient world, symmetry was synonymous with perfection. What could be better than a circle or a sphere? The Sun and the planets were supposed to circle the Earth. It took a long time to get to the apparently less than perfect ellipses!

Of course most shapes in the natural world display little or no symmetry, but many are almost symmetric. An orange is close to a perfect sphere; Humans are almost symmetric about their vertical axis, but not quite, and ancient man must have been aware of this. Could this lack of exact symmetry have been viewed as a sign of imperfection, imperfection that humans need to atone for?

It must have been clear that highly symmetric objects were special, but it is a curious fact that the mathematical structures which generate symmetrical patterns were not systematically studied until the nineteenth century. That is not to say that symmetry patterns were unknown or neglected, witness the Moors in Spain who displayed the seventeen different ways to tile a plane on the walls of their palaces!

Évariste Galois in his study of the roots of polynomials of degree larger than four, equated the problem to that of a set of substitutions which form that mathematical structure we call a group. In physics, the study of crystals elicited wonderfully regular patterns which were described in terms of their symmetries. In the Twentieth Century, with the advent of Quantum Mechanics, symmetries have assumed a central role in the study of Nature.

The importance of symmetries is reinforced by the Standard Model of elementary particle physics, which indicates that Nature displays more symmetries in the small than in the large. In Cosmological terms, this means that our Universe emerged from the Big Bang as a highly symmetrical structure, although most of its symmetries are no longer evident today. Like an ancient piece of pottery, some of its parts may not have survived the eons, leaving us today with its shards. This is a very pleasing concept that resonates with the old Greek ideal of perfection. Did our universe emerge at the Big Bang with perfect symmetry that was progressively shattered by cosmological evolution, or was it born with internal defects that generated the breaking of its symmetries? It is a profound question which some physicists try to answer today by using conceptual models of a perfectly symmetric universe, e.g. Superstrings.

Some symmetries of the Natural world are so commonplace, that they are difficult to identify. The outcome of an experiment performed by undergraduates should not depend on the time and location of the bench on which it was performed. Their results should be impervious to shifts in time and space, as consequences of time and space translation invariances, respectively. But there are more subtle manifestations of symmetries. The great Galileo Galilei made something of a "trivial" observation: when your ship glides on a smooth sea with a steady wind, you can close your eyes and not "feel" that you are moving. Better yet, you can perform experiments whose outcomes are the same as if you were standing still! Today, you can leave your glass of wine on an airplane at cruising altitude without fear of spilling. The great genius that he was elevated this to his principle of Relativity: the laws of Physics do not depend on whether you are at rest or move with constant velocity! However if the velocity changes, you can feel it (a little turbulence will spill your wine). Our experience of the everyday world appears complicated by the fact that it is dominated by frictional forces; in a situation where their effect can be neglected, simplicity and symmetries (in some sense analogous concepts) are revealed.

According to Quantum Mechanics, Physics takes place in Hilbert spaces. Bizarre as this notion might be, we have learned to live with it as it continues to be verified whenever experimentally tested. Surely, this abstract identification of a physical system with a state vector in Hilbert space will eventually be found to be incomplete, but in a presently unimaginable way, which will involve some other weird mathematical structure. That Nature uses the same mathematical structures invented by mathematicians is a profound mystery hinting at the way our brains are wired. Whatever the root cause, mathematical structures which find natural representations in Hilbert spaces have assumed enormous physical interest. Prominent among them are *groups* which, subject to specific axioms, describe transformations in these spaces.

Since physicists are mainly interested in how groups operate in Hilbert spaces, we will focus mostly in the study of their representations. Mathematical concepts will be introduced as we go along in the form of *scholia* sprinkled throughout the text. Our approach will be short on proofs, which can be found in many excellent textbooks. From representations, we will focus on their products and show how to build group invariants for possible physical applications. We will also discuss the embeddings of the representations of a subgroup inside those of the group. Numerous tables will be included.

This book begins with the study of *finite* groups, which as the name indicates, have a finite number of symmetry operations. The smallest finite group has only two elements, but there is no limit as to their number of elements: the permutations on n letters forms a finite group with $n!$ elements. Finite groups have found numerous applications in physics, mostly in crystallography and in the behavior of new materials. In elementary particle physics, only small finite groups have found applications, but in a world with extra dimensions, and three mysterious families of elementary particles, this situation is bound to change. Notably, the sporadic groups, an exceptional set of twenty six finite groups, stand mostly as mathematical curiosities waiting for an application.

We then consider *continuous* symmetry transformations, such as rotations by arbitrary angles, or open-ended time translation, to name a few. Continuous transformations can be thought of as repeated applications of infinitesimal steps, stemming from generators. Typically these generators form algebraic structures called Lie algebras. Our approach will be to present the simplest continuous groups and their associated Lie algebras, and build from them to the more complicated cases. Lie algebras will be treated *à la Dynkin*, using both Dynkin notation and diagrams. Special attention will be devoted to exceptional groups and their representations. In particular, the magic square and magic triangle will be discussed. We will link back to finite groups, as most can be understood as subgroups of continuous groups.

Some non-compact symmetries are discussed, especially the representations of space-time symmetries, such as the Poincaré and Conformal groups. Group-theoretic aspects of the Standard Model and Grand-Unification are presented as well. The algebraic construction of the five Exceptional Lie algebras is treated in detail. Two generalizations of Lie algebras are also discussed, Super-Lie algebras and their classification, and infinite-dimensional affine Kac-Moody algebras.

I would like to express my gratitude to the Institute for Advanced Study, and the Aspen Center for Physics for their hospitality, where a good part of this book was written. I would like to thank Professors L. Brink, and J. Patera, as well as Drs. D. Belyaev, Sung-Soo Kim, and C. Luhn, for their critical reading of the manuscript, and many useful suggestions. Finally, I owe much to my wife Lillian, whose patience, encouragements, and understanding made this book possible.