Homework 8:

1.(a) Let $z = \sqrt{\frac{m\omega}{\hbar}}\, x = \frac{x}{\xi}$

$$\rightarrow \frac{d}{dz} = \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}$$

$$\rightarrow \langle x|n\rangle = \frac{1}{\sqrt{n!}}\frac{1}{2^{n/2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,x - \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}\right)^n \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \frac{1}{\sqrt{n!}}\frac{1}{2^{n/2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(z - \frac{d}{dz}\right)^n e^{-\frac{1}{2}z^2}$$

This derivative will have the form

$$\langle x|n\rangle = \left(z - \frac{d}{dz}\right)^n e^{-\frac{1}{2}z^2} = H_n(z)\, e^{-\frac{1}{2}z^2},$$

Where $H_n(z)$ is a polynomial. We can derive a recursion relation among the $H_n(z)$:

$$\left(z - \frac{d}{dz}\right)^n e^{-\frac{1}{2}z^2} = \left(z - \frac{d}{dz}\right) H_{n-1}(z)\, e^{-\frac{1}{2}z^2}$$

$$= \left(z H_{n-1} - \frac{dH_{n-1}}{dz} + z H_{n-1}\right) e^{-\frac{1}{2}z^2}$$

$$\Rightarrow H_n = \left(2z - \frac{d}{dz}\right) H_{n-1}$$

These polynomials are called the Hermite polynomials.

$$\langle x|n\rangle = \frac{1}{\pi^{1/4}}\left(\frac{m\omega}{\hbar}\right)^{1/4}\frac{1}{\sqrt{n!}}\frac{1}{2^{n/2}} H_n(z)\, e^{-\frac{1}{2}z^2}$$

$$H_0 = 1$$

$$H_1 = \left(2z - \frac{d}{dz}\right)1 = 2z$$

$$H_2 = \left(2z - \frac{d}{dz}\right)2z = 4z^2 - 2$$

$$H_3 = \left(2z - \frac{d}{dz}\right)(4z^2 - 2) = 8z^3 - 4z - 8z = 8z^3 - 12z$$

Summary:

$$\varphi_0(x) = \langle x|0\rangle = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{\xi}} \underbrace{\frac{1}{\sqrt{0!}} \frac{1}{2^{0/2}}}_{1} \underbrace{H_0\left(\frac{x}{\xi}\right)}_{1} e^{-\frac{1}{2}\left(\frac{x}{\xi}\right)^2}$$

$$\varphi_1(x) = \langle x|1\rangle = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{\xi}} \frac{1}{\sqrt{1!}} \frac{1}{2^{1/2}} H_1\left(\frac{x}{\xi}\right) e^{-\frac{1}{2}\left(\frac{x}{\xi}\right)^2}$$

$$\varphi_2(x) = \langle x|2\rangle = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{\xi}} \frac{1}{\sqrt{2!}} \frac{1}{2^{2/2}} H_2\left(\frac{x}{\xi}\right) e^{-\frac{1}{2}\left(\frac{x}{\xi}\right)^2}$$

$$= \langle x|3\rangle = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{\xi}} \frac{1}{\sqrt{3!}} \frac{1}{2^{3/2}} H_3\left(\frac{x}{\xi}\right) e^{-\frac{1}{2}\left(\frac{x}{\xi}\right)^2}$$

(b),(c) See plots on following pages. The red lines are the maximum classical positions.

(d) Since $\varphi_n(x) = \varphi_n(-x)$ for n even, the eigenvalue of the parity operator is 1 for $n = 0, 2$.
Since $\varphi_n(x) = -\varphi_n(-x)$ for n odd, the eigenvalue of the parity operator is -1 for $n = 1, 3$.

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% Matlab file for plots in Homework Assignment 8.

z = -5:0.01:5;

H0 = 1 + 0*z;
H1 = 2*z;
H2 = 4*z.^2 - 2;
H3 = 8*z.^3 - 12*z;

phi0 = (pi^-0.25)*H0.*exp(-0.5*z.^2);
phi1 = (pi^-0.25)*(2^-0.5)*H1.*exp(-0.5*z.^2);
phi2 = (pi^-0.25)*(factorial(2)^-0.5)*(2^-1.0)*H2.*exp(-0.5*z.^2);
phi3 = (pi^-0.25)*(factorial(3)^-0.5)*(2^-1.5)*H3.*exp(-0.5*z.^2);

prob0 = phi0.^2;
prob1 = phi1.^2;
prob2 = phi2.^2;
prob3 = phi3.^2;

figure(1)
subplot(2,2,1); plot(z,phi0,'b',[1 1],[-1 1],'r',[-1 -1],[-1 1],'r');
   xlabel('x/\xi'); ylabel('\xi ^{1/2} \phi_0')
subplot(2,2,2); plot(z,phi1,'b',[sqrt(3) sqrt(3)],[-1 1],'r',[-sqrt(3) -sqrt(3)],[-1 1],'r');
   xlabel('x/\xi'); ylabel('\xi ^{1/2} \phi_1')
subplot(2,2,3); plot(z,phi2,'b',[sqrt(5) sqrt(5)],[-1 1],'r',[-sqrt(5) -sqrt(5)],[-1 1],'r');
   xlabel('x/\xi'); ylabel('\xi ^{1/2} \phi_2')
subplot(2,2,4); plot(z,phi3,'b',[sqrt(7) sqrt(7)],[-1 1],'r',[-sqrt(7) -sqrt(7)],[-1 1],'r');
   xlabel('x/\xi'); ylabel('\xi ^{1/2} \phi_3')

figure(2)
subplot(2,2,1); plot(z,prob0,'b',[1 1],[0 0.6],'r',[-1 -1],[0 0.6],'r');
   xlabel('x/\xi'); ylabel('\xi |\phi_0|^2');axis([-5 5 0 0.6])
subplot(2,2,2); plot(z,prob1,'b',[sqrt(3) sqrt(3)],[0 0.6],'r',[-sqrt(3) -sqrt(3)],[0 0.6],'r');
   xlabel('x/\xi'); ylabel('\xi |\phi_1|^2');axis([-5 5 0 0.6])
subplot(2,2,3); plot(z,prob2,'b',[sqrt(5) sqrt(5)],[0 0.6],'r',[-sqrt(5) -sqrt(5)],[0 0.6],'r');
   xlabel('x/\xi'); ylabel('\xi |\phi_2|^2');axis([-5 5 0 0.6])
subplot(2,2,4); plot(z,prob3,'b',[sqrt(7) sqrt(7)],[0 0.6],'r',[-sqrt(7) -sqrt(7)],[0 0.6],'r');
   xlabel('x/\xi'); ylabel('\xi |\phi_3|^2');axis([-5 5 0 0.6])
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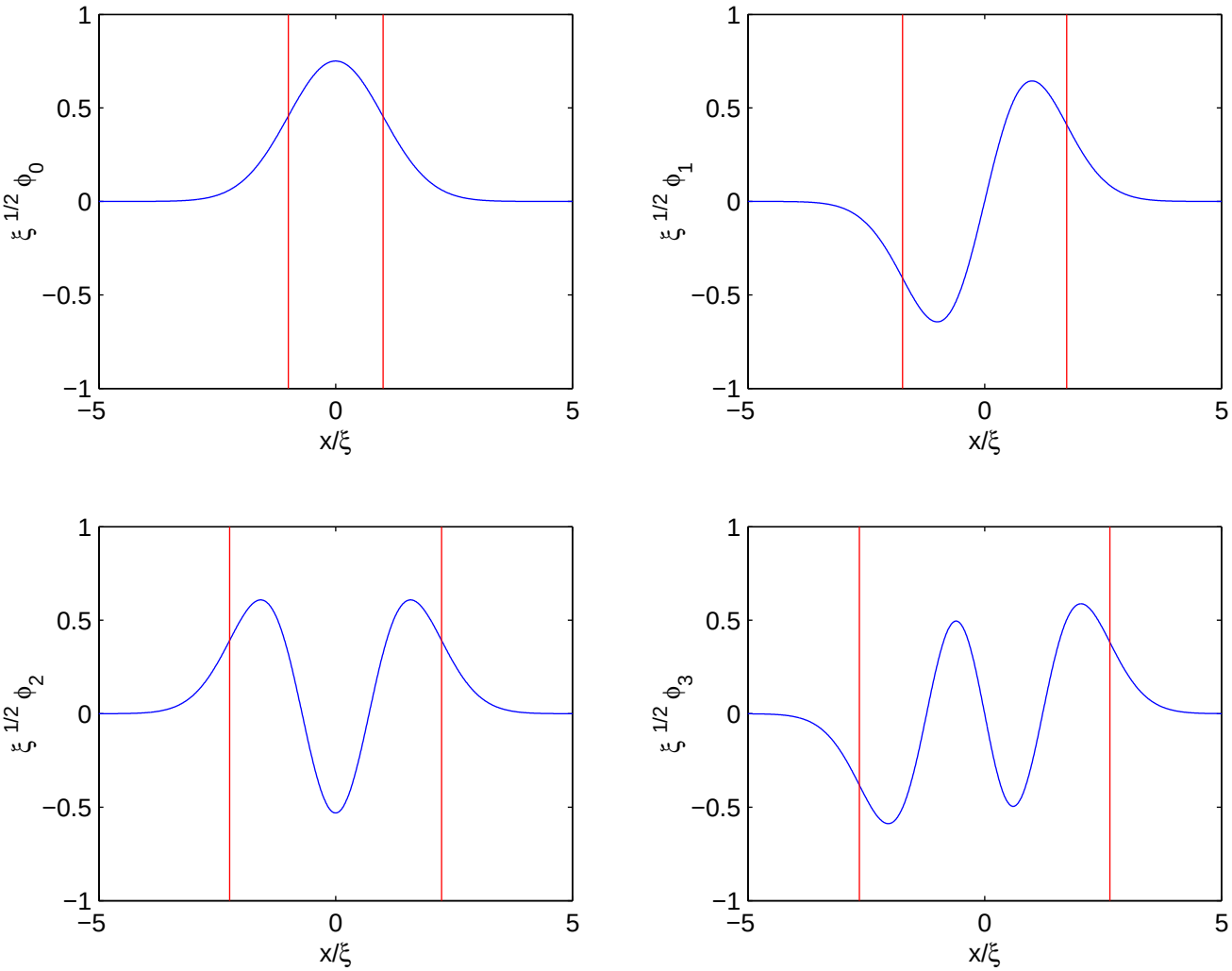
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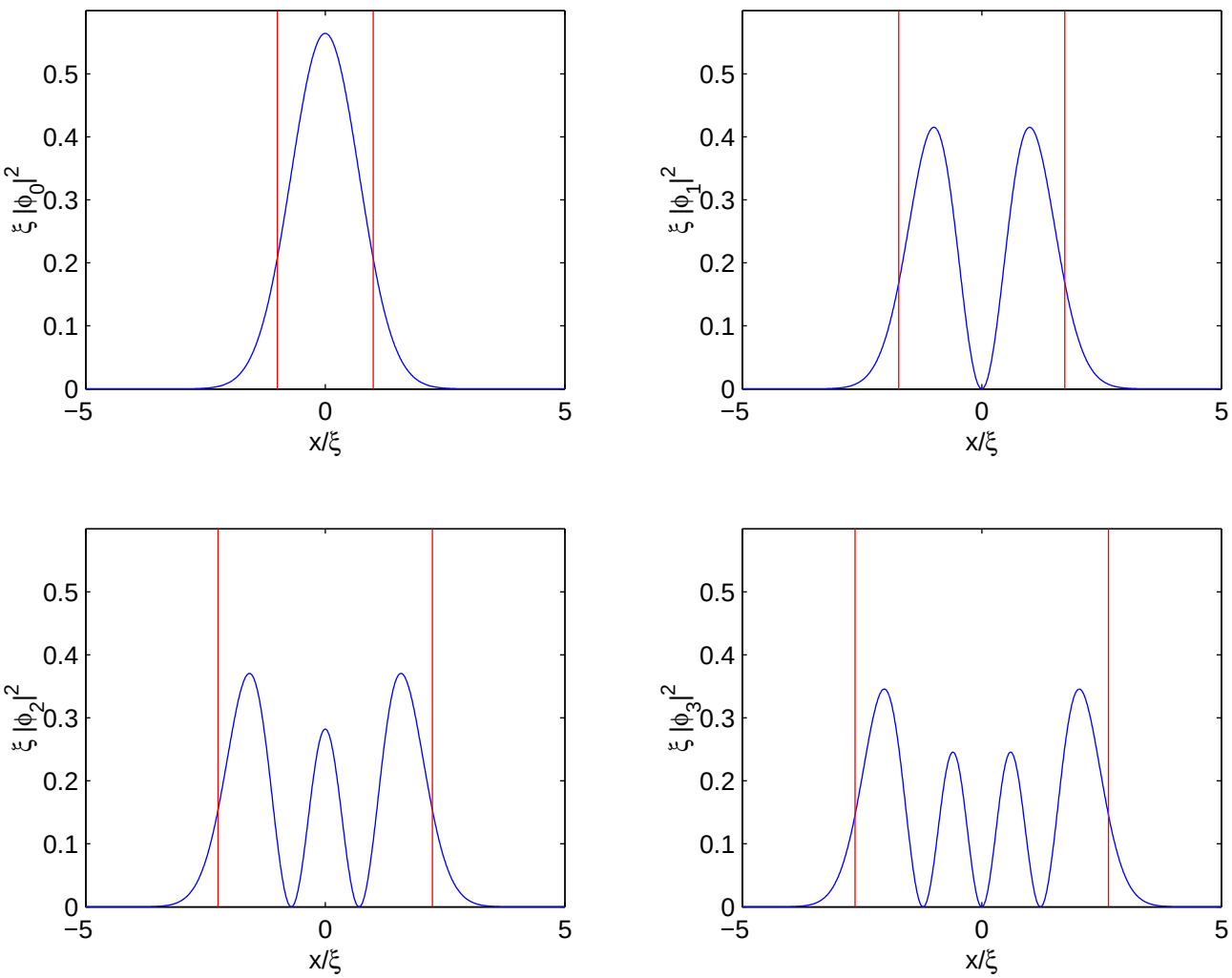
−1
−5
0
5
x/ξ
ξ 1/2 φ0
ξ 1/2 φ1
ξ 1/2 φ2
ξ 1/2 φ3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0.4
0.3
0.2
0.1
0
−5
0
5
x/ξ
ξ |φ0|2
ξ |φ1|2
ξ |φ2|2
ξ |φ3|2

2. Let $|\psi_1(0)\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$

$|\psi_2(0)\rangle = \cos\theta\,|1\rangle + \sin\theta\,|2\rangle$

$|\psi_3(0)\rangle = \cos\theta\,|0\rangle + \sin\theta\,|2\rangle$.

$$\rightarrow \quad |\psi_1(t)\rangle = \cos\theta\, e^{-i\omega(0+\frac{1}{2})t}\,|0\rangle + \sin\theta\, e^{-i\omega(1+\frac{1}{2})t}\,|1\rangle$$

$$|\psi_2(t)\rangle = \cos\theta\, e^{-i\omega(1+\frac{1}{2})t}\,|1\rangle + \sin\theta\, e^{-i\omega(2+\frac{1}{2})t}\,|2\rangle$$

$$|\psi_3(t)\rangle = \cos\theta\, e^{-i\omega(0+\frac{1}{2})t}\,|0\rangle + \sin\theta\, e^{-i\omega(2+\frac{1}{2})t}\,|2\rangle .$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a)$$

$$p = \sqrt{\frac{m\hbar\omega}{2}}\, i\,(a^\dagger - a)$$

$$\rightarrow \langle\psi_1(t)|x|\psi_1(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\cos\theta\sin\theta\left\{\langle 0|(a^\dagger+a)|1\rangle e^{-i\omega t} + \langle 1|(a^\dagger+a)|0\rangle e^{i\omega t}\right\}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,\cos\theta\sin\theta\; 2\cos\omega t$$

$$\langle\psi_2(t)|x|\psi_2(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\cos\theta\sin\theta\left\{\langle 1|(a^\dagger+a)|2\rangle e^{-i\omega t} + \langle 2|(a^\dagger+a)|1\rangle e^{i\omega t}\right\}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,\cos\theta\sin\theta\; 2\sqrt{2}\cos\omega t$$

$$\langle\psi_3(t)|x|\psi_3(t)\rangle = 0$$

$$\langle \psi_1(t)|p|\psi_1(t)\rangle = \sqrt{\frac{m\hbar\omega}{2}}\, i \cos\theta \sin\theta \left\{ \langle 0|(a^+ - a)|1\rangle e^{-i\omega t} + \langle 1|(a^+ - a)|0\rangle e^{i\omega t} \right\}$$

$$= \sqrt{\frac{m\hbar\omega}{2}}\, i \cos\theta \sin\theta \left\{ -e^{-i\omega t} + e^{i\omega t} \right\}$$

$$= -\sqrt{\frac{m\hbar\omega}{2}} \cos\theta \sin\theta \, 2\sin(\omega t)$$

$$\langle \psi_2(t)|p|\psi_2(t)\rangle = \sqrt{\frac{m\hbar\omega}{2}}\, i \cos\theta \sin\theta \left\{ \langle 1|(a^+ - a)|2\rangle e^{-i\omega t} \langle 2|(a^+ - a)|1\rangle e^{i\omega t} \right\}$$

$$= -\sqrt{\frac{m\hbar\omega}{2}} \cos\theta \sin\theta \, 2\sqrt{2} \sin(\omega t)$$

$$\langle \psi_3(t)|p|\psi_3(t)\rangle = 0$$

The periods for ψ_1 and ψ_2 are both $2\pi/\omega$. ψ_3 has no time dependence in $\langle x\rangle(t)$. The amplitudes for the ψ_1 and ψ_2 cases are

$$2\sqrt{\frac{\hbar}{2m\omega}} \cos\theta \sin\theta \quad \text{... for } \psi_1$$

$$2\sqrt{2}\sqrt{\frac{\hbar}{2m\omega}} \cos\theta \sin\theta \quad \text{... for } \psi_2 .$$