Homework 2 Solution:

1. Example: $\psi(x) = Ae^{ikx} + A'e^{-ikx}$

$$\Rightarrow \quad \frac{\partial \psi}{\partial x}(x) = ikAe^{ikx} - ikA'e^{-ikx}$$

$$\psi^*(x) = A^*e^{-ikx} + A'^*e^{ikx}$$

$$\frac{\partial \psi^*}{\partial x}(x) = -ikA^*e^{-ikx} + ikA'^*e^{ikx}$$

$$\psi^*\frac{\partial \psi}{\partial x} = (A^*e^{-ikx} + A'^*e^{ikx})(ikAe^{ikx} - ikA'e^{-ikx})$$
$$= ik|A|^2 - ik|A'|^2 + ikA'^*Ae^{2ikx} - ikA^*A'e^{-2ikx}$$

$$\psi\frac{\partial \psi^*}{\partial x} = -ik|A|^2 + ik|A'|^2 + ikA'^*Ae^{2ikx} - ikA^*A'e^{-2ikx}$$

$$\Rightarrow \psi^*\frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x} = 2ik|A|^2 - 2ik|A'|^2$$

$$\Rightarrow j = \frac{\hbar k}{m}|A|^2 - \frac{\hbar k}{m}|A'|^2$$

Example: $\psi(x) = Be^{\rho x} + B'e^{-\rho x}$

$$\frac{\partial \psi}{\partial x} = \rho B e^{\rho x} - \rho B' e^{-\rho x}$$

$$\psi^* = B^* e^{\rho x} + B'^* e^{-\rho x}$$

$$\frac{\partial \psi^*}{\partial x} = \rho B^* e^{\rho x} - \rho B'^* e^{-\rho x}$$

$$\psi^* \frac{\partial \psi}{\partial x} = \rho |B|^2 e^{2\rho x} - \rho |B'|^2 e^{-2\rho x} + \rho BB'^* - \rho B' B^*$$

$$\psi \frac{\partial \psi^*}{\partial x} = \rho |B|^2 e^{2\rho x} - \rho |B'|^2 e^{-2\rho x} + \rho B^* B' - \rho B'^* B$$

$$\Rightarrow \psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} = 2\rho BB'^* - 2\rho B' B^*$$

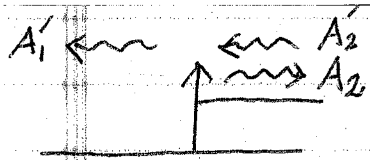
$$\Rightarrow j = \frac{\hbar}{mi} (\rho BB'^* - \rho B' B^*)$$

$$= \frac{\hbar}{mi} \rho \, 2i \, Im\{BB'^*\}$$

$$= \frac{\hbar \rho}{m} 2 \, Im\{BB'^*\}$$

If one of B or B' is zero, then j is zero.

2.



$$A_1 = 0 \rightarrow \quad A_1' = A_2 + A_2'$$

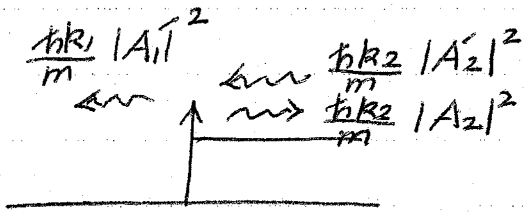
$$-\frac{k_1}{k_2} A_1' = A_2 - A_2'$$

$$\rightarrow \quad A_2 = \frac{1}{2}\left(1 - \frac{k_1}{k_2}\right) A_1'$$

$$A_2' = \frac{1}{2}\left(1 + \frac{k_1}{k_2}\right) A_1'$$

$$\frac{A_2}{A_2'} = \frac{1 - k_1/k_2}{1 + k_1/k_2} = \frac{k_2 - k_1}{k_2 + k_1}$$

$$\frac{A_1'}{A_2'} = \frac{2}{1 + k_1/k_2} = \frac{2k_2}{k_1 + k_2}$$



$$R = \frac{|A_2|^2}{|A_2'|^2} = \frac{(k_2 - k_1)^2}{(k_2 + k_1)^2} = 1 - \frac{4 k_1 k_2}{(k_1 + k_2)^2}$$

$$T = \frac{|A_1'|^2}{|A_2'|^2} \frac{k_1}{k_2} = \frac{k_1}{k_2}\left(\frac{2k_2}{k_1 + k_2}\right)^2 = \frac{4 k_1 k_2}{(k_1 + k_2)^2} \qquad \checkmark \; R + T = 1$$

Note: This solution makes heavy use of my notes on potential barriers. You should have started from scratch.

3. For $E < V_0$, the solution in region II is:

$$\varphi(x) = B_2 e^{\rho_2 x} + B_2' e^{-\rho_2 x}, \text{ where}$$

$$\rho_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}.$$

In the previous expressions take $k_2 \to i\rho_2$:

RIGHT MOVING WAVE

$$\frac{A_1'}{A_1} = \frac{i\,\frac{-\rho_2^2 - k_1^2}{2k_1 i\rho_2}\,\frac{\sinh(\rho_2 \ell)}{i}}{\cosh(\rho_2\ell) - i\,\frac{k_1^2-\rho_2^2}{2k_1 i\rho_2}\,\frac{\sinh(\rho_2\ell)}{i}}$$

$$= \frac{-i\,\frac{\rho_2^2+k_1^2}{2k_1\rho_2}\sinh(\rho_2\ell)}{\cosh(\rho_2\ell) - i\,\frac{k_1^2-\rho_2^2}{2k_1\rho_2}\sinh(\rho_2\ell)}$$

$$\frac{A_3}{A_1} = \frac{e^{-ik_1\ell}}{\cosh(\rho_2\ell) - i\,\frac{k_1^2-\rho_2^2}{2k_1\rho_2}\sinh(\rho_2\ell)}$$

$$\Rightarrow R = \frac{\left(\frac{\rho_2^2+k_1^2}{2k_1\rho_2}\right)^2\sinh^2(\rho_2\ell)}{\cosh^2(\rho_2\ell) + \left(\frac{k_1^2-\rho_2^2}{2k_1\rho_2}\right)^2\sinh^2(\rho_2\ell)}$$

$$T = \frac{1}{\cosh^2(\rho_2\ell) + \left(\frac{k_1^2-\rho_2^2}{2k_1\rho_2}\right)^2\sinh^2(\rho_2\ell)}$$

Since $\cosh^2(\rho_2\ell) - \sinh^2(\rho_2\ell) = 1$, the denominator can be written as:

$$1 + \sinh^2(\rho_2\ell) + \left(\frac{k_1^2-\rho_2^2}{2k_1\rho_2}\right)^2\sinh^2(\rho_2\ell) = 1 + \left(\frac{k_1^2+\rho_2^2}{2k_1\rho_2}\right)^2\sinh^2(\rho_2\ell)$$

and again $R+T=1$.

$$T = \frac{1}{1 + \left(\frac{k_1^2+\rho_2^2}{2k_1\rho_2}\right)^2 \sinh^2(\rho_2 l)} \quad , \; R = 1 - T$$

For a thick barrier, $\rho_2 l \gg 1$ and

$$\sinh^2(\rho_2 l) \approx \frac{1}{4} e^{2\rho_2 l} \gg 1$$

$$\rightarrow T \approx \frac{1}{\left(\frac{k_1^2+\rho_2^2}{2k_1\rho_2}\right)^2 \frac{1}{4} e^{2\rho_2 l}} \propto e^{-2\rho_2 l} \qquad \text{(tunneling)}$$