Name: Solution

**Exam 2 - PHY 4604 - Fall 2001**
October 30, 2001
6:15-8:05PM, NPB 1002

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

1. **Short answer section**

(a) What is the commutator of the position and momentum operators?

$$[x, p] = i\hbar$$

(b) What differential equations do the expectation values of the position and momentum operators satisfy for the hamiltonian $H = \frac{p^2}{2m} + V(x)$?

$$\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m}$$

$$\frac{d\langle p\rangle}{dt} = -\left\langle \frac{dV}{dx} \right\rangle$$

(c) Express the solution of the time-dependent Schrodinger equation at time $t$ in terms of the wave function at $t = 0$ and the eigenvectors of the hamiltonian.

Let $|\psi(t=0)\rangle = \sum_n c_n |\phi_n\rangle$, where $H|\phi_n\rangle = E_n|\phi_n\rangle$.

Then $|\psi(t)\rangle = \sum_n c_n e^{-i\frac{E_n t}{\hbar}} |\phi_n\rangle$.

(d) Explain the difference between the Heisenberg and Schrodinger pictures.

| | wave function | operators |
|---|---|---|
| Schrodinger pict. | time dep. | time indep. |
| Heisenberg pict. | time indep. | time dep. |

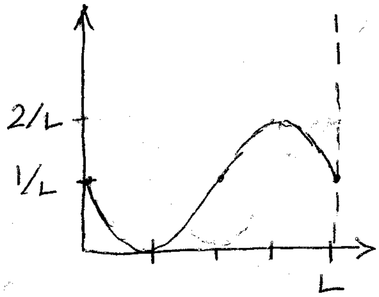
$$\langle A\rangle = \underbrace{\langle\psi(t)|A|\psi(t)\rangle}_{\text{Schrodinger}} = \underbrace{\langle\psi(t_0)|A_H(t)|\psi(t_0)\rangle}_{\text{Heisenberg}}$$

(e) What is the commutation relation between the raising and lowering operators?

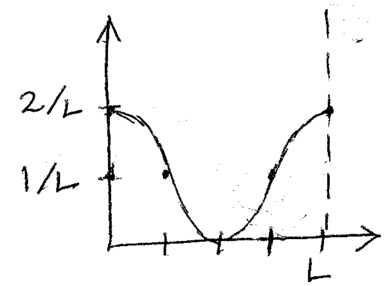
$$[a, a^\dagger] = 1$$

(d) Sketch the probability density at $\tau = 0$, $\tau/4$, $\tau/2$, and $3\tau/4$.

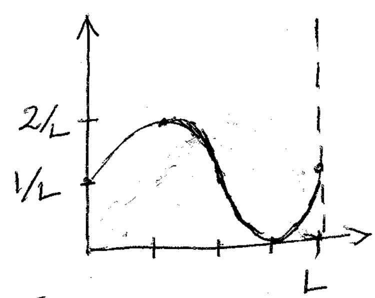
$|\psi(x,0)|^2 = \frac{1}{L}(1-\sin(kx))$



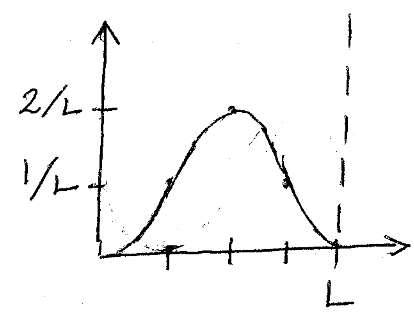

$|\psi(x,\frac{\tau}{4})|^2 = \frac{1}{L}(1-\sin(kx-\frac{\pi}{2}))$




$|\psi(x,\frac{\tau}{2})|^2 = \frac{1}{L}(1-\sin(kx-\pi))$




$|\psi(x,\frac{3\tau}{4})|^2 = \frac{1}{L}(1-\sin(kx-\frac{3\pi}{2}))$




(e) If the position of the particle is measured at $t = \frac{\tau}{4}$, what is the probability that the particle is between 0 and $L/2$, i.e., $0 \le x \le L/2$?

$$\text{Probability} = \int_0^{L/2} |\psi(x,\tfrac{\tau}{4})|^2 dx$$

$$= \int_0^{L/2} \frac{1}{L}\left(1+\cos\left(\frac{2\pi x}{L}\right)\right) dx$$

$$= \frac{1}{L}\left(\frac{L}{2} + \frac{L}{2\pi}\sin\left(\frac{2\pi x}{L}\right)\Big|_0^{L/2}\right)$$

$$= \frac{1}{2}$$

3. **Spin 1/2**

(a) At time $t = 0$, the spin in the z-direction, $S_z$, is measured and found to be $-\hbar/2$. What is the state vector, $|\psi(0)\rangle$, immediately after the measurement?

$$|\psi(0)\rangle = |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b) Immediately after this measurement, a uniform magnetic field is applied in the x-direction: $\mathbf{B} = B_o\mathbf{x}$. Determine the wave function at time t: $|\psi(t)\rangle$.

$$H = -\vec{M}\cdot\vec{B} = -\gamma\vec{S}\cdot\vec{B} = -\frac{\gamma B_o\hbar}{2}\sigma_x$$

$$U(t,0) = exp\left(-\frac{iHt}{\hbar}\right) = exp\left(i\frac{\gamma B_o t}{2}\sigma_x\right)$$

$$= \cos\left(\frac{\gamma B_o}{2}t\right)\mathbb{1} + i\sin\left(\frac{\gamma B_o}{2}t\right)\sigma_x$$

$$|\psi(t)\rangle = exp\left(\frac{-iHt}{\hbar}\right)|\psi(0)\rangle$$

$$= \begin{pmatrix} \cos\left(\frac{\gamma B_o}{2}t\right) & i\sin\left(\frac{\gamma B_o}{2}t\right) \\ i\sin\left(\frac{\gamma B_o}{2}t\right) & \cos\left(\frac{\gamma B_o}{2}t\right) \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi(t)\rangle = \begin{pmatrix} i\sin\left(\frac{\gamma B_o}{2}t\right) \\ \cos\left(\frac{\gamma B_o}{2}t\right) \end{pmatrix}$$

(c) At this time $t$, we measure $S_y$. What values can we find and with what probabilities?

$$S_y = +\frac{\hbar}{2}, \quad \text{Prob.} = |{}_y\langle + | \psi(t)\rangle|^2 = \left| \frac{1}{\sqrt{2}} (1 \ \ -i) \begin{pmatrix} i\sin(\frac{\gamma B_0}{2}t) \\ \cos(\frac{\gamma B_0}{2}t) \end{pmatrix} \right|^2$$

$$= \frac{1}{2} \left| i\sin(\tfrac{\gamma B_0}{2}t) - i\cos(\tfrac{\gamma B_0}{2}t) \right|^2 = \frac{1}{2}\left(\sin(\tfrac{\gamma B_0}{2}t) - \cos(\tfrac{\gamma B_0}{2}t)\right)^2$$

$$= \frac{1}{2}(1 - \sin(\gamma B_0 t))$$

$$S_y = -\frac{\hbar}{2}, \quad \text{Prob.} = |{}_y\langle - | \psi(t)\rangle|^2 = \left| \frac{1}{\sqrt{2}} (1 \ \ i) \begin{pmatrix} i\sin(\frac{\gamma B_0}{2}t) \\ \cos(\frac{\gamma B_0}{2}t) \end{pmatrix} \right|^2$$

$$= \frac{1}{2}\left(\sin(\tfrac{\gamma B_0}{2}t) + \cos(\tfrac{\gamma B_0}{2}t)\right)^2$$

$$= \frac{1}{2}(1 + \sin(\gamma B_0 t))$$

(d) What relations must exist between $B_o$ and $t$ for the result of the measurement to be certain? Give a physical interpretation of this condition.

$$\text{Prob.}\{S_y = \tfrac{\hbar}{2}\} = 1 \quad \text{for} \quad \gamma B_0 t = \frac{3\pi}{2} + 2\pi n$$

$$\text{Prob.}\{S_y = -\tfrac{\hbar}{2}\} = 1 \quad \text{for} \quad \gamma B_0 t = \frac{\pi}{2} + 2\pi n$$

Since $\vec{B} = B_0 \hat{x}$, the spin precesses about the $x$-axis. Since the spin is initially in the $-\hat{z}$ direction, it precesses in the $y$-$z$ plane → Periodically it points in the $+\hat{y}$ and $-\hat{y}$ directions.

(e) What is the expectation values of $S_y$ as a function of time? Give a physical interpretation of this result and compare it to the results of (c) and (d) above.

$$\langle S_y \rangle = \langle \psi(t) | S_y | \psi(t) \rangle$$

$$= \frac{\hbar}{2} \left(-i\sin(\tfrac{\gamma B_0}{2}t) \ \ \cos(\tfrac{\gamma B_0}{2}t)\right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} i\sin(\frac{\gamma B_0}{2}t) \\ \cos(\frac{\gamma B_0}{2}t) \end{pmatrix}$$

$$= \frac{\hbar}{2} \left(-i\sin(\tfrac{\gamma B_0}{2}t) \ \ \cos(\tfrac{\gamma B_0}{2}t)\right) \begin{pmatrix} -i\cos(\frac{\gamma B_0}{2}t) \\ -\sin(\frac{\gamma B_0}{2}t) \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(-2\sin(\tfrac{\gamma B_0}{2}t)\cos(\tfrac{\gamma B_0}{2}t)\right) = \boxed{-\frac{\hbar}{2}\sin(\gamma B_0 t) = \langle S_y \rangle}$$

$\langle S_y \rangle = -\frac{\hbar}{2}$ when $\text{Prob.}\{S_y = -\frac{\hbar}{2}\} = 1$ and
$\langle S_y \rangle = \frac{\hbar}{2}$ when $\text{Prob.}\{S_y = \frac{\hbar}{2}\} = 1$. ✓

4. **Harmonic oscillator**

(a) At $t = 0$ a harmonic oscillator is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle.$$

What is the wave function at an arbitrary time $t$?

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = \frac{1}{\sqrt{2}} e^{-iE_0 t/\hbar}|0\rangle + \frac{i}{\sqrt{2}} e^{-iE_1 t/\hbar}|1\rangle$$

(b) For the harmonic oscillator the position and momentum operators may be written as

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$$
$$p = \sqrt{\frac{m\hbar\omega}{2}}\, i\,(a^\dagger - a).$$

Compute the expectation value of the position and momentum as a function of time for the above wave function. Do the results for $\langle x\rangle$ and $\langle p\rangle$ make sense when compared to the classical equations of motion?

$$\langle x\rangle = \langle\psi(t)|x|\psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\Big(\tfrac{1}{2}\langle 0|(a^\dagger+a)|0\rangle + \tfrac{1}{2}\langle 1|(a^\dagger+a)|1\rangle$$
$$+\frac{i e^{-\frac{i(E_1-E_0)t}{\hbar}}}{2}\langle 0|(a^\dagger+a)|1\rangle$$
$$-\frac{i e^{\frac{i(E_1-E_0)t}{\hbar}}}{2}\langle 1|(a^\dagger+a)|0\rangle\Big)$$

$$\boxed{\langle x\rangle(t) = \sqrt{\frac{\hbar}{2m\omega}}\sin(\omega t)}\text{, where } E_n = \hbar\omega(n+\tfrac{1}{2})$$

$$\langle p\rangle = \langle\psi(t)|x|\psi(t)\rangle = \sqrt{\frac{m\hbar\omega}{2}}\Big(\tfrac{1}{2}\langle 0|i(a^\dagger-a)|0\rangle + \tfrac{1}{2}\langle 1|i(a^\dagger-a)|1\rangle$$
$$+\frac{i e^{-i\omega t}}{2}\langle 0|i(a^\dagger-a)|1\rangle - \frac{i e^{i\omega t}}{2}\langle 1|i(a^\dagger-a)|0\rangle\Big)$$

$$\boxed{\langle p\rangle(t) = \sqrt{\frac{m\hbar\omega}{2}}\cos(\omega t)}$$

Like the classical equations of motion: $\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m}$

$$\frac{d\langle p\rangle}{dt} = -m\omega^2\langle x\rangle.$$

(c) Compute the expectation values of $x^2$ and $p^2$ as as a function of time. How are these related to the energy?

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\Big(\tfrac{1}{2}\langle 0|(a^\dagger+a)^2|0\rangle + \tfrac{1}{2}\langle 1|(a^\dagger+a)^2|1\rangle + \frac{ie^{-i\omega t}}{2}\langle 0|(a^\dagger+a)^2|1\rangle - \frac{ie^{i\omega t}}{2}\langle 1|(a^\dagger+a)^2|0\rangle\Big)$$

$$= \frac{\hbar}{2m\omega}\Big(\tfrac{1}{2}\langle 0|a^\dagger a+aa^\dagger|0\rangle + \tfrac{1}{2}\langle 1|a^\dagger a+aa^\dagger|1\rangle\Big)$$

$$= \frac{\hbar}{2m\omega}\Big(\frac{2\cdot 0+1}{2} + \frac{2\cdot 1+1}{2}\Big) = \boxed{\frac{\hbar}{m\omega} = \langle x^2\rangle}$$

$$\langle p^2\rangle = \frac{m\hbar\omega}{2}\Big(\tfrac{1}{2}\langle 0|-(a^\dagger-a)^2|0\rangle + \tfrac{1}{2}\langle 1|-(a^\dagger-a)^2|1\rangle + \frac{ie^{-i\omega t}}{2}\langle 0|-(a^\dagger-a)^2|1\rangle - \frac{ie^{i\omega t}}{2}\langle 1|-(a^\dagger-a)^2|0\rangle\Big)$$

$$= \frac{m\hbar\omega}{2}\Big(\tfrac{1}{2}\langle 0|a^\dagger a+aa^\dagger|0\rangle + \tfrac{1}{2}\langle 1|a^\dagger a+aa^\dagger|1\rangle\Big) = \boxed{m\hbar\omega = \langle p^2\rangle}$$

The expectation value of the energy is $\frac{\langle p^2\rangle}{2m} + \frac{1}{2}m\omega^2\langle x^2\rangle = \hbar\omega$. ☆

(d) Finally, using the results from (b) and (c) above, check the uncertainty principle for this wave function.

$$(\Delta x)^2 = \langle\psi|(x-\langle x\rangle)^2|\psi\rangle = \langle\psi|x^2|\psi\rangle - 2\langle x\rangle\langle\psi|x|\psi\rangle + \langle x\rangle^2$$

$$= \langle x^2\rangle - \langle x\rangle^2$$

$$= \frac{\hbar}{m\omega} - \frac{\hbar}{2m\omega}\sin^2(\omega t)$$

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = m\hbar\omega - \frac{m\hbar\omega}{2}\cos^2(\omega t)$$

$$\rightarrow \Delta x = \sqrt{\frac{\hbar}{m\omega}}\sqrt{1-\tfrac{1}{2}\sin^2(\omega t)}$$

$$\Delta p = \sqrt{m\hbar\omega}\sqrt{1-\tfrac{1}{2}\cos^2(\omega t)}$$

$$\rightarrow \Delta x\Delta p = \hbar\Big[\Big(1-\frac{\sin^2(\omega t)}{2}\Big)\Big(1-\frac{\cos^2(\omega t)}{2}\Big)\Big]^{1/2}$$

$$= \hbar\Big[1-\frac{1}{2}+\frac{\sin^2(\omega t)}{2}\frac{\cos^2(\omega t)}{2}\Big]^{1/2} \geq \hbar\Big[\frac{1}{2}\Big]^{1/2} > \frac{\hbar}{2}. \checkmark$$

☆ Note: $\hbar\omega$ is in between $E_0 = \frac{\hbar\omega}{2}$ and $E_1 = \frac{3\hbar\omega}{2}$.