Name: Solution

Final Exam - PHY 4604 - Fall 2001
December 3, 2001
3:00-5:00PM, 1101 New Physics Building

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

1. **Short answer section**

(a) Write down the time dependent Schrodinger equation.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2 \nabla^2}{2m}\psi + V(\vec{r})\psi \quad \text{in 3D}$$

(b) What is the meaning of the magnitude squared of the wave function?

$|\psi(x,t)|^2 dx$ is the probability of finding a particle between x & $x+dx$ at time t.

(c) What is the commutation relation between the raising and lowering operators of the harmonic oscillator?

$$[a, a^{\dagger}] = 1$$

(d) What are the commutators of L_x, L_y, L_z and L^2?

$$[L_x, L_y] = i\hbar L_z$$
$$[L_y, L_z] = i\hbar L_x$$
$$[L_z, L_x] = i\hbar L_y$$

$[L_\alpha, L^2] = 0$ for $\alpha = x, y, z$

(e) What are the bound state energies of the hydrogen atom?

$$E = -E_I/n^2, \quad E_I = 13.6\ \text{eV}$$

2. **One-dimensional Schrodinger Equation**

(a) Consider the following potential in one dimension:

$$V(x) = V_o \text{ for } x < 0$$
$$= 0 \text{ for } x > 0.$$

Assuming that $E > V_o > 0$ find the form of the solutions to the Schrodinger equation for $x > 0$ and for $x < 0$.

Let $k_1 = \sqrt{\frac{2m(E-V_o)}{\hbar^2}}$ and $k_2 = \sqrt{\frac{2mE}{\hbar^2}}$.

$\psi(x<0) = Ae^{ik_1x} + Be^{-ik_1x}$

$\psi(x>0) = Ce^{ik_2x} + De^{-ik_2x}$

(b) Matching boundary conditions at $x = 0$, find the stationary state wave function for a particle moving from left to right. What is the transmission probability?

For a particle moving from left to right $D = 0$.

$\psi(0) = A + B = C$

$\psi'(0) = ik_1(A-B) = ik_2C$

$\rightarrow A - B = \frac{k_2}{k_1}C$

$\rightarrow 2A = \left(1 + \frac{k_2}{k_1}\right)C \rightarrow \frac{C}{A} = \frac{2}{1+\frac{k_2}{k_1}} = \boxed{\frac{2k_1}{k_1+k_2} = \frac{C}{A}}$

$\rightarrow \frac{B}{A} = \frac{C}{A} - 1 = \boxed{\frac{k_1-k_2}{k_1+k_2} = \frac{B}{A}}$

$R = \left|\frac{B}{A}\right|^2 = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$

Note that $T+R=1$.

$\boxed{T = \frac{k_2}{k_1}\left|\frac{C}{A}\right|^2 = \frac{4k_1k_2}{(k_1+k_2)^2}}$

3. **Wave Function Space & Principles of Quantum Mechanics**

Consider an infinite square well between $x = 0$ and $x = L$. The energy eigenstates of the Hamiltonian are

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right)$$

for $n = 1, 2, \ldots$ These states form a complete orthonormal basis.

(a) Suppose at $t = 0$ the wave function of the system is given by

$$\psi(x, 0) = 1 \text{ if } x < L/2 \text{ and } 0 \text{ if } x > L/2.$$

Express $\psi(x, 0)$ in terms of the $\phi_n(x)$.

$|\psi(t=0)\rangle = \sum_{n=1}^{\infty} |\phi_n\rangle\langle\phi_n|\psi(t=0)\rangle$, where

$$\langle\phi_n|\psi(t=0)\rangle = \int_0^L dx\, \phi_n^*(x)\, \psi(x, t=0)$$

$$= \int_0^{L/2} dx \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right) = \sqrt{\frac{2}{L}}\, \frac{L}{n\pi}\, -\cos\left(\frac{\pi n x}{L}\right)\Big|_0^{L/2}$$

$$= \frac{\sqrt{2L}}{n\pi}\left(1 - \cos\left(\frac{\pi n}{2}\right)\right)$$

(b) Based on your results from part (a), compute the wave function at an arbitrary time, $\psi(x, t)$.

$$|\psi(t)\rangle = \sum_{n=1}^{\infty} \frac{\sqrt{2L}}{n\pi}\left(1 - \cos\left(\frac{\pi n}{2}\right)\right) e^{-\frac{iE_n t}{\hbar}} |\phi_n\rangle,$$

where $E_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{L}\right)^2$ because

$$H\phi_n = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right) = \frac{\hbar^2}{2m}\left(\frac{\pi n}{L}\right)^2 \phi_n.$$

4. **Spin 1/2**

(a) A spin 1/2 particle is in a magnetic field pointing in the z-direction, $B_o\mathbf{z}$. If at $t = 0$, the spin of the particle is measured to point in the +x direction, what is the wave function as a function of time?

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

$$|\psi(t)\rangle = exp\left(\frac{-iHt}{\hbar}\right)|\psi(0)\rangle, \text{ where } H = -\vec{M}\cdot\vec{B}$$

$$= -\gamma\vec{S}\cdot\vec{B}$$

$$= -\frac{\gamma\hbar B}{2}\sigma_z$$

$$= exp\left(\frac{i\gamma B}{2}t\,\sigma_z\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\gamma Bt/2}\\e^{-i\gamma Bt/2}\end{pmatrix}$$

(b) At time, t, the spin of the particle is measured in the $\mathbf{y}$ direction. What are the probabilities for the different possible measured values of the spin?

The eigenvectors of S_y are

$$S_y\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

$$S_y\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix} = -\frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}.$$

$$\rightarrow \text{Prob.}\left\{S_y = \frac{\hbar}{2}\right\} = \left|\frac{1}{\sqrt{2}}(1\ \ -i)\frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\gamma Bt/2}\\e^{-i\gamma Bt/2}\end{pmatrix}\right|^2$$

$$= \frac{1}{4}\left|e^{i\gamma Bt/2} + e^{-i\gamma Bt/2}(-i)\right|^2$$

$$= \frac{1}{4}\left(2 - ie^{-i\gamma Bt} + ie^{i\gamma Bt}\right) = \frac{1}{2}(1 - \sin(\gamma Bt))$$

$$\text{Prob.}\left\{S_y = -\frac{\hbar}{2}\right\} = \left|\frac{1}{\sqrt{2}}(1\ \ i)\frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\gamma Bt/2}\\e^{-i\gamma Bt/2}\end{pmatrix}\right|^2$$

$$= \frac{1}{2}(1 + \sin(\gamma Bt))$$

5. **Harmonic Oscillator**

(a) At $t = 0$ a harmonic oscillator is in the state

$$|\psi(0)\rangle = \frac{2}{5}|0\rangle + \frac{3i}{5}|1\rangle.$$

What is the wave function at an arbitrary time t?

$$|\psi(t)\rangle = \frac{2}{5} e^{-iE_0 t/\hbar} |0\rangle + \frac{3i}{5} e^{-iE_1 t/\hbar} |1\rangle,$$

where $E_n = \hbar\omega(n+\frac{1}{2})$.

(b) For the harmonic oscillator the position operator may be written as

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a).$$

Compute the expectation value of the position as a function of time for the above wave function.

$$\langle\psi(t)|x|\psi(t)\rangle =$$

$$= \left(\frac{2}{5} e^{\frac{iE_0 t}{\hbar}} \langle 0| - \frac{3i}{5} e^{\frac{iE_1 t}{\hbar}} \langle 1|\right) x \left(\frac{2}{5} e^{-\frac{iE_0 t}{\hbar}} |0\rangle + \frac{3i}{5} e^{-\frac{iE_1 t}{\hbar}} |1\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \frac{2}{5}\frac{3}{5}\left(e^{\frac{i(E_0-E_1)t}{\hbar}} i\langle 0|a|1\rangle + e^{\frac{i(E_1-E_0)t}{\hbar}} -i\langle 1|a|0\rangle\right)$$

$$= \frac{6}{25}\sqrt{\frac{\hbar}{2m\omega}}\left(ie^{-i\omega t} - ie^{i\omega t}\right)$$

$$= \frac{12}{25}\sqrt{\frac{\hbar}{2m\omega}} \sin(\omega t)$$

6. **Angular Momentum**

(a) Compute the commutator $[J_x^2, J_y]$.

$$[J_x^2, J_y] = J_x^2 J_y - J_y J_x^2 - J_x J_y J_x + J_x J_y J_x$$

$$= J_x [J_x, J_y] + [J_x, J_y] J_x$$

$$= i\hbar \left(J_x J_z + J_z J_x \right)$$

(b) What are the non-zero matrix elements of J_- for $j = 3/2$?

Since $J_- |j, m\rangle = \hbar \sqrt{j(j+1) - m(m-1)}\, |j, m-1\rangle$, the non-zero matrix elements are:

$$\langle \tfrac{3}{2}, \tfrac{1}{2} | J_- | \tfrac{3}{2}, \tfrac{3}{2} \rangle = \hbar \sqrt{\tfrac{3}{2} \cdot \tfrac{5}{2} - \tfrac{3}{2} \cdot \tfrac{1}{2}} = \tfrac{\hbar}{2} \sqrt{12} = \hbar \sqrt{3}$$

$$\langle \tfrac{3}{2}, -\tfrac{1}{2} | J_- | \tfrac{3}{2}, \tfrac{1}{2} \rangle = \hbar \sqrt{\tfrac{3}{2} \cdot \tfrac{5}{2} - \tfrac{1}{2} \cdot \left(-\tfrac{1}{2}\right)} = \tfrac{\hbar}{2} \sqrt{16} = \hbar\, 2$$

$$\langle \tfrac{3}{2}, -\tfrac{3}{2} | J_- | \tfrac{3}{2}, -\tfrac{1}{2} \rangle = \hbar \sqrt{\tfrac{3}{2} \cdot \tfrac{5}{2} + \tfrac{1}{2} \cdot \left(\tfrac{-3}{2}\right)} = \tfrac{\hbar}{2} \sqrt{12} = \hbar \sqrt{3}$$

7. **Central Potentials**

Consider the three dimensional harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2.$$

(a) What is the radial Schrodinger equation for $u(r)$ with angular momentum l?

$$\left(\frac{-\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + \frac{1}{2}m\omega^2 r^2 \right) u(r) = E u(r)$$

(b) Show that $u(r) = C\exp(-Ar^2)$ is a solution to the differential equation for $l = 0$. Is this a physical solution? Explain your answer.

For $l = 0$, $\left(\frac{-\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{1}{2}m\omega^2 r^2 \right) u = E u$.

$$\frac{d}{dr} Ce^{-Ar^2} = -2ArCe^{-Ar^2}$$

$$\frac{d^2}{dr^2} Ce^{-Ar^2} = 4A^2r^2Ce^{-Ar^2} \quad -2A\,Ce^{-Ar^2}$$

$$\rightarrow \left(-\frac{\hbar^2}{2m}4A^2r^2 + \frac{\hbar^2}{2m}2A + \frac{1}{2}m\omega^2 r^2 \right) u = E u$$

This is a solution to the differential equation provided that:

$$\frac{\hbar^2}{2m}4A^2 = \frac{1}{2}m\omega^2 \rightarrow A^2 = \frac{m^2\omega^2}{4\hbar^2} \rightarrow A = \frac{m\omega}{2\hbar}$$

and $\frac{\hbar^2}{m}A = E = \frac{\hbar\omega}{2}$;

however, this <u>is not</u> a physical solution because $u(r) \sim r^{l+1}$ as $r \rightarrow 0$.