Homework 11
(due Wednesday, Nov. 28)

This is the last homework assignment! It appears long because I have provided details to guide you through the problems. There are enough details that you should even be able to finish it this weekend before we have covered all the material for this assignment.

1. **Spherical harmonics as a complete basis:**

   The spherical harmonics, $Y_l^m(\theta,\phi)$, are orthonormal:

   $$\langle l,m|l',m'\rangle = \int_0^\pi d\theta \sin(\theta) \int_0^{2\pi} d\phi Y_l^{m*}(\theta,\phi) Y_{l'}^{m'}(\theta,\phi) = \delta_{l,l'}\delta_{m,m'}.$$

   They are also complete on the unit sphere. In other words any function, $f(\theta,\phi)$, may be expressed in terms of the spherical harmonics.

   $$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{l,m} Y_l^m(\theta,\phi).$$

   (a) Determine the $c_{l,m}$ for $l = 0$, 1, and 2 (and all m) for the function

   $$\begin{aligned} f(\theta,\phi) &= 1 \text{ for } 0 \le \theta \le \frac{\pi}{2} \\ &= 0 \text{ for } \frac{\pi}{2} < \theta \le \pi. \end{aligned}$$

   (b) Determine the $c_{l,m}$ for $l = 0$, 1, and 2 (and all m) for the function

   $$\begin{aligned} f(\theta,\phi) &= 1 \text{ for } 0 \le \phi \le \pi \\ &= 0 \text{ for } \pi < \phi \le 2\pi. \end{aligned}$$

2. **Symmetry:**

   Here we consider a periodic potential in one dimension:

   $$H = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x),$$

   where $V(x+a) = V(x)$.

   (a) Show that the translation operator, T, commutes with this hamiltonian.

   $$T\psi(x) = \psi(x+a)$$

   (This is done by computing $TH\psi$ and $HT\psi$ and showing that they are equal.)

   (b) Because $[T,H] = 0$, there are simultaneous eigenstates of T and H. Let the eigenvalue of T be λ,

   $$T\psi(x) = \lambda\psi(x) = \psi(x+a).$$

   Show that $\psi(x+na) = \lambda^n\psi(x)$. Provide an argument based on the normalization of the wave function that $|\lambda| = 1$.

(c) Since $|\lambda| = 1$, it may be written in the form $\lambda = e^{ika}$ for some real k. Define

$$u(x) = e^{-ikx}\psi(x).$$

Show that $u(x)$ is periodic: $u(x) = u(x + a)$.
The statement that for periodic potentials the wave function may be written as $\psi(x) = e^{ikx}u(x)$, where $u(x)$ is periodic, is called Bloch's theorem. It is one of the cornerstones of solid state physics.

3. **Central potentials:**

Solve for the energy, $E < 0$, and wave function for the $l = 0$ state(s) of the central potential

$$\begin{aligned} V(r) &= -V_o \text{ for } r < r_o \\ &= 0 \text{ for } r > r_o, \end{aligned}$$

where $V_o > 0$. The solution of this Schrodinger equation for a spherically symmetrical potential has the form

$$\psi(r, \theta, \phi) = R(r)Y_l^m(\theta, \phi) = \frac{u(r)}{r}Y_l^m(\theta, \phi),$$

where the function $u(r)$ satisfies

$$\begin{aligned} \left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right)u(r) &= Eu(r) \\ \int_0^\infty (u(r))^2 &= 1. \end{aligned}$$

(a) Solve the differential equation for $u(r)$ for $0 < r < r_o$ with the above $V(r)$ and $l = 0$. There will be two independent solutions since this is a second order differential equation.

(b) Apply the boundary condition $u(0) = 0$ to reduce this to one solution.

(c) Solve the differential equation for $u(r)$ for $r > r_o$ with the above $V(r)$ and $l = 0$. Again there will be two independent solutions.

(d) Apply the condition that $u(r)$ must be normalizable to reduce this to one independent solution.

(e) Match the boundary conditions that $u(r)$ and its derivative are continuous at $r = r_o$. Show that this leads to the condition that

$$\tan(kr_o) = -\frac{(kr_o)}{\sqrt{\frac{2mV_o r_o^2}{\hbar^2} - (kr_o)^2}}, \text{ where } k = \sqrt{\frac{2m(E + V_o)}{\hbar^2}}.$$

(f) Solve this boundary condition by plotting both the right hand side and left hand side of the above equation as a function of $kr_o > 0$. Mark the solutions on your graph. Express the energy, E, in terms of k.

(g) You will note that the number of solutions depends on the dimensionless parameter $2mV_o r_o^2/\hbar^2$. What are the conditions on this dimensionless parameter for there to be no solution, one solution, or two solutions? Note: the solution for $k = 0$ implies that $u = 0$, which is not a physical solution.

(h) Determine the prefactor of $u(r)$ so that it is normalized.

4. **Hydrogen atom:**

Solve for the radial wave function of the hydrogen atom for $n = 2$. There are two possible l values for $n = 2$: $l = 0$ and $l = 1$.

(a) The first step is to solve for the function $y(\rho)$,

$$y(\rho) = \rho^{l+1} \sum_{q=0}^{\infty} c_q \rho^q,$$

where $\rho = r/a_o$ and the c_q are determined by the recursion relation

$$c_q = \frac{2(\lambda(q+l) - 1)}{q(q+2l+1)} c_{q-1}$$

with $\lambda = 1/n$. For $n = 2$ and $l = 1$ show that $c_1 = 0$. For $n = 2$ and $l = 0$ show that $c_2 = 0$, and determine the ratio of c_1/c_0. This will allow you to express $y(\rho)$ in terms of the c_o in both cases.

(b) From $y(\rho)$ determine the function $u(r)$ by

$$u(r) = y(r/a_o) e^{-\lambda r/a_o}.$$

(c) Determine c_o by the normalization condition

$$\int_0^\infty (u(r))^2 = 1.$$

(d) Finally compute $R(r) = u(r)/r$ and compare your results to those in the book on page 797 (in the book's notation $k = n - l$).