Homework 5:

1. a) $H\phi_n = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi_n(x) = -\frac{\hbar^2}{2m} \times -\left(\frac{\pi n}{L}\right)^2 \phi_n$

$$= \frac{\hbar^2}{2m}\left(\frac{\pi n}{L}\right)^2 \phi_n \rightarrow E_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{L}\right)^2$$

b) $i\hbar \frac{\partial \psi}{\partial t} = \sum_{n=1}^{\infty} \hbar\omega_n c_n e^{-i\omega_n t}\phi_n(x)$

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = \sum_{n=1}^{\infty} c_n e^{-i\omega_n t} E_n \phi_n(x)$$

Since $E_n = \hbar\omega_n$, $i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}$

c) $\psi(x, t=0) = \sum_{n=1}^{\infty} c_n \phi_n(x)$

$$\rightarrow \int_0^L dx\, \phi_m^*(x)\psi(x,0) = \sum_{n=1}^{\infty} c_n \int_0^L dx\, \phi_m^*(x)\phi_n(x) = c_m,$$

i.e. $c_m = \langle \phi_m | \psi(t=0)\rangle = \int_0^L dx\, \phi_m^*(x)\psi(x,0)$

2. a) $c_n = \int_0^{1/2} dx\, \sqrt{2}\sin(\pi n x)\sqrt{2} = \frac{2}{\pi n} - \cos(\pi n x)\Big|_0^{1/2}$

$$= \frac{2}{\pi n}\left(1 - \cos\left(\frac{\pi n}{2}\right)\right)$$

$$\Rightarrow \psi(x,t) = \sum_{n=1}^{\infty} \frac{2}{\pi n}\left(1 - \cos\left(\frac{\pi n}{2}\right)\right) e^{-i\frac{E_n}{\hbar}t}\phi_n(x)$$

$t = 0.01\frac{\hbar}{E_1} \rightarrow \frac{E_n}{\hbar}t = 0.01\frac{E_n}{E_1} = 0.01 n^2,$

and similarly for $t = 0.1\frac{\hbar}{E_1}$, $t = \frac{\hbar}{E_1}$, $t = 10\frac{\hbar}{E_1}$.

Plots of $|\psi(x,t)|^2$ are shown on the following pages.

b) $\psi(x,t) = \frac{1}{\sqrt{2}} e^{-i\omega_1 t}\phi_1(x) + \frac{1}{\sqrt{2}} e^{-i\omega_2 t}\phi_2(x)$

$$|\psi(x,t)|^2 = \frac{1}{2}|\phi_1(x)|^2 + \frac{1}{2}|\phi_2(x)|^2$$

$$+\frac{1}{2}\phi_1(x)\,\phi_2(x)\left(e^{-i(\omega_1-\omega_2)t} + e^{-i(\omega_2-\omega_1)t}\right)$$

$\phi$'s are real.

$$= \frac{1}{2}\{\phi_1(x) + \phi_2(x) + 2\phi_1(x)\phi_2(x)\cos((\omega_2-\omega_1)t)\}$$

This has period $\tau = \frac{2\pi}{\omega_2-\omega_1}$.

For $t=0$, $|\psi(x,t)|^2 = \frac{1}{2}(\phi_1(x)+\phi_2(x))^2$.

For $t=\frac{\tau}{4}$, $|\psi(x,t)|^2 = \frac{1}{2}(\phi_1^2(x)+\phi_2^2(x))$.

For $t=\frac{\tau}{2}$, $|\psi(x,t)|^2 = \frac{1}{2}(\phi_1(x)-\phi_2(x))^2$.

For $t=\frac{3\tau}{4}$, $|\psi(x,t)|^2 = \frac{1}{2}(\phi_1^2(x)+\phi_2^2(x))$.

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% Matlab file to produce plots needed for problem 2 in homework 5.

npts = 500;     % The actual number of points in one more than this.
nterms = 20;
x = (0:npts)/npts;
n = (1:nterms)';
ratioenergies = (n.*n)';
angle = pi*n*x;
basis = sqrt(2)*sin(angle);

% step function
coefstep = 2*(1 - cos(pi*(n')/2))./(pi*(n'));
coefstep2 = coefstep.*exp(-i*ratioenergies*0.01);
coefstep3 = coefstep.*exp(-i*ratioenergies*0.1);
coefstep4 = coefstep.*exp(-i*ratioenergies*1.0);
coefstep5 = coefstep.*exp(-i*ratioenergies*10.0);
psi1 = coefstep*basis;
psi2 = coefstep2*basis;
psi3 = coefstep3*basis;
psi4 = coefstep4*basis;
psi5 = coefstep5*basis;
step   = sqrt(2)*(x < 0.5);

figure(1)
plot(x,abs(psi1).^2,x,abs(psi2).^2,x,abs(psi3).^2,x,abs(psi4).^2,x,abs(psi5).^2)
legend('t = 0','t = 0.01 hbar/E_1','t = 0.1 hbar/E_1','t = 1.0 hbar/E_1','t = 10 hbar/E
_1')
xlabel('x')
ylabel('|\psi(x,t)|^2')
title('Problem 2.a.')

figure(2)
dens1 = 0.5*(basis(1,:) + basis(2,:)).^2;
dens2 = 0.5*(basis(1,:).^2 + basis(2,:).^2);
dens3 = 0.5*(basis(1,:) - basis(2,:)).^2;
plot(x,dens1,x,dens2,x,dens3)
legend('t = 0','t = 0.25\tau, 0.75\tau','t = 0.5\tau')
xlabel('x')
ylabel('|\psi(x,t)|^2')
title('Problem 2.b.')
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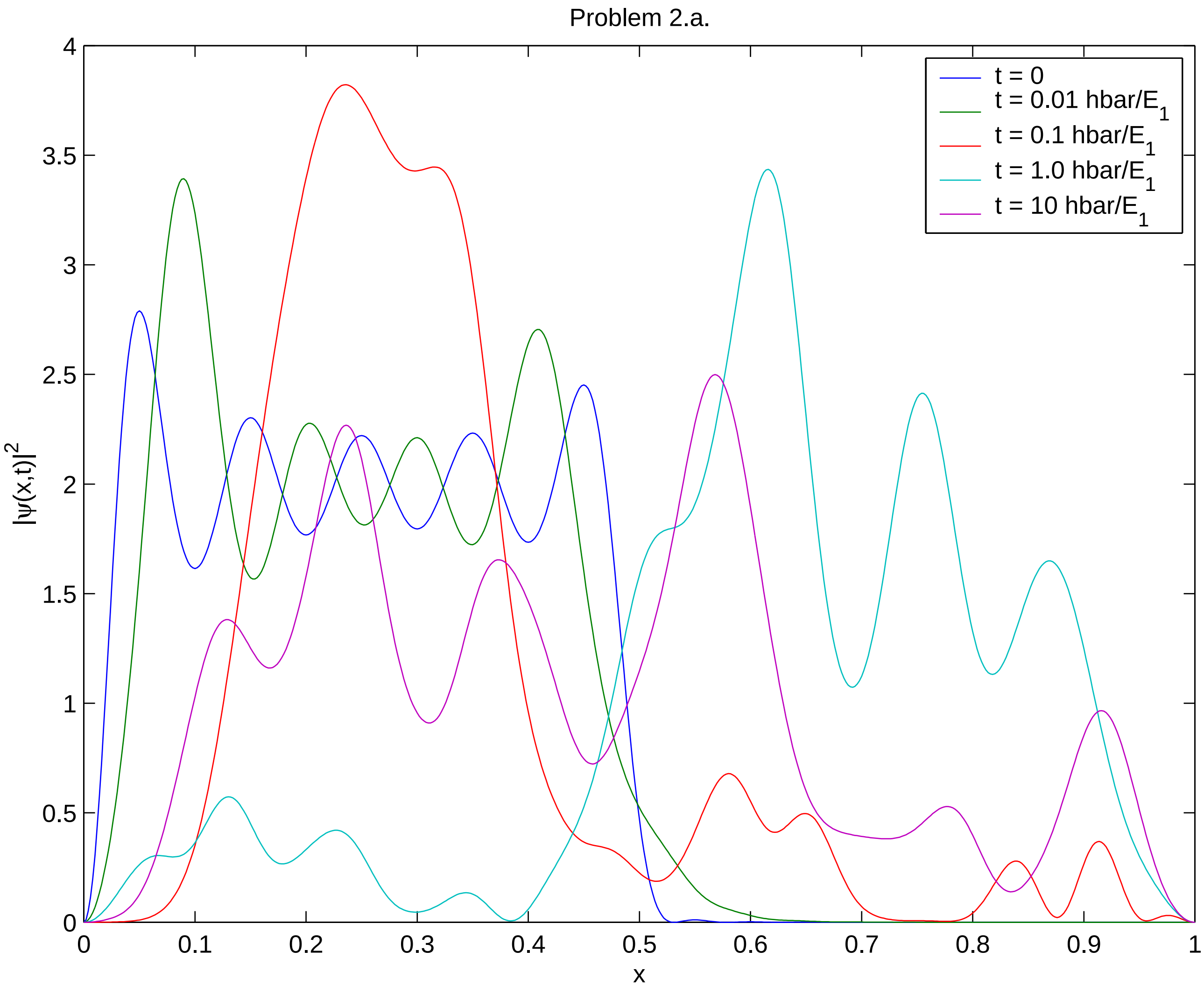
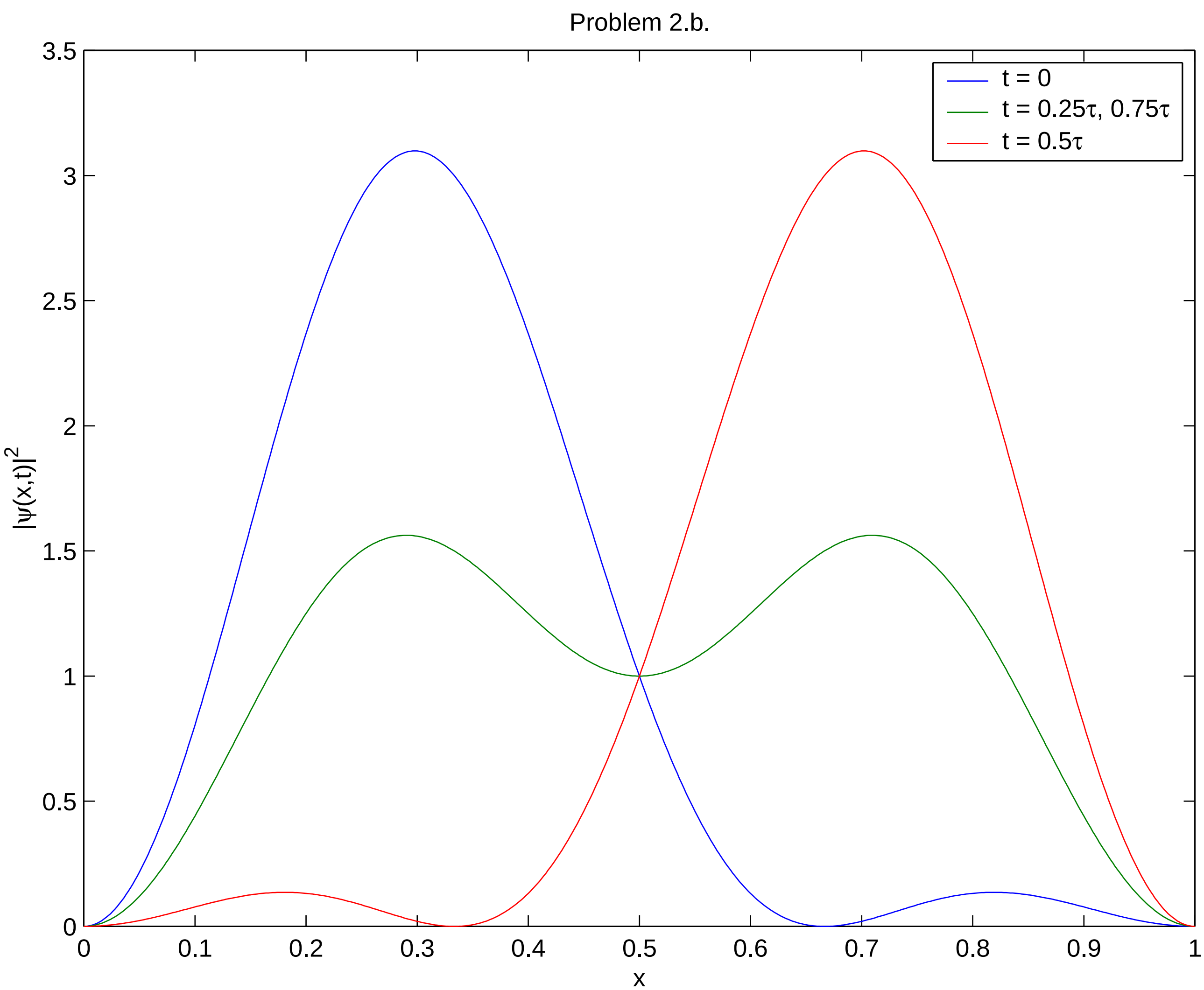

Problem 2.a.
t = 0
t = 0.01 hbar/E1
t = 0.1 hbar/E1
t = 1.0 hbar/E1
t = 10 hbar/E1
|ψ(x,t)|2
x
0
0.5
1
1.5
2
2.5
3
3.5
4
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9



Problem 2.b.
t = 0
t = 0.25τ, 0.75τ
t = 0.5τ
|ψ(x,t)|2
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
1.5
2
2.5
3
3.5

$$\langle \phi_n | x | \phi_n \rangle = \int_0^1 dx\, 2x \sin^2(\pi n x) \qquad \cdots (i)$$

$$\langle \phi_n | x^2 | \phi_n \rangle = \int_0^1 dx\, 2x^2 \sin^2(\pi n x) \qquad \cdots (ii)$$

$$\langle \phi_n | p | \phi_n \rangle = \int_0^1 dx\, 2\left(\frac{\hbar}{i}\pi n\right) \sin(\pi n x) \cos(\pi n x) \qquad \cdots (iii)$$

$$\langle \phi_n | p^2 | \phi_n \rangle = \int_0^1 dx\, 2 - \left(\frac{\hbar}{i}\pi n\right)^2 \sin^2(\pi n x) \qquad \cdots (iv)$$

$$= (\hbar \pi n)^2 \langle \phi_n | \phi_n \rangle = (\hbar \pi n)^2$$

$$\sin(2\pi n x) = 2\sin(\pi n x)\cos(\pi n x)$$

$$\cos(2\pi n x) = 1 - 2\sin^2(\pi n x)$$

$$\rightarrow 1 - \cos(2\pi n x) = 2\sin^2(\pi n x)$$

$$\langle \phi_n | x | \phi_n \rangle = \int_0^1 dx\, x\,(1 - \cos(2\pi n x))$$

$$= \frac{1}{2} - \int_0^1 dx\, x \cos(2\pi n x)$$

$$\langle \phi_n | x^2 | \phi_n \rangle = \int_0^1 dx\, x^2 (1 - \cos(2\pi n x))$$

$$= \frac{1}{3} - \int_0^1 dx\, x^2 \cos(2\pi n x)$$

$$\langle \phi_n | p | \phi_n \rangle = \int_0^1 dx \left(\frac{\hbar \pi n}{i}\right) \cos(2\pi n x)$$

$$= \left(\frac{\hbar \pi n}{i}\right) - \frac{\sin(2\pi n x)}{2\pi n}\Bigg|_0^1 = 0$$

$$\frac{d}{dx}\, x\sin(ax) = ax\cos(ax) + \sin(ax)$$

$$\frac{d}{dx}\, \frac{\cos(ax)}{a} = \qquad -\sin(ax)$$

$$\frac{d}{dx}\left[\frac{x}{a}\sin(ax) + \frac{1}{a^2}\cos(ax)\right] = x\cos(ax)$$

$$\Rightarrow \int_0^1 dx\; x\cos(2\pi n x) = \frac{x}{2\pi n}\sin(2\pi n x) + \frac{1}{(2\pi n)^2}\cos(2\pi n x)\Big|_0^1 = 0$$

$$\Rightarrow \langle \phi_n | x | \phi_n \rangle = \frac{1}{2}$$

$$\frac{d}{dx}\, x^2\sin(ax) = ax^2\cos(ax) + 2x\sin(ax)$$

$$\frac{d}{dx}\, \frac{2x\cos(ax)}{a} = \qquad -2x\sin(ax) + \frac{2}{a}\cos(ax)$$

$$\frac{d}{dx}\, -\frac{2}{a^2}\sin(ax) = \qquad -\frac{2}{a}\cos(ax)$$

$$\Rightarrow \frac{d}{dx}\left[\frac{x^2}{a}\sin(ax) + \frac{2x}{a^2}\cos(ax) - \frac{2}{a^3}\sin(ax)\right] = x^2\cos(ax)$$

$$\Rightarrow \int_0^1 dx\; x^2\cos(2\pi n x) = \frac{2x}{(2\pi n)^2}\cos(2\pi n x)\Big|_0^1 = \frac{2}{(2\pi n)^2}$$

$$\Rightarrow \langle \phi_n | x^2 | \phi_n \rangle = \frac{1}{3} - \frac{2}{(2\pi n)^2}$$

$$(\Delta x)^2 = \frac{1}{3} - \frac{2}{(2\pi n)^2} - \frac{1}{4} = \frac{1}{12} - \frac{1}{2\pi^2 n^2}$$

$$(\Delta p)^2 = (\hbar \pi n)^2$$

$$(\Delta x)(\Delta p) = \sqrt{\frac{1}{12} - \frac{1}{2\pi^2 n^2}}\ \hbar \pi n$$

$$= \frac{\hbar}{2} \cdot \pi n \cdot \sqrt{\frac{1}{3} - \frac{2}{\pi^2 n^2}}$$

This is smallest for $n = 1$:

$$(\Delta x)(\Delta p) = \frac{\hbar}{2} \cdot \pi \sqrt{\frac{1}{3} - \frac{2}{\pi^2}} \approx 1.1357 \frac{\hbar}{2} > \frac{\hbar}{2}. \checkmark$$