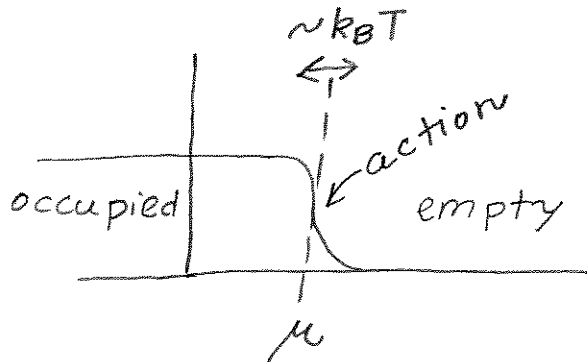


Fermi-Dirac distribution $f(E) = \frac{1}{1 + e^{\beta(E - \mu)}}$



Periodic boundary conditions:

$$\psi(r) = \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{\sqrt{Vol}}$$

$$\psi(x+L, y, z) = \psi(x, y, z) \rightarrow e^{i k_x L} = 1$$

same for y & z directions

$$\rightarrow k_x = \frac{2\pi n_x}{L}, \Delta k = \frac{2\pi}{L}$$

$$\frac{1}{Vol} \sum_{\mathbf{k}} = \frac{1}{Vol} \sum_{n_x, n_y, n_z} = \frac{1}{Vol} \left(\frac{L}{2\pi}\right)^3 \sum_{\mathbf{k}} (\Delta k)^3$$

$$\boxed{\frac{1}{Vol} \sum_{\mathbf{k}} = \int \frac{d^3 k}{(2\pi)^3}}$$

Physical quantities:

$$\langle \vec{p} \rangle = \langle \frac{\hbar}{i} \vec{\nabla} \rangle = \hbar \vec{k}$$

$$\vec{v} = \hbar \vec{k} / m$$

$$E = \frac{\hbar^2 k^2}{2m}$$

Orders of magnitude:

$$T = 0 \text{ or } T \ll E_F$$

$$n = 2 \int_{\text{Spin } k < k_F} \frac{d^3 k}{(2\pi)^3} = 2 \frac{1}{(2\pi)^3} \frac{4\pi}{3} k_F^3 = \frac{k_F^3}{3\pi^2}$$

Table 2.1: $k_F \sim 10^8 \text{ cm}^{-1} = 1/\text{\AA}$

$$v_F = \frac{\hbar k_F}{m} \sim 1 \text{ to } 2 \times 10^8 \frac{\text{cm}}{\text{s}} \sim \frac{1}{100} c$$

$$E_F = \frac{\hbar^2 k_F^2}{2m} \sim 1 \text{ to } 10 \text{ eV}$$

$$T_F = E_F / k_B \sim 10^4 \text{ to } 10^5 \text{ K} \quad \gg \text{room temp.} \quad (1 \text{ eV} \sim 10^4 \text{ K})$$

Total energy:

$$\frac{E}{\text{Vol.}} = 2 \int_{k < k_F} \frac{d^3 k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} = \frac{1}{\pi^2} \frac{\hbar^2 k_F^2}{10m}$$

$$\frac{E}{N} = \frac{\frac{1}{\pi^2} \frac{\hbar^2 k_F^2}{10m}}{\frac{k_F^3}{3\pi^2}} = \boxed{\frac{3}{5} E_F = \frac{E}{N}}$$

Free electron Bulk modulus

$$\begin{aligned}
 P = \text{pressure} &= - \left(\frac{\partial E}{\partial V} \right)_N \\
 E &= \frac{3}{5} N \mathcal{E}_F \\
 \mathcal{E}_F \propto k_F^2 \propto n^{2/3} &= \left(\frac{N}{V} \right)^{2/3}
 \end{aligned}
 \left. \vphantom{\begin{aligned} P = \text{pressure} \\ E \\ \mathcal{E}_F \end{aligned}} \right\} P = \frac{2}{3} \frac{E}{V}$$

$$B = \text{bulk modulus} = \frac{1}{\underset{\substack{\uparrow \\ \text{compressibility}}}{K}} = -V \frac{\partial P}{\partial V} \quad (\text{like } \frac{\Delta P}{\Delta V/V})$$

$$\begin{aligned}
 E \propto V^{-2/3}, \quad P \propto V^{-5/3} &\rightarrow B = \frac{5}{3} P = \frac{5}{3} \cdot \frac{2}{3} \frac{E}{V} \frac{N}{N} \\
 &= \frac{5}{3} \cdot \frac{2}{3} \cdot \frac{3}{5} \cdot n \mathcal{E}_F
 \end{aligned}$$

$$\boxed{B = \frac{2}{3} n \mathcal{E}_F}$$

Table 2.2 correct
order of magnitude.

Density of states:

$$2 \int \frac{d^3k}{(2\pi)^3} F(\epsilon_k) = \int d\epsilon F(\epsilon) \underbrace{2 \int \frac{d^3k}{(2\pi)^3} \delta(\epsilon - \epsilon_k)}_{\text{density of states} \equiv g(\epsilon)}$$

$$g(\epsilon) = \frac{2}{(2\pi)^3} 4\pi \int k^2 dk \delta\left(\epsilon - \frac{\hbar^2 k^2}{2m}\right),$$

$$= \frac{2}{(2\pi)^3} 4\pi \frac{k^2}{\hbar^2 k/m} = \frac{1}{\hbar^2 \pi^2} m k, \text{ where } \frac{\hbar^2 k^2}{2m} = \epsilon$$

$$\text{or } k = \sqrt{\frac{2m\epsilon}{\hbar^2}}$$

$$\rightarrow g(\epsilon) = \frac{m}{\hbar^2 \pi^2} k = \frac{m}{\hbar^2 \pi^2} \sqrt{\frac{2m\epsilon}{\hbar^2}}$$

note $\sqrt{\epsilon}$ in 3D

$$g(\epsilon_F) = \frac{m k_F}{\hbar^2 \pi^2} = \frac{3}{2} \frac{n}{\epsilon_F}$$

Specific heat & temperature dependence:

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

$$u = \frac{U}{\text{Vol}} = 2 \int \frac{d^3k}{(2\pi)^3} \epsilon_k f(\epsilon_k) \quad \swarrow \text{Fermi fnt.}$$

$$n = 2 \int \frac{d^3k}{(2\pi)^3} f(\epsilon_k) \quad \text{held fixed } (\mu \text{ varies w/ temperature})$$

$$u = \int d\epsilon g(\epsilon) \epsilon f(\epsilon)$$

$$n = \int d\epsilon g(\epsilon) f(\epsilon)$$

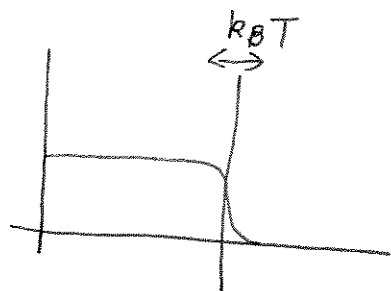
Specific heat comparison:

$$C_V = \frac{\pi^2}{3} k_B T g(E_F) \quad , \quad g(E_F) = \frac{3}{2} \frac{n}{E_F} \propto m v$$

$$= \frac{\pi^2}{2} \left(\frac{k_B T}{E_F} \right) n k_B \quad \text{vs. classical } \frac{3}{2} n k_B$$

$$\frac{k_B T}{E_F} \sim \frac{100}{10^4} \sim 10^{-2}$$

Simple argument:



Fraction of e/s. $k_B T / E_F$
moved $k_B T$

$$u \sim n \frac{(k_B T)^2}{E_F}$$

$$C_V \sim n k_B \frac{k_B T}{E_F}$$

Sizes:

$$\gamma \equiv \frac{C_V}{T} \sim \text{few} \times 10^{-4} \text{ cal/mole/K}^2$$

$\gamma_{\text{meas.}} / \gamma_{\text{free elec}} \sim \text{few w/exceptions (Table 2.3)}$

Commonly plotted as

$$C_V = \gamma T + AT^3 \rightarrow \frac{C_V}{T} = \gamma + AT^2$$

$$C_V / T \text{ vs } T^2$$

Sommerfeld model = Drude + Fermi-Dirac statistics

$\lambda = v_F \tau$ $\frac{k_B T}{m} \ll v_F^2$ by $\frac{k_B T}{E_F}$
larger smaller

$$K = \frac{1}{3} v_F^2 \tau C_V$$

$$\sigma = \frac{n e^2 \tau}{m}$$

$$\left. \begin{array}{l} K \\ \sigma \end{array} \right\} \frac{K}{\sigma T} = \frac{1}{3} \frac{m v_F^2 e^2 C_V}{n E_F} = \frac{1}{3} \frac{\pi^2}{2} \left(\frac{k_B T}{E_F} \right) k_B$$

$$= \frac{\pi^2}{3} \left(\frac{k_B}{e} \right)^2 \sim 2.44 \times 10^{-8} \text{ W}\Omega/\text{K}^2$$

(Drude: $\frac{K}{\sigma T} = \frac{3}{2} \left(\frac{k_B}{e} \right)^2 \sim 1.11 \times 10^{-8} \text{ W}\Omega/\text{K}^2$)

Thermopower:

$$Q = -\frac{C_V}{3 n e} = -\frac{1}{3 e} \frac{\pi^2}{2} \left(\frac{k_B T}{E_F} \right) k_B$$

$$= -\frac{\pi^2}{6} \frac{k_B}{e} \left(\frac{k_B T}{E_F} \right) \text{ vs. } -\frac{k_B}{2e}$$

Drude

Big puzzles: Why are some elements not metals?

(Chpt. 3)

How can the electrons be treated as free or weakly interacting?