Homework 4:

1. In the first homework assignment you computed the eigenvalues & eigenvectors of the 2×2 matrices, σ_x, σ_y, and σ_z. Normalize the eigenvectors:

$$\underbrace{\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}}_{\sigma_x,\ \sigma_y \text{ or } \sigma_z} \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{\text{eigenvector}} = \underbrace{\lambda}_{\text{eigenvalue}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

The normalized eigenvector is

$$|\psi\rangle = \frac{1}{\sqrt{|c_1|^2 + |c_2|^2}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

and the "bra" for this ket is

$$\langle\psi| = \frac{1}{\sqrt{|c_1|^2 + |c_2|^2}} \begin{pmatrix} c_1^* & c_2^* \end{pmatrix}. \quad \rightarrow \langle\psi|\psi\rangle = 1$$

Compute all $2 * 3 = 6$ normalized eigenvectors.

2. (a) For the eigenvectors of σ_z verify the completeness relation: $\sum_{n=1}^{2} |\psi_n\rangle\langle\psi_n| = \mathbb{1}$. Note a 2×1 matrix (like $|\psi_n\rangle$) times a 1×2 matrix (like $\langle\psi_n|$) is a 2×2 matrix.
(b) Do the same for the eigenvectors of σ_x.
(c) Do the same for the eigenvectors of σ_y.

3. (a) For the eigenvectors of σ_z, show that σ_z can be represented as

$$\sigma_z = \sum_{n=1}^{2} \lambda_n |\psi_n\rangle\langle\psi_n| .$$

(b) Similarly, for the eigenvectors of σ_x, show that σ_x can be represented as

$$\sigma_x = \sum_{n=1}^{2} \lambda_n |\psi_n\rangle\langle\psi_n|$$

(c) Do the same for σ_y.

4. (a) Show that the eigenvectors of σ_z are orthogonal: $\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_1\rangle = 0$.
(b) Show that the eigenvectors of σ_x are orthogonal.
(c) Show that the eigenvectors of σ_y are orthogonal.

5. (a) From the results of the first homework assignment show that

$$[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x = 2i\sigma_z$$
$$[\sigma_y, \sigma_z] = \sigma_y\sigma_z - \sigma_z\sigma_y = 2i\sigma_x$$
$$[\sigma_z, \sigma_x] = \sigma_z\sigma_x - \sigma_x\sigma_z = 2i\sigma_y .$$

(b) The exponent of an operator or matrix, A, is defined as

$$\exp(A) = \mathbb{1} + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

Using $(\sigma_x)^2 = \mathbb{1}$, show that

$$\exp(i\theta\sigma_x) = \cos(\theta)\mathbb{1} + i\sin(\theta)\sigma_x .$$

(c) What are the eigenvalues and eigenvectors of $\exp(i\theta\sigma_x)$? Do they make sense?