Homework 10

(due Friday, Nov. 16)

This is the first homework assignment on angular momentum. It is designed to familiarize you with the commutation relations, matrix elements, and spherical harmonics. It is straight forward; however, be forewarned that there is a considerable amount of algebra involved.

1. Commutators:
   (a) Show that the commutator $[L^2, L_z]$ is equal to zero.
   (b) Compute the commutator $[L_x^2 L_y, L_x]$.
   (c) Compute the commutator $[L_y^3, L_x]$.
   (d) Compute the commutator $[L_z^2 L_y, L_x]$.
   (e) Add the results from (b)-(d) to compute $[L^2 L_y, L_x]$. Does the result make sense?

2. Matrix elements:
   (a) For $j = 1$ compute the three matrices, $\langle j', m'|J_\alpha|j, m\rangle$ for $\alpha = x, y, z$.
   (b) Check that these matrices obey the angular momentum commutation relations: $[J_x, J_y] = i\hbar J_z$, $[J_y, J_z] = i\hbar J_x$, and $[J_z, J_x] = i\hbar J_y$.
   (c) For $j = 3/2$ compute the three matrices, $\langle j', m'|J_\alpha|j, m\rangle$ for $\alpha = x, y, z$.
   (d) Check that these matrices obey the angular momentum commutation relations: $[J_x, J_y] = i\hbar J_z$, $[J_y, J_z] = i\hbar J_x$, and $[J_z, J_x] = i\hbar J_y$.

3. Spherical harmonics:
   (a) Using the lowering operator
   $$L_- = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot(\theta)\frac{\partial}{\partial\phi}\right)$$
   and the spherical harmonic for $m = l$
   $$Y_l^l(\theta,\phi) = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}(\sin\theta)^l e^{il\phi}$$
   determine all the spherical harmonics, $Y_l^m(\theta,\phi)$, for $l = 0$, 1, and 2. You can check your results with those on page 682.
   (b) Verify the normalization of the $l = 2$ spherical harmonics:
   $$\langle l = 2, m|l = 2, m\rangle = \int_0^\pi \sin(\theta)d\theta \int_0^{2\pi} d\phi |Y_2^m(\theta,\phi)|^2 = 1.$$
   Hint: do a change of variables to $u = \cos(\theta)$.
   (c) Verify the orthogonality of the $l = 1$ spherical harmonics:
   $$\langle l = 1, m'|l = 1, m\rangle = \int_0^\pi \sin(\theta)d\theta \int_0^{2\pi} d\phi Y_1^{m'*}(\theta,\phi)Y_1^m(\theta,\phi) = 0.$$
   for $m \neq m'$.