Homework 3

1. $\varphi(x) = \begin{cases} 0 & 0<x<\frac{1}{2} \\ 1 & \frac{1}{2}<x<1 \end{cases}$

$$\langle \psi_n | \varphi \rangle = \int_{1/2}^{1} dx \, \sqrt{2} \sin(\pi n x)$$

$$= -\sqrt{2} \frac{\cos(\pi n x)}{\pi n} \Big|_{1/2}^{1}$$

$$= \frac{\sqrt{2}}{\pi n} \left(\cos\left(\frac{\pi n}{2}\right) - \cos(\pi n) \right)$$

$\varphi(x) = x$

$$\langle \psi_n | \varphi \rangle = \int_0^1 dx \, \sqrt{2} \sin(\pi n x) \cdot x$$

$$= \sqrt{2} \left(-\frac{\cos(\pi n x) \cdot x}{\pi n} + \frac{\sin(\pi n x)}{(\pi n)^2} \right) \Big|_0^1$$

$$= -\frac{\sqrt{2}}{\pi n} \cos(\pi n)$$

$$\varphi(x) = \cos(2\pi x)$$

$$\langle \psi_n | \varphi \rangle = \int_0^1 dx\, \sqrt{2}\, \sin(\pi n x) \cos(2\pi x)$$

$$= \frac{\sqrt{2}}{2} \int_0^1 dx \left(\sin(\pi(n+2)x) + \sin(\pi(n-2)x) \right)$$

$$= \frac{\sqrt{2}}{2} \left(-\frac{\cos(\pi(n+2)x)}{\pi(n+2)} - \frac{\cos(\pi(n-2)x)}{\pi(n-2)} \right) \Bigg|_0^1$$

$$= \frac{\sqrt{2}}{2} \left(\frac{1-\cos(\pi(n+2))}{\pi(n+2)} + \frac{1-\cos(\pi(n-2))}{\pi(n-2)} \right)$$

$$= 0 \text{ for } n \text{ even}$$

$$= \frac{\sqrt{2}}{2} \left(\frac{2}{\pi(n+2)} + \frac{2}{\pi(n-2)} \right) \text{ for } n \text{ odd}$$

$$= \frac{\sqrt{2}}{\pi} \left(\frac{1}{n+2} + \frac{1}{n-2} \right) \text{ for } n \text{ odd}$$

$$\varphi(x) = \delta(x - \tfrac{1}{4})$$

$$\langle \psi_n | \varphi \rangle = \int_0^1 dx\, \sqrt{2}\, \sin(\pi n x)\, \delta(x - \tfrac{1}{4})$$

$$= \sqrt{2} \sin\left(\frac{\pi n}{4}\right)$$

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%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% For the second part of this assignment, I have used matlab              %
% to make the plots.  I include my code here; however, you should use     %
% whatever program you are comfortable with.  You are not encouraged      %
% (or discouraged) from using what I use.                                 %
%                                                                         %
% There are five plots following this page.  The first plot is just to    %
% check that I have enough data points to make the curves look smooth.    %
% (I use 500 points.)                                                     %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% L = 1
npts = 500;     % The actual number of points in one more than this.
nterms = 20;
x = (0:npts)/npts;
n = (1:nterms)';
angle = pi*n*x;
basis = sqrt(2)*sin(angle);

% smoothness check
figure(1)
plot(x(1:50),basis(20,1:50))
xlabel('x')
ylabel('\psi_{20}(x)')
title('Smoothness Check')

% step function
coefstep = sqrt(2)*(cos(pi*(n')/2) - cos(pi*(n'))).;/(pi*(n'));
step20 = coefstep*basis;
step10 = coefstep(:,1:10)*basis(1:10,:);
step5  = coefstep(:,1:5)*basis(1:5,:);
step   = (x > 0.5);

figure(2)
plot(x,step5,x,step10,x,step20,x,step)
legend('5 terms','10 terms','20 terms','exact',2)
xlabel('x')
title('\phi(x) = \theta(x-0.5)')

% linear function
coeflinef = -sqrt(2)*cos(pi*(n'))./(pi*(n'));
linef20 = coeflinef*basis;
linef10 = coeflinef(:,1:10)*basis(1:10,:);
linef5  = coeflinef(:,1:5)*basis(1:5,:);
linef   = x;

figure(3)
plot(x,linef5,x,linef10,x,linef20,x,linef)
legend('5 terms','10 terms','20 terms','exact',2)
xlabel('x')
title('\phi(x) = x')

% cosine function
nodd = 1:2:nterms;
coefcosf = (sqrt(2)/pi)*(1./(nodd + 2) + 1./(nodd - 2));
cosf20 = coefcosf*basis(1:2:20,:);
cosf10 = coefcosf(:,1:5)*basis(1:2:10,:);
cosf5  = coefcosf(:,1:3)*basis(1:2:5,:);
cosf   = cos(2*pi*x);

figure(4)
plot(x,cosf5,x,cosf10,x,cosf20,x,cosf)
legend('5 terms','10 terms','20 terms','exact',0)
xlabel('x')
title('\phi(x) = cos(x)')

% delta function
coefdelta = sqrt(2)*sin(pi*(n')/4);
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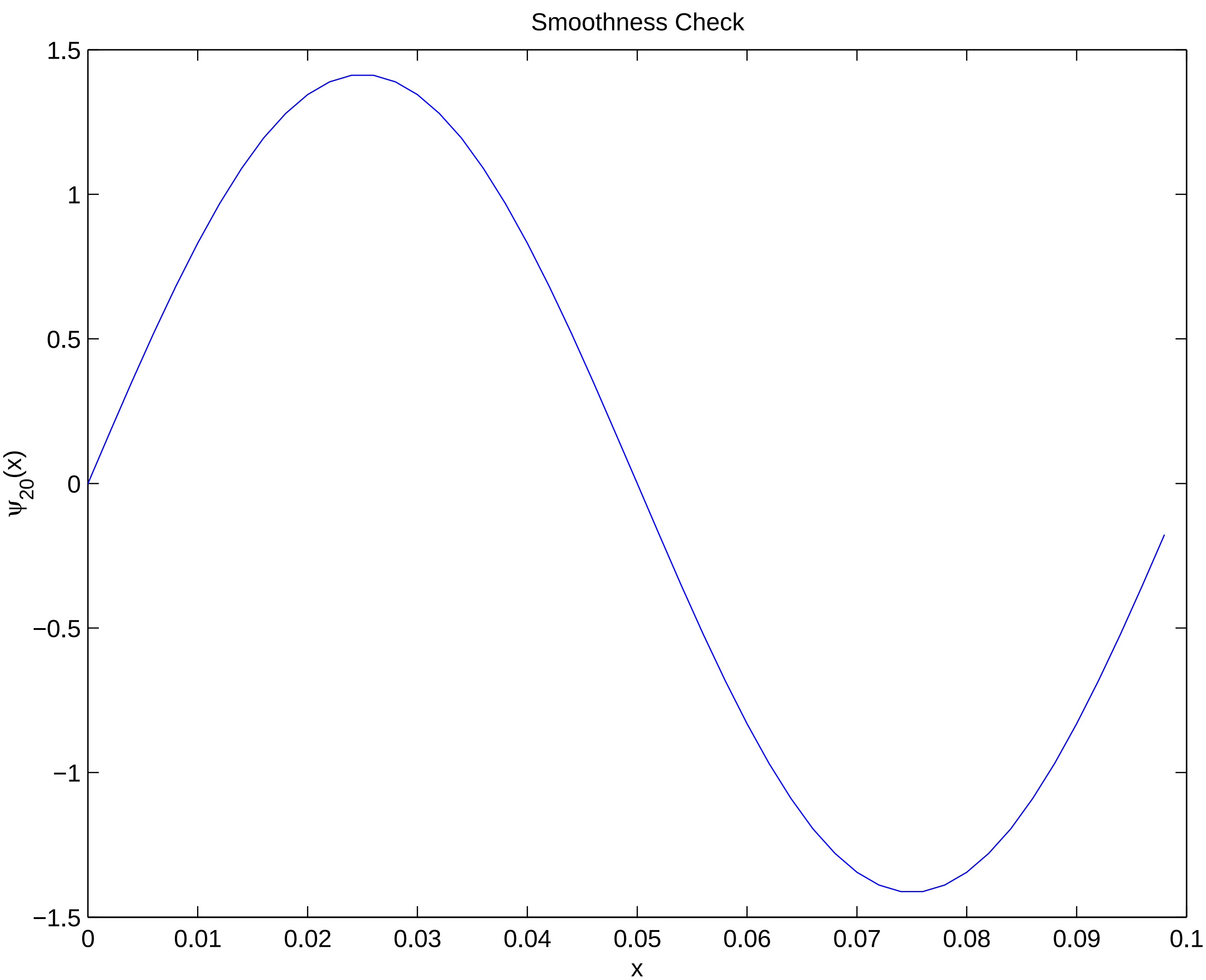
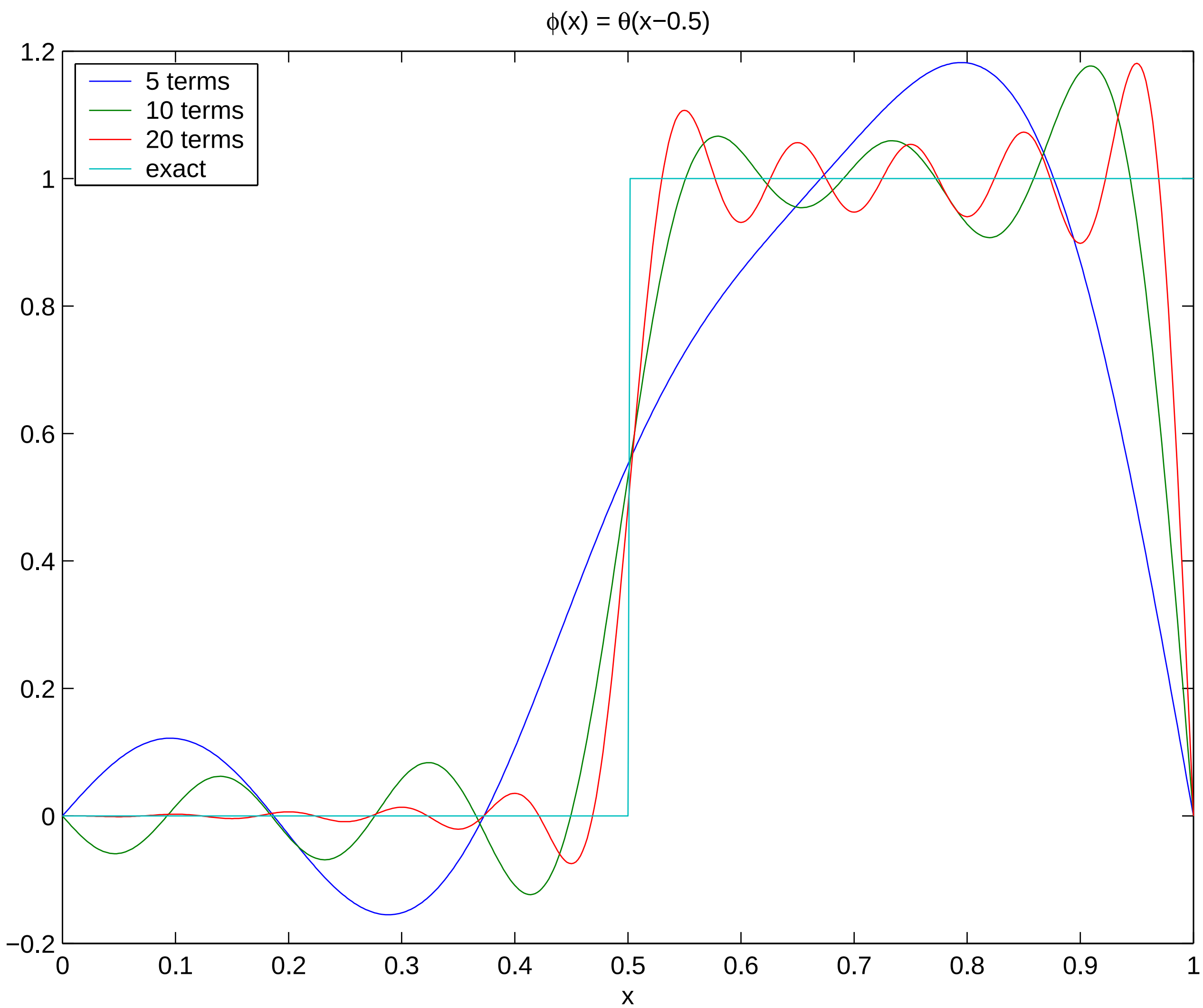
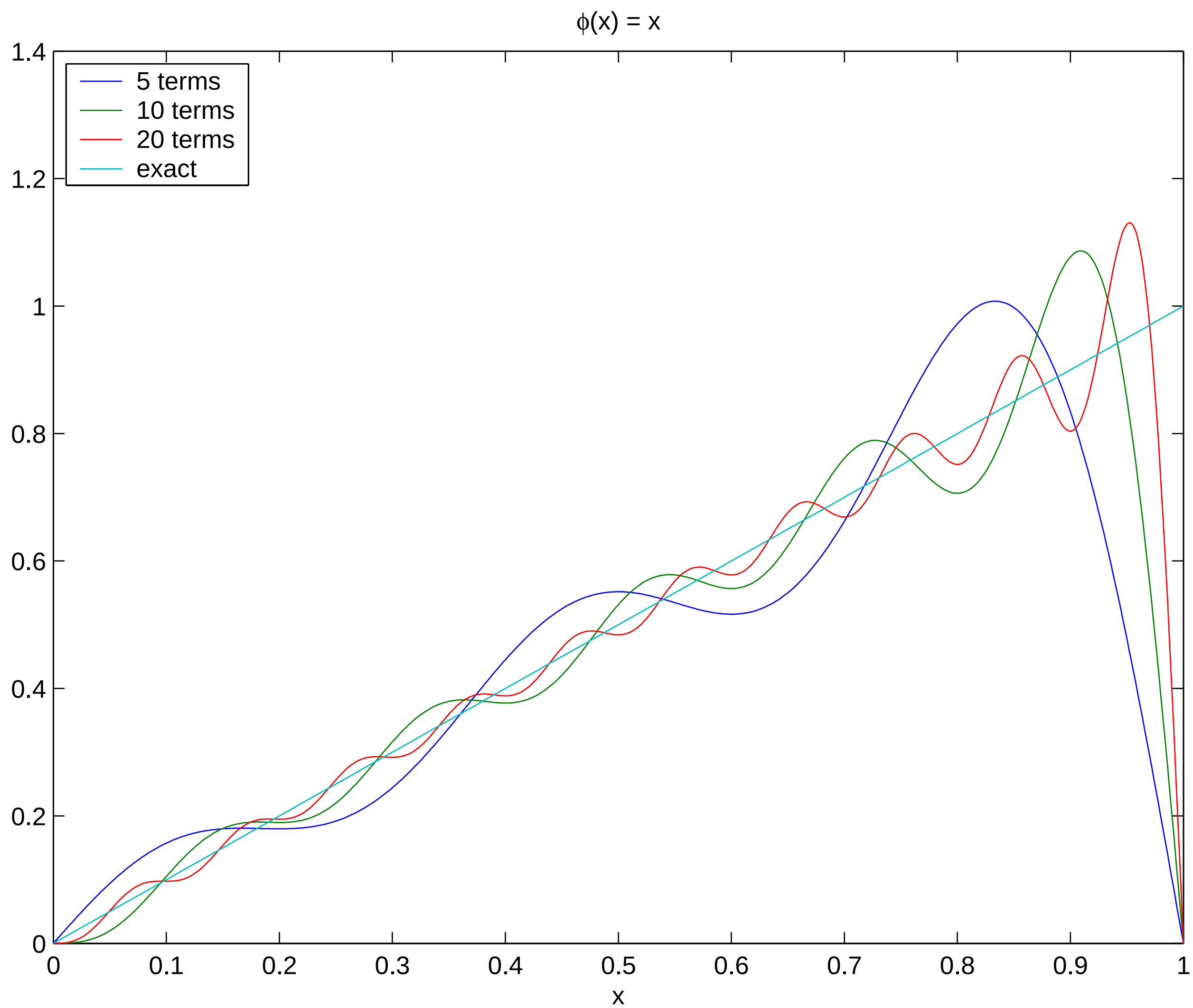
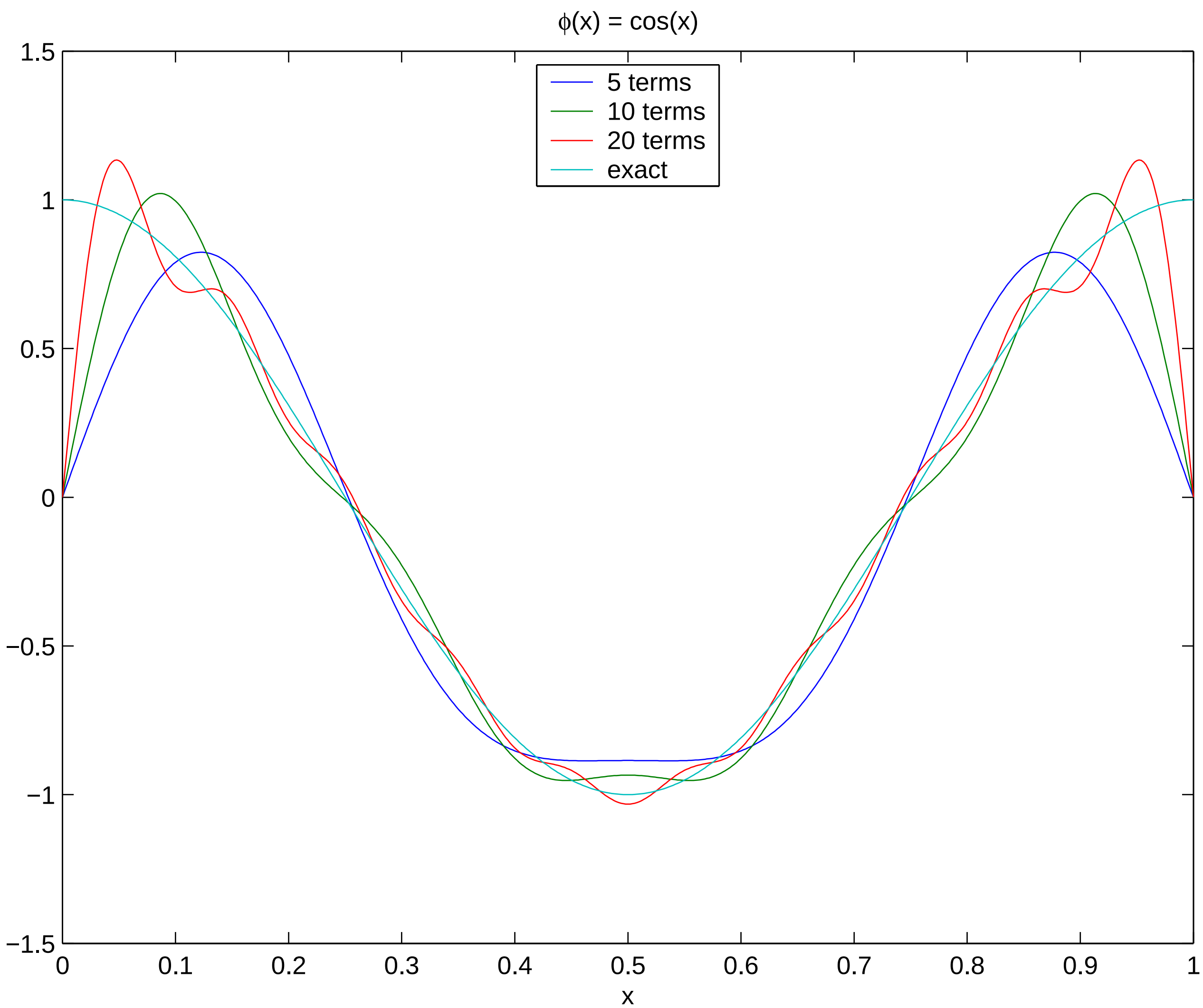
```
delta20 = coefdelta*basis;
delta10 = coefdelta(:,1:10)*basis(1:10,:);
delta5  = coefdelta(:,1:5)*basis(1:5,:);

figure(5)
plot(x,delta5,x,delta10,x,delta20,[0.25 0.25],[0 25])
legend('5 terms','10 terms','20 terms','exact',1)
xlabel('x')
title('\phi(x) = \delta(x-0.25)')
```


Smoothness Check
1.5
1
0.5
0
−0.5
−1
−1.5
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
x
$\psi_{20}(x)$

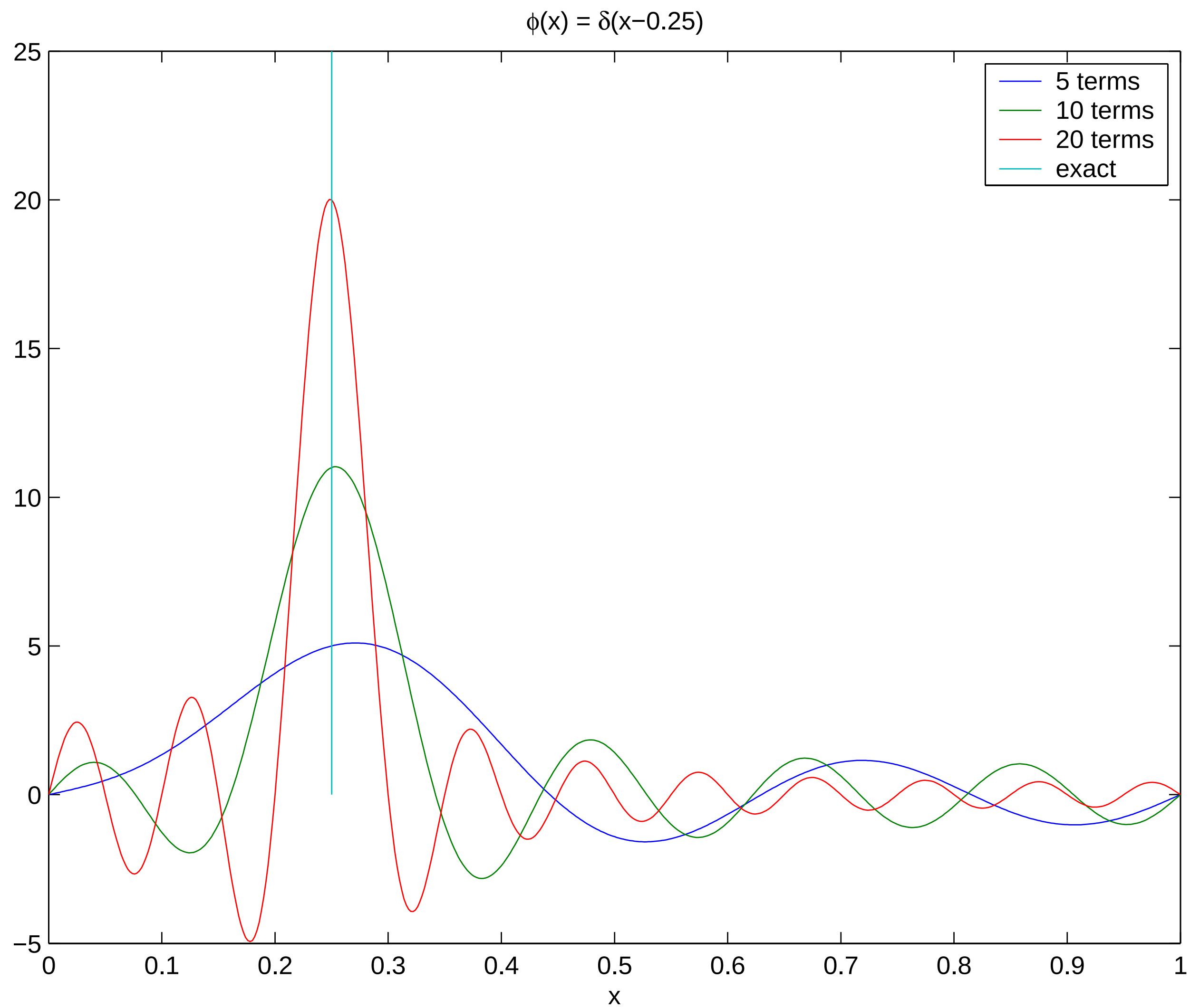

φ(x) = θ(x−0.5)
5 terms
10 terms
20 terms
exact
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x



φ(x) = x
5 terms
10 terms
20 terms
exact
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x



φ(x) = cos(x)
5 terms
10 terms
20 terms
exact
1.5
1
0.5
0
−0.5
−1
−1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x



φ(x) = δ(x−0.25)
5 terms
10 terms
20 terms
exact
25
20
15
10
5
0
−5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x