Homework 6 Solution:

1. a) $H = -\gamma \frac{\hbar}{2} B \sigma_z$. Let $\omega_0 = -\gamma B$, Then

$$H = \frac{\hbar\omega_0}{2}\sigma_z$$

$$H|+\rangle = H\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar\omega_0}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar\omega_0}{2}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar\omega_0}{2}|+\rangle$$

$$H|-\rangle = \frac{\hbar\omega_0}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar\omega_0}{2}\begin{pmatrix}0\\-1\end{pmatrix} = -\frac{\hbar\omega_0}{2}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar\omega_0}{2}|-\rangle$$

$$\Rightarrow E_+ = \frac{\hbar\omega_0}{2} \text{ and } E_- = -\frac{\hbar\omega_0}{2}.$$

b) If $|\psi(t=0)\rangle = c_1|+\rangle + c_2|-\rangle$, then

$$|\psi(t)\rangle = c_1 e^{-i\frac{E_+}{\hbar}t}|+\rangle + c_2 e^{-i\frac{E_-}{\hbar}t}|-\rangle$$

$$= c_1 e^{-i\frac{\omega_0}{2}t}|+\rangle + c_2 e^{i\frac{\omega_0}{2}t}|-\rangle.$$

c) $$\langle\psi(t)|S_x|\psi(t)\rangle = \left(c_1^* e^{+i\frac{\omega_0}{2}t} \;\; c_2^* e^{-i\frac{\omega_0}{2}t}\right)\frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}c_1 e^{-i\frac{\omega_0}{2}t}\\ c_2 e^{i\frac{\omega_0}{2}t}\end{pmatrix}$$

$$= \frac{\hbar}{2}\left(c_1^* e^{i\frac{\omega_0}{2}t} \;\; c_2^* e^{-i\frac{\omega_0}{2}t}\right)\begin{pmatrix}c_2 e^{i\frac{\omega_0}{2}t}\\ c_1 e^{-i\frac{\omega_0}{2}t}\end{pmatrix}$$

$$\boxed{\langle S_x\rangle(t) = \frac{\hbar}{2}\left(c_1^* c_2 e^{i\omega_0 t} + c_1 c_2^* e^{-i\omega_0 t}\right)}$$

$$\langle\psi(t)|S_y|\psi(t)\rangle = \left(c_1^* e^{i\frac{\omega_0}{2}t} \;\; c_2^* e^{-i\frac{\omega_0}{2}t}\right)\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} c_1 e^{-i\frac{\omega_0}{2}t} \\ c_2 e^{+i\frac{\omega_0}{2}t} \end{pmatrix}$$

$$= \left(c_1^* e^{i\frac{\omega_0}{2}t} \;\; c_2^* e^{-i\frac{\omega_0}{2}t}\right)\frac{\hbar}{2}\begin{pmatrix} -i c_2 e^{i\frac{\omega_0}{2}t} \\ +i c_1 e^{-i\frac{\omega_0}{2}t} \end{pmatrix}$$

$$\boxed{\langle S_y\rangle(t) = \frac{\hbar}{2} i\left(-c_1^* c_2 e^{i\omega_0 t} + c_1 c_2^* e^{-i\omega_0 t}\right)}$$

$$\langle\psi(t)|S_z|\psi(t)\rangle = \left(c_1^* e^{i\frac{\omega_0}{2}t} \;\; c_2^* e^{-i\frac{\omega_0}{2}t}\right)\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c_1 e^{-i\frac{\omega_0}{2}t} \\ c_2 e^{i\frac{\omega_0}{2}t} \end{pmatrix}$$

$$= \left(c_1^* e^{i\frac{\omega_0}{2}t} \;\; c_2^* e^{-i\frac{\omega_0}{2}t}\right)\frac{\hbar}{2}\begin{pmatrix} c_1 e^{-i\frac{\omega_0}{2}t} \\ -c_2 e^{i\frac{\omega_0}{2}t} \end{pmatrix}$$

$$\boxed{\langle S_z\rangle(t) = \frac{\hbar}{2}\left(|c_1|^2 - |c_2|^2\right)}$$

d) $$\frac{d\langle S_x\rangle}{dt} = \omega_0 \frac{\hbar}{2} i\left(c_1^* c_2 e^{i\omega_0 t} - c_1 c_2^* e^{-i\omega_0 t}\right) = \omega_0 \langle S_y\rangle$$

$$\frac{d\langle S_y\rangle}{dt} = -\omega_0 \frac{\hbar}{2}\left(c_1^* c_2 e^{i\omega_0 t} + c_1 c_2^* e^{-i\omega_0 t}\right) = -\omega_0 \langle S_x\rangle$$

$$\frac{d\langle S_z\rangle}{dt} = 0$$

These are the x, y, z components of the differential equation:

$$\frac{d}{dt}\left(\langle S_x\rangle\hat{x} + \langle S_y\rangle\hat{y} + \langle S_z\rangle\hat{z}\right) = \omega_0\left(\langle S_x\rangle\hat{x} + \langle S_y\rangle\hat{y} + \langle S_z\rangle\hat{z}\right) \times \vec{B}.$$

2. a) $$\frac{d}{dt}\left(e^{\frac{iHt}{\hbar}} S_x e^{-\frac{iHt}{\hbar}}\right) = \frac{iH}{\hbar}\left(e^{\frac{iHt}{\hbar}} S_x e^{-\frac{iHt}{\hbar}}\right) + \left(e^{\frac{iHt}{\hbar}} S_x e^{-\frac{iHt}{\hbar}}\right)\left(\frac{-iH}{\hbar}\right)$$

$$= \frac{i}{\hbar}\left[H, e^{\frac{iHt}{\hbar}} S_x e^{-\frac{iHt}{\hbar}}\right]$$

$$= \frac{-i}{\hbar}\left[e^{\frac{iHt}{\hbar}} S_x e^{-\frac{iHt}{\hbar}}, H\right]$$

$$\rightarrow i\hbar \frac{d}{dt}\left(e^{\frac{iHt}{\hbar}} S_x e^{-\frac{iHt}{\hbar}}\right) = \left[e^{\frac{iHt}{\hbar}} S_x e^{-\frac{iHt}{\hbar}}, H\right], \checkmark$$

and similarly for S_y and S_z.

b) $$\exp\left(\frac{iHt}{\hbar}\right) = \exp\left(i\frac{\omega_0}{2}\sigma_z t\right) = \cos\left(\frac{\omega_0 t}{2}\right)\mathbb{1} + i\sin\left(\frac{\omega_0 t}{2}\right)\sigma_z$$

$$\exp\left(\frac{-iHt}{\hbar}\right) = \exp\left(-i\frac{\omega_0 \sigma_z t}{2}\right) = \cos\left(\frac{\omega_0 t}{2}\right)\mathbb{1} - i\sin\left(\frac{\omega_0 t}{2}\right)\sigma_z$$

c) $$S_{x,H}(t) = e^{i\frac{Ht}{\hbar}} S_x e^{-\frac{iHt}{\hbar}}$$

$$= \left\{\cos\left(\frac{\omega_0 t}{2}\right)\mathbb{1} + i\sin\left(\frac{\omega_0 t}{2}\right)\sigma_z\right\}\sigma_x\left\{\cos\left(\frac{\omega_0 t}{2}\right)\mathbb{1} - i\sin\left(\frac{\omega_0 t}{2}\right)\sigma_z\right\}\frac{\hbar}{2}$$

$$= \left\{\cos\left(\frac{\omega_0 t}{2}\right)\mathbb{1} + i\sin\left(\frac{\omega_0 t}{2}\right)\sigma_z\right\}\left\{\cos\left(\frac{\omega_0 t}{2}\right)\sigma_x - \sin\left(\frac{\omega_0 t}{2}\right)\sigma_y\right\}\frac{\hbar}{2}$$

$$= \left\{\cos^2\left(\frac{\omega_0 t}{2}\right)\sigma_x - 2\cos\left(\frac{\omega_0 t}{2}\right)\sin\left(\frac{\omega_0 t}{2}\right)\sigma_y - \sin^2\left(\frac{\omega_0 t}{2}\right)\sigma_x\right\}\frac{\hbar}{2}$$

$$\boxed{S_{x,H}(t) = \left\{\cos(\omega_0 t)\sigma_x - \sin(\omega_0 t)\sigma_y\right\}\frac{\hbar}{2}}$$

$$S_{y,H}(t) = e^{i\frac{Ht}{\hbar}} S_y e^{-i\frac{Ht}{\hbar}}$$

$$= \left\{\cos\left(\frac{\omega_0 t}{2}\right)\mathbb{1} + i\sin\left(\frac{\omega_0 t}{2}\right)\sigma_z\right\}\sigma_y\left\{\cos\left(\frac{\omega_0 t}{2}\right)\mathbb{1} - i\sin\left(\frac{\omega_0 t}{2}\right)\sigma_z\right\}\frac{\hbar}{2}$$

$$= \left\{\cos^2\left(\frac{\omega_0 t}{2}\right)\sigma_y - \sin^2\left(\frac{\omega_0 t}{2}\right)\sigma_y + 2\sin\left(\frac{\omega_0 t}{2}\right)\cos\left(\frac{\omega_0 t}{2}\right)\sigma_x\right\}\frac{\hbar}{2}$$

$$\boxed{S_{y,H}(t) = \frac{\hbar}{2}\left\{\cos(\omega_0 t)\sigma_y + \sin(\omega_0 t)\sigma_x\right\}}$$

$$S_{z,H}(t) = e^{i\frac{Ht}{\hbar}} S_z e^{-i\frac{Ht}{\hbar}}$$

$$= \frac{\hbar}{2}\left\{\cos\left(\frac{\omega_0 t}{2}\right)\mathbb{1} + i\sin\left(\frac{\omega_0 t}{2}\right)\sigma_z\right\}\sigma_z\left\{\cos\left(\frac{\omega_0 t}{2}\right)\mathbb{1} - i\sin\left(\frac{\omega_0 t}{2}\right)\sigma_z\right\}$$

$$\boxed{S_{z,H}(t) = \frac{\hbar}{2}\sigma_z}$$

d) In part c) we have shown that

$$S_{x,H}(t) = \cos(\omega_0 t)S_{x,H}(0) - \sin(\omega_0 t)S_{y,H}(0)$$
$$S_{y,H}(t) = \cos(\omega_0 t)S_{y,H}(0) + \sin(\omega_0 t)S_{x,H}(0)$$
$$S_{z,H}(t) = S_{z,H}(0) \quad .$$

Since $\langle\psi(t=0)|S_{x,H}(0)|\psi(t=0)\rangle = \langle\psi(t=0)|S_x|\psi(t=0)\rangle$

$$= (c_1^* \; c_2^*)\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$$

$$= \frac{\hbar}{2}(c_1 c_2^* + c_1^* c_2)$$

$$\langle\psi(t=0)|S_{y,H}(0)|\psi(t=0)\rangle = \frac{\hbar}{2}i(c_1 c_2^* - c_1^* c_2)$$

$$\langle\psi(t=0)|S_{z,H}(0)|\psi(t=0)\rangle = \frac{\hbar}{2}(|c_1|^2 - |c_2|^2)$$

$$\rightarrow \langle\psi(0)|S_{x,H}(t)|\psi(0)\rangle = \cos(\omega_0 t)\frac{\hbar}{2}(c_1 c_2^* + c_1^* c_2)$$
$$- \sin(\omega_0 t)\frac{\hbar}{2} i(c_1 c_2^* - c_1^* c_2)$$
$$= \frac{\hbar}{2}\left(c_1 c_2^* e^{-i\omega_0 t} + c_1^* c_2 e^{i\omega_0 t}\right) \checkmark$$

$$\langle\psi(0)|S_{y,H}(t)|\psi(0)\rangle = \cos(\omega_0 t)\frac{\hbar}{2} i(c_1 c_2^* - c_1^* c_2)$$
$$+ \sin(\omega_0 t)\frac{\hbar}{2}(c_1 c_2^* + c_1^* c_2)$$
$$= \frac{\hbar}{2} i\left(c_1 c_2^* e^{-i\omega_0 t} - c_1^* c_2 e^{i\omega_0 t}\right) \checkmark$$

$$\langle\psi(0)|S_{z,H}(t)|\psi(0)\rangle = \frac{\hbar}{2}(|c_1|^2 - |c_2|^2) \checkmark$$

e) $$\frac{dS_{x,H}(t)}{dt} = -\omega_0 \sin(\omega_0 t) S_{x,H}(0) - \omega_0 \cos(\omega_0 t) S_{y,H}(0)$$
$$= -\omega_0 S_{y,H}(t)$$

$$\frac{dS_{y,H}(t)}{dt} = -\omega_0 \sin(\omega_0 t) S_{y,H}(0) + \omega_0 \cos(\omega_0 t) S_{x,H}(0)$$
$$= \omega_0 S_{x,H}(t)$$

$$\frac{dS_{z,H}(t)}{dt} = 0$$