Homework 4 solution:

1. For σ_x, $\lambda = +1$, $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $\langle\psi| = \frac{1}{\sqrt{2}}(1\ \ 1)$.

For σ_x, $\lambda = -1$, $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, $\langle\psi| = \frac{1}{\sqrt{2}}(1\ \ -1)$.

For σ_y, $\lambda = +1$, $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\+i\end{pmatrix}$, $\langle\psi| = \frac{1}{\sqrt{2}}(1\ \ -i)$.

For σ_y, $\lambda = -1$, $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$, $\langle\psi| = \frac{1}{\sqrt{2}}(1\ \ +i)$.

For σ_z, $\lambda = +1$, $|\psi\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $\langle\psi| = (1\ \ 0)$.

For σ_z, $\lambda = -1$, $|\psi\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, $\langle\psi| = (0\ \ 1)$.

2 (a) For σ_z, $\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}(1\ \ 0) + \frac{1}{2}\begin{pmatrix}0\\-1\end{pmatrix}(0\ \ -1)$

$$= \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \mathbb{1}$$

(b) For σ_x, $\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}(1\ \ 1) + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}(1\ \ -1)$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix} = \mathbb{1}$$

(c) For σ_y, $\frac{1}{2}\begin{pmatrix}1\\i\end{pmatrix}(1\ \ -i) + \frac{1}{2}\begin{pmatrix}1\\-i\end{pmatrix}(1\ \ i)$

$$= \frac{1}{2}\begin{pmatrix}1 & -i\\+i & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & +i\\-i & 1\end{pmatrix} = \mathbb{1}$$

3. (a) $(1)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + (-1)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \sigma_z$

(b) $(1)\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + (-1)\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \sigma_x$

(c) $(1)\frac{1}{2}\begin{pmatrix}1 & -i\\ +i & 1\end{pmatrix} + (-1)\frac{1}{2}\begin{pmatrix}1 & +i\\ -i & 1\end{pmatrix} = \begin{pmatrix}0 & -i\\ +i & 0\end{pmatrix} = \sigma_y$

4. (a) $(1\ 0)\begin{pmatrix}0\\ 1\end{pmatrix} = 0$

(b) $\frac{1}{\sqrt{2}}(1\ 1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = 0$

(c) $\frac{1}{\sqrt{2}}(1\ -i)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} = 0$

5. (a) We have from homework assignment 1,

$$\sigma_x\sigma_y = i\sigma_z = -\sigma_y\sigma_x \longrightarrow [\sigma_x, \sigma_y] = i\sigma_z$$
$$\sigma_y\sigma_z = i\sigma_x = -\sigma_z\sigma_y \longrightarrow [\sigma_y, \sigma_z] = i\sigma_x$$
$$\sigma_z\sigma_x = i\sigma_y = -\sigma_x\sigma_z \longrightarrow [\sigma_z, \sigma_x] = i\sigma_y .$$

(b) $\exp(i\theta\sigma_x) = \mathbb{1} + i\theta\sigma_x - \frac{\theta^2}{2!}(\sigma_x)^2 - \frac{i\theta^3}{3!}(\sigma_x)^3 + \frac{\theta^4}{4!}(\sigma_x)^4 + \cdots$

$$= \mathbb{1}\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\cdots\right) + i\sigma_x\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$

$$= \cos\theta\,\mathbb{1} + i\sigma_x\sin\theta$$

(c) $\cos\theta \mathbb{1} + i\sin\theta\,\sigma_x = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$.

Find the eigenvalues:

$$0 = \begin{vmatrix} \cos\theta - \lambda & i\sin\theta \\ i\sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta$$

$$= 1 - 2\cos\theta\lambda + \lambda^2$$

$$\Rightarrow \lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{-\sin^2\theta}$$

$$\Rightarrow \lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}.$$

Find the eigenvectors:

$\lambda = e^{i\theta}$: $\begin{pmatrix} \cos\theta - e^{i\theta} & i\sin\theta \\ i\sin\theta & \cos\theta - e^{i\theta} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$

$\rightarrow \begin{pmatrix} -i\sin\theta & i\sin\theta \\ i\sin\theta & -i\sin\theta \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \rightarrow c_1 = c_2$

$\lambda = e^{-i\theta}$: $\begin{pmatrix} \cos\theta - e^{-i\theta} & i\sin\theta \\ i\sin\theta & \cos\theta - e^{-i\theta} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$

$\begin{pmatrix} i\sin\theta & i\sin\theta \\ i\sin\theta & i\sin\theta \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \rightarrow c_1 = -c_2$

Thus, the $\lambda = 1$ eigenvector of σ_x is also an eigenvector of $\exp(i\theta\sigma_x)$ with eigenvalue $e^{i\theta(+1)}$, and the $\lambda = -1$ eigenvector of σ_x is an eigenvector of $\exp(i\theta\sigma_x)$ with eigenvalue $e^{i\theta(-1)}$.