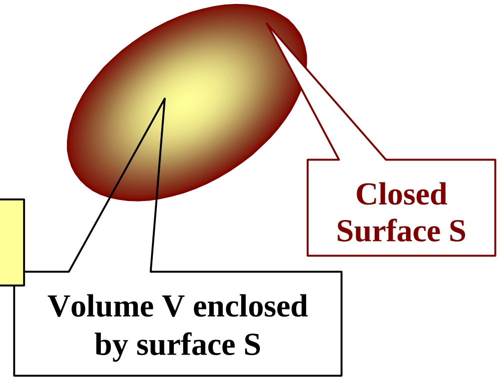
Divergence Theorem

The divergence theorem states that the integral of the divergence of a vector function over a volume, V, is equal to the flux, Φ_F, of the vector function through the closed surface, S, that encloses the volume V:



$$\int_V (\vec{\nabla}\cdot\vec{F})dV = \oint_S \vec{F}\cdot d\vec{A}$$

Divergence Theorem

Proof (*sketch*):

$$\Phi_F = \oint_S \vec{F}\cdot d\vec{A} = \sum_{i=1}^{N}\oint_{S_i}\vec{F}\cdot d\vec{A}_i = \sum_{i=1}^{N} V_i\left[\frac{1}{V_i}\oint_{S_i}\vec{F}\cdot d\vec{A}_i\right] \underset{\substack{N\to\infty \\ V_i\to 0}}{\Rightarrow} \int_V (\vec{\nabla}\cdot\vec{F})dV$$

The Laplacian Operator

Suppose that the vector function, $\vec{F}(x,y,z)$, is the gradient of the scalar function f(x,y,z), $\vec{F} = \vec{\nabla}f$. Now suppose we construct a new scalar function g(x,y,z) that is the divergence of $\vec{F}(x,y,z)$, $g = \vec{\nabla}\cdot\vec{F}$. Then g(x,y,z) is the divergence of the gradient of f(x,y,z) as follows:

$$g = \vec{\nabla}\cdot\vec{\nabla}f = \nabla^2 f$$

In cartesian (or retangular) coordinates:

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \qquad \vec{\nabla}\cdot\vec{\nabla} = \nabla^2$$

Laplacian Operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$g = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$