Name: Solution

Exam 3 - PHY 4604 - Fall 2001

December 3, 2001

6:15-8:05PM, 310 Larsen Hall

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

Spherical Harmonics

$$
\begin{aligned}
Y_0^0(\theta,\phi) &= \sqrt{\frac{1}{4\pi}} \\
Y_1^{\pm 1}(\theta,\phi) &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \\
Y_1^0(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_2^{\pm 2}(\theta,\phi) &= \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi} \\
Y_2^{\pm 1}(\theta,\phi) &= \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi} \\
Y_2^0(\theta,\phi) &= \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)
\end{aligned}
$$

1. **Short answer section**

(a) What is J_+ acting on $|j,m\rangle$? What is J_- acting on $|j,m\rangle$? What is J_z acting on $|j,m\rangle$?

$J_+|j,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}\,|j,m+1\rangle$

$J_-|j,m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\,|j,m-1\rangle$

$J_z|j,m\rangle = \hbar m|j,m\rangle$

(b) What is a symmetry in quantum mechanics? Give some examples of symmetries we have studied.

A symmetry is an operation which leaves the hamiltonian unchanged. Examples:

$H = \frac{p^2}{2m}$, translation ; $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$, inversion

$H = \frac{p^2}{2m} + V(|r|)$, rotation

(c) What is the form of the solution to the Schrodinger equation for a central potential?

$\psi(r,\theta,\varphi) = R(r)Y_\ell^m(\theta,\varphi)$

(d) What is the Bohr radius (numerically)?

$a_0 \approx 0.529\,\text{Å}$

(e) What is the asymptotic behavior of the eigenstates of the hydrogen atom for energies less than zero?

$R(r) \sim e^{-\lambda r/a_0}$, where $\lambda = \frac{1}{n}$.

2. **Angular momentum**

(a) Compute the commutator $[[L_x, L_z], L_z]$.

$= [-i\hbar L_y, L_z] = (-i\hbar)(i\hbar L_z) = \hbar^2 L_x$

(b) Expand the function, $f(\theta, \phi) = (1 + \cos(\theta))^2$ in terms of the spherical harmonics. Do not leave your final answer in terms of integrals.

$$f(\theta,\varphi) = 1 + 2\cos\theta + \cos^2\theta$$

$$= \frac{4}{3} + 2\cos\theta + \left(\cos^2\theta - \frac{1}{3}\right)$$

$$= \frac{4}{3}\sqrt{4\pi}\, Y_0^0 + 2\sqrt{\frac{4\pi}{3}}\, Y_1^0 + \frac{1}{3}\sqrt{\frac{16\pi}{5}}\, Y_2^0$$

(c) For $l = 1$ compute the matrix elements of the operator $(L_x)^2$ and express it as a matrix.

$$L_x = \frac{1}{2}(L_+ + L_-) \rightarrow (L_x)^2 = \frac{1}{4}(L_+^2 + L_+L_- + L_-L_+ + L_-^2)$$

$$\left.\begin{aligned}
&L_+|1,0\rangle = \hbar\sqrt{2}\,|1,1\rangle \rightarrow L_-L_+|1,0\rangle = 2\hbar^2|1,0\rangle \\
&L_+|1,-1\rangle = \hbar\sqrt{2}\,|1,0\rangle \rightarrow L_-L_+|1,-1\rangle = 2\hbar^2|1,-1\rangle \\
&L_-|1,0\rangle = \hbar\sqrt{2}\,|1,-1\rangle \rightarrow L_+L_-|1,0\rangle = 2\hbar^2|1,0\rangle \\
&L_-|1,+1\rangle = \hbar\sqrt{2}\,|1,0\rangle \rightarrow L_+L_-|1,1\rangle = 2\hbar^2|1,1\rangle \\
&\text{and} \quad (L_+)^2|1,-1\rangle = 2\hbar^2|1,1\rangle \\
&\qquad\quad (L_-)^2|1,1\rangle = 2\hbar^2|1,-1\rangle
\end{aligned}\right\} \text{All other terms are zero.}$$

$$\rightarrow (L_x)^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

3. **Central potentials**

Consider the following central potential, where $V_o > 0$.

$$\begin{aligned} V(r) &= -V_o \text{ for } r < r_o \\ &= +V_o \text{ for } r > r_o, \end{aligned}$$

Note that is similar *but not identical* to the one in homework assignment 11.

(a) Solve the radial Schrodinger equation for $u(r)$ in the region $0 < r < r_o$ with the above $V(r)$ and $l = 0$. Assume that $E < V_o$. Apply the boundary condition at $r = 0$ to reduce this to one solution.

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} - V_o u = Eu \rightarrow \frac{d^2u}{dr^2} = -\frac{2m}{\hbar^2}(E+V_o)u$$

For $E < -V_o$ there will not be a solution consistent with $u(0) = 0$ and $\int_0^\infty dr\,(u(r))^2 = 1$. $\rightarrow E > -V_o \rightarrow E + V_o > 0$.

Let $k = \sqrt{\frac{2m}{\hbar^2}(E+V_o)}$.

$$\rightarrow \frac{d^2u}{dr^2} = -k^2 u \rightarrow u = e^{\pm ikr}$$

Since $u(0) = 0$, $u(r) = A\sin(kr)$.

(b) Solve the radial Schrodinger equation for $u(r)$ in the region $r > r_o$ again assuming that $E < V_o$ and $l = 0$. Apply the condition that $u(r)$ must be normalizable to reduce this to one solution.

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V_o u = Eu \rightarrow \frac{d^2u}{dr^2} = \frac{2m}{\hbar^2}(V_o - E)u$$

Since $V_o > E$, $V_o - E > 0$, let $K = \sqrt{\frac{2m}{\hbar^2}(V_o - E)}$.

$$\rightarrow \frac{d^2u}{dr^2} = K^2 u \rightarrow u = e^{\pm Kr}$$

Since $\int_0^\infty u(r)^2\,dr = 1 < \infty$, $u(r) = Be^{-Kr}$

(c) Match the boundary conditions for $u(r)$ at $r = r_o$ and derive a condition for there being a bound state.

$$u(r_0) = A \sin(kr_0) = Be^{-Kr_0}$$

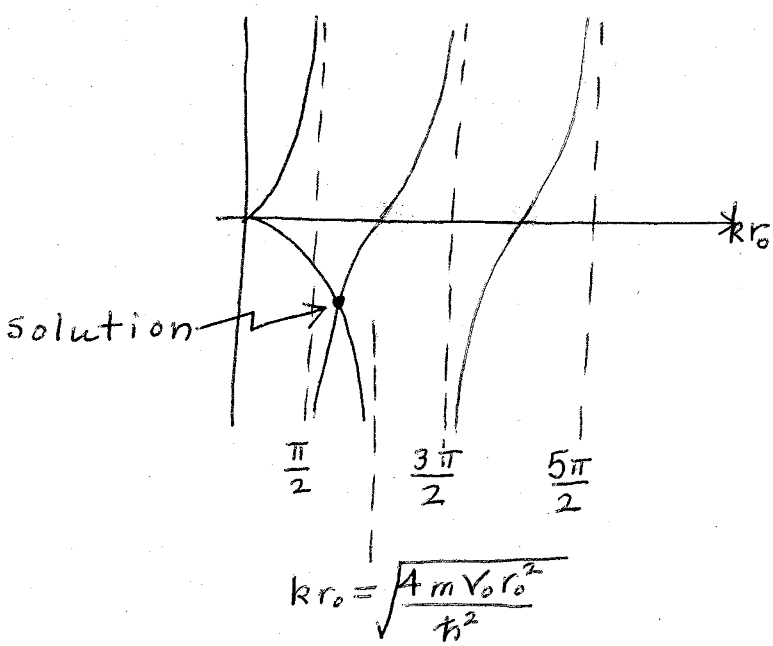
$$u'(r_0) = +kA\cos(kr_0) = -KBe^{-Kr_0}$$

$$\rightarrow \tan(kr_0) = -k/K$$

Because $K^2 = \frac{2m}{\hbar^2}(V_0 - E) = \frac{4m}{\hbar^2}V_0 - \frac{2m}{\hbar^2}(V_0 + E) = \frac{4mV_0}{\hbar^2} - k^2$, this can be written as

$$\tan(kr_0) = \frac{-(kr_0)}{\sqrt{\frac{4mV_0r_0^2}{\hbar^2} - (kr_0)^2}},$$

(d) Illustrate this condition graphically. What is the minimum value of V_o for there to be at least one bound state for $E < V_o$?



For there to be at least one bound state,

$$\sqrt{\frac{4mV_0r_0^2}{\hbar^2}} > \frac{\pi}{2}.$$

4. **Symmetry, Rotations, and the Hydrogen Atom**

Consider the following Hamiltonian on the unit circle:

$$H = -A\frac{d^2}{d\theta^2},$$

where the constant A has the units of energy. The wave functions on the unit circle satisfy $\psi(\theta) = \psi(\theta + 2\pi)$.

(a) What is a symmetry of this Hamiltonian? What is the operator corresponding to this symmetry?

H is invariant under the operation

$$T\psi(\theta) = \psi(\theta + a).$$

This may be regarded as either a translation or a rotation.

(b) Find the eigenvectors of H which are also eigenvectors of the symmetry operator in part (a). What are the corresponding eigenvalues of the Hamiltonian and the symmetry operator?

$\psi(\theta) = Ce^{ik\theta}$ has energy $= Ak^2$,

and T eigenvalue $= e^{ika}$,

where $k = 2\pi n$ for ψ to be single valued.

(c) Express the symmetry operator in terms of its infinitesimal generator.

$$\psi(\theta + \delta\theta) \approx \left(\mathbb{1} + \frac{d}{d\theta}\delta\theta\right)\psi(\theta)$$

$\rightarrow$ For large N, $\psi(\theta + a) \approx \left(\mathbb{1} + \frac{a}{N}\frac{d}{d\theta}\right)^N \psi(\theta)$

$$\rightarrow \exp\left(a\frac{d}{d\theta}\right)\psi(\theta) \text{ as } N \rightarrow \infty,$$

(d) Below are shown plots of the radial wave, $R(r)$, for the hydrogen atom for $n = 1$ and $n = 2$. From what you know about the behavior as $r \to 0$ and $r \to \infty$ label the graphs with the appropriate n and l indices.