Homework 5

1. The solutions to the time independent Schrodinger equation for an infinite well between $x = 0$ and $x = L$ are

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right).$$

   (a) Show that the energies of these states are

$$E_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{L}\right)^2.$$

   (b) Let $\omega_n = E_n/\hbar$. Show that the following wave function is a solution to the time dependent Schrodinger equation.

$$\psi(x,t) = \sum_{n=1}^{\infty} c_n e^{-i\omega_n t}\phi_n(x)$$

   Here, the c_n's are constants. The above equation is a solution for any choice of c_n, and it is the general form of the solution to this Schrodinger equation.

   (c) Suppose that you are give the wave function at $t = 0$, $\psi(x,0)$. Determine the c_n from $\psi(x,0)$.

2. Using the results of part 1, plot the following for $L = 1$.

   (a) Suppose that the wave function at $t = 0$ is $\sqrt{2}$ for $0 < x < 0.5$ and zero for $0.5 < x < 1$. Plot the probability density, $|\psi(x,t)|^2$, for $t = 0.01\hbar/E_1$, $t = 0.1\hbar/E_1$, $t = \hbar/E_1$, and $t = 10\hbar/E_1$ by keeping the first 20 terms in the series, i.e., $n = 1, 2, \ldots, 20$.

   (b) Consider the case when $c_1 = 1/\sqrt{2}$ and $c_2 = 1/\sqrt{2}$. Determine analytically the probability density as a function of time, $|\psi(x,t)|^2$. What is the period in time, τ, of $|\psi(x,t)|^2$? Plot $|\psi(x,t)|^2$ for $t = 0$, $\tau/4$, $\tau/2$, and $3\tau/4$.

3. Determine the following expectation values for the wave functions $\phi_n(x)$.

$$\begin{aligned}
\langle\phi_n|x|\phi_n\rangle &= \int_0^L dx\phi_n^*(x)x\phi_n(x) \\
\langle\phi_n|x^2|\phi_n\rangle &= \int_0^L dx\phi_n^*(x)x^2\phi_n(x) \\
\langle\phi_n|p|\phi_n\rangle &= \int_0^L dx\phi_n^*(x)\left(\frac{\hbar}{i}\frac{d}{dx}\right)\phi_n(x) \\
\langle\phi_n|p^2|\phi_n\rangle &= \int_0^L dx\phi_n^*(x)\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2\phi_n(x)
\end{aligned}$$

   Compute the uncertainty in x and p:

$$\begin{aligned}
(\Delta x)^2 &= \langle\phi_n|x^2|\phi_n\rangle - \langle\phi_n|x|\phi_n\rangle^2 \\
(\Delta p)^2 &= \langle\phi_n|p^2|\phi_n\rangle - \langle\phi_n|p|\phi_n\rangle^2
\end{aligned}$$

   Are these results consistent with the uncertainty principle?