Homework 6

This homework will illustrate the time evolution of states and operators discussed in Chapter 3 using the example of spin 1/2 particles discussed in Chapter 4. In addition, the specific case of precessing magnetic moments will be used in treating magnetic resonance.

1. A spin 1/2 particle is placed in an external magnetic field in the z-direction. The hamiltonian for this system is $H = -\mathbf{M} \cdot \mathbf{B} = -\gamma \mathbf{S} \cdot \mathbf{B} = -\gamma B S_z = -\frac{1}{2}\hbar\gamma B \sigma_z$.

   (a) Show that the states $|+\rangle$ and $|-\rangle$ are eigenstates of the hamiltonian and find their energies.

   (b) Suppose that at $t = 0$ the state of the system is $|\psi(t)\rangle = c_1|+\rangle + c_2|-\rangle$, where c_1 and c_2 are *complex* numbers with $|c_1|^2 + |c_2|^2 = 1$. Solve for $|\psi(t)\rangle$.

   (c) Compute the expectation values of S_x, S_y, and S_z: $\langle S_x\rangle = \langle\psi(t)|S_x|\psi(t)\rangle$, $\langle S_y\rangle = \langle\psi(t)|S_y|\psi(t)\rangle$, $\langle S_z\rangle = \langle\psi(t)|S_z|\psi(t)\rangle$.

   (d) Show that the expectation value of $\mathbf{S} = S_x\mathbf{x} + S_y\mathbf{y} + S_z\mathbf{z}$ satisfies the classical precession equation:
   $$\frac{d\langle\mathbf{S}\rangle}{dt} = \gamma\langle\mathbf{S}\rangle \times \mathbf{B}.$$

2. Now we do the same calculation in the Heisenberg picture instead of the Schrodinger picture of part 1.

   (a) Show that the Heisenberg operators $S_{x,H}(t) = e^{iHt/\hbar} S_x e^{-iHt/\hbar}$, $S_{y,H}(t) = e^{iHt/\hbar} S_y e^{-iHt/\hbar}$, and $S_{z,H}(t) = e^{iHt/\hbar} S_z e^{-iHt/\hbar}$ satisfy the equation of time evolution in the Heisenberg picture.

   (b) Compute $\exp(iHt/\hbar)$ and $\exp(-iHt/\hbar)$. (Hint: You may wish to look at homework assignment 4.)

   (c) Evaluate $S_{x,H}(t)$, $S_{y,H}(t)$, and $S_{z,H}(t)$.

   (d) Evaluate $\langle\psi(t=0)|S_{\alpha,H}(t)|\psi(t=0)\rangle$ for $\alpha = x$, y, z, and show that you get the same result as part 1 for $\langle\psi(t)|S_\alpha|\psi(t)\rangle$.

   (e) Show that the Heisenberg operator, $\mathbf{S}_H(t)$, satisfies the precession equation,
   $$\begin{aligned} \frac{dS_{x,H}(t)}{dt} &= \gamma B S_{y,H}(t) \\ \frac{dS_{y,H}(t)}{dt} &= -\gamma B S_{x,H}(t) \\ \frac{dS_{z,H}(t)}{dt} &= 0. \end{aligned}$$