Boltzmann Equation

No scattering:

$$\frac{d}{dt} f(k(t), r(t)) = \dot{k} \cdot \nabla_k f + \dot{r} \cdot \nabla_r f = 0$$

With scattering:

$$\dot{k} \cdot \nabla_k f + \dot{r} \cdot \nabla_r f = \left(-\frac{\partial f}{\partial t} \right)_{collision}$$

Allow for separate time dependence if f:

$$\boxed{\frac{\partial f}{\partial t} + \frac{\vec{F}}{\hbar} \cdot \vec{\nabla}_k f + \vec{v} \cdot \vec{\nabla}_r f = \left(-\frac{\partial f}{\partial t} \right)_{coll.}}$$

where $\vec{F}$ is the force, e.g. $\vec{F} = -e(\vec{E} + \vec{v} \times \vec{B})$.

Relaxation time approximation:

$$\boxed{\left(-\frac{\partial f}{\partial t} \right)_{coll.} = -\frac{f(k,r,t) - f_{eq}(k)}{\tau(\varepsilon_k)},}$$

where $f_{eq}(k) = \dfrac{1}{1 + e^{\beta(\varepsilon_k - \mu)}}$.

Holds for each band in the relaxation time approx.

1. Multiply by (−e) and integrate over k:

$$-e\int\frac{d^3k}{4\pi^3}\frac{\partial f}{\partial t} = \frac{\partial \rho}{\partial t}$$

$$-e\int\frac{d^3k}{4\pi^3}\frac{\vec{F}}{\hbar}\cdot\vec{\nabla}_k f = 0$$ ← (periodic in k-space)

(With a magnitic field, $\vec{F} = -e\frac{1}{\hbar}\vec{\nabla}_k \mathcal{E}_k \times \vec{B}$, after integrating by part, one will have $\vec{\nabla}_k \times B\cdot\vec{\nabla}_k f$ which is zero.)

$$-e\int\frac{d^3k}{4\pi^3}\vec{v}\cdot\vec{\nabla}_r f = \vec{\nabla}\cdot\vec{j}$$

$$-e\int\frac{d^3k}{4\pi^3}\left(-\frac{\partial f}{\partial t}\right)_{coll.} = 0$$

$$\leftrightarrow \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0}$$

2. Uniform electric field: $\left(\frac{\partial f}{\partial t} = 0,\ \vec{\nabla}_r f = 0,\ \vec{F} = -e\vec{E}\right)$

$$-\frac{e\vec{E}}{\hbar}\cdot\vec{\nabla}_k f = -\frac{f-f_{eq}}{\tau} = -\frac{f}{\tau} + \frac{f_{eq}}{\tau}$$

$$f = \tau\frac{e\vec{E}}{\hbar}\cdot\vec{\nabla}_k f + f_{eq}$$

In linear response

$$f = f_{eq} + \tau\frac{e\vec{E}}{\hbar}\cdot\vec{\nabla}_k f_{eq}(\mathcal{E}_k)$$

$$= f_{eq} + \tau\frac{e\vec{E}}{\hbar}\cdot\vec{\nabla}_k \mathcal{E}_k\frac{\partial f_{eq}}{\partial \mathcal{E}_k}$$

$$= f_{eq} - \tau e\vec{E}\cdot\vec{V}_k\left(-\frac{\partial f_{eq}}{\partial \mathcal{E}_k}\right)$$

$$j_\mu = -e \int \frac{d^3k}{4\pi^3} v_\mu f(k)$$

No contribution from equilibrium f: f_{eq}

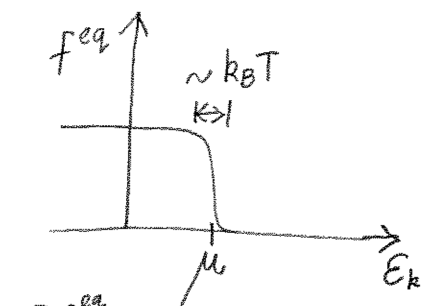
because $\vec{v}_k = -\vec{v}_{-k}$ but $\mathcal{E}_k = \mathcal{E}_{-k}$.

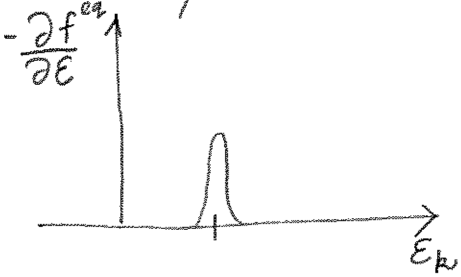
$$\longrightarrow j_\mu = -e \int \frac{d^3k}{4\pi^3} v_\mu \left(-\tau e \vec{E} \cdot \vec{v}\right)\left(-\frac{\partial f_{eq}}{\partial \mathcal{E}_k}\right)$$

$$= \sigma_{\mu\nu} E_\nu \text{ , where}$$

$$\boxed{\sigma_{\mu\nu} = e^2 \int \frac{d^3k}{4\pi^3} \tau\, v_\mu v_\nu \left(-\frac{\partial f_{eq}}{\partial \mathcal{E}_k}\right)}$$

Like Eq. (13.25) in A&M.





$$\int_{-\infty}^{+\infty} d\mathcal{E} \left(-\frac{\partial f_{eq}}{\partial \mathcal{E}}\right) = -f_{eq}\Big|_{-\infty}^{+\infty} = 0 - -1 = 1$$

$$\rightarrow \boxed{\left(-\frac{\partial f_{eq}}{\partial \mathcal{E}}\right) \approx \delta(\mathcal{E}-\mu) \text{ or } \delta(\mathcal{E}-\mathcal{E}_F)}$$

Recovering the free electron result:

Suppose $\mathcal{E}_k = \frac{\hbar^2 k^2}{2m^*}$ + constant in a band.

$$\rightarrow v_k = \frac{\hbar k}{m^*}$$

$$\rightarrow \sigma_{\mu\nu} = e^2 \int \frac{d^3k}{4\pi^3}\, \tau\, \frac{\hbar k_\mu}{m^*} \frac{\hbar k_\nu}{m^*}\, \delta(\mathcal{E}_k - \mathcal{E}_F)$$

$$= e^2 \int d\mathcal{E}\, g(\mathcal{E}) \int \frac{d\Omega_k}{4\pi}\, \tau \left(\frac{\hbar k}{m^*}\right)^2 \hat{k}_\mu \hat{k}_\nu\, \delta(\mathcal{E} - \mathcal{E}_F)$$

$$= e^2 \int d\mathcal{E}\, g(\mathcal{E})\, \tau \left(\frac{\hbar k}{m^*}\right)^2 \frac{1}{3} \delta_{\mu\nu}\, \delta(\mathcal{E} - \mathcal{E}_F)$$

$$= e^2 g(\mathcal{E}_F)\, \tau \left(\frac{\hbar k_F}{m^*}\right)^2 \frac{1}{3} \delta_{\mu\nu}$$

For $\mathcal{E}_k = \frac{\hbar^2 k^2}{2m^*}$, $g(\mathcal{E}_F) = \frac{3}{2}\frac{n}{\mathcal{E}_F}$. $\frac{\hbar^2 k_F^2}{m^{*2}} = \frac{2\mathcal{E}_F}{m^*}$

$$\rightarrow \sigma_{\mu\nu} = e^2 \cancel{\frac{3}{2}} \frac{n}{\cancel{\mathcal{E}_F}}\, \tau\, \cancel{\frac{2\mathcal{E}_F}{m^*}}\, \cancel{\frac{1}{3}}\, \delta_{\mu\nu}$$

$$\boxed{\sigma_{\mu\nu} = \frac{n e^2 \tau}{m^*} \delta_{\mu\nu}}$$

For $\mathcal{E}_k \approx -\frac{\hbar^2 k^2}{2m^*}$ + constant in a band, $v_k = -\frac{\hbar k}{m^*}$.

Again $\sigma_{\mu\nu} = e^2 g(\mathcal{E}_F)\, \tau \left(\frac{\hbar k_F}{m^*}\right)^2 \frac{1}{3} \delta_{\mu\nu}$.

Using $g(\mathcal{E}_F) = \frac{3}{2}\frac{n}{\mathcal{E}_F}$ with n being the density of <u>holes</u>,

we again get $\sigma_{\mu\nu} = \frac{n e^2 \tau}{m^*} \delta_{\mu\nu}$.

3. Time dependent electric field $(\vec{\nabla}_r f = 0, \vec{F} = -e\vec{E})$

$$\frac{\partial f}{\partial t} + \frac{-e\vec{E}}{\hbar} \cdot \vec{\nabla}_k f = -\frac{(f - f_{eq})}{\tau}$$

Linear response:

$$\frac{\partial f}{\partial t} - \frac{e\vec{E}}{\hbar} \cdot \vec{\nabla}_k f_{eq} = -\frac{(f - f_{eq})}{\tau}$$

Fourier transform:

$$-i\omega f - \frac{e\vec{E}}{\hbar} \cdot \vec{\nabla}_k \varepsilon \frac{\partial f_{eq}}{\partial \varepsilon} = -\frac{f}{\tau}$$

$$\left(\frac{1}{\tau} - i\omega\right) f = e\vec{E} \cdot \vec{v} \left(\frac{\partial f_{eq}}{\partial \varepsilon}\right)$$

$$\rightarrow j_\mu = -e \int \frac{d^3k}{4\pi^3} v_\mu \frac{-e\vec{E} \cdot \vec{v}}{\frac{1}{\tau} - i\omega} \left(-\frac{\partial f_{eq}}{\partial \varepsilon_k}\right)$$

$$= \sigma_{\mu\nu} E_\nu$$

$$\boxed{\sigma_{\mu\nu} = e^2 \int \frac{d^3k}{4\pi^3} \frac{1}{\frac{1}{\tau} - i\omega} v_\mu v_\nu \left(\frac{-\partial f_{eq}}{\partial \varepsilon_k}\right)}$$