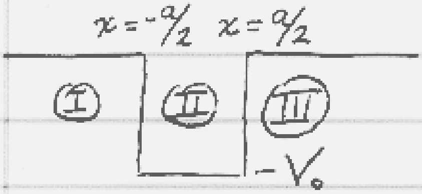
Potential Well:



$-V_0 < E < 0$

$$\text{I} \quad \varphi(x) = B_1' e^{+\rho x} \qquad , \rho = \sqrt{\frac{2m(0-E)}{\hbar^2}}$$

$$\text{II} \quad \varphi(x) = A_2 e^{ikx} + A_2' e^{-ikx}$$

$$\text{III} \quad \varphi(x) = B_3 e^{-\rho x} \qquad , k = \sqrt{\frac{2m(E--V_0)}{\hbar^2}}$$

$$\varphi\left(\tfrac{a}{2}\right) = B_3 e^{-\rho\frac{a}{2}} = A_2 e^{ik\frac{a}{2}} + A_2' e^{-ik\frac{a}{2}}$$

$$\varphi'\left(\tfrac{a}{2}\right) = -\rho B_3 e^{-\rho\frac{a}{2}} = ikA_2 e^{ik\frac{a}{2}} - ikA_2' e^{-ik\frac{a}{2}}$$

$$\Rightarrow \frac{\varphi'\left(\frac{a}{2}\right)}{\varphi\left(\frac{a}{2}\right)} = -\rho = ik \frac{A_2 e^{ik\frac{a}{2}} - A_2' e^{-ik\frac{a}{2}}}{A_2 e^{ik\frac{a}{2}} + A_2' e^{-ik\frac{a}{2}}} \qquad (i)$$

$$\varphi\left(\tfrac{-a}{2}\right) = B_1' e^{-\rho\frac{a}{2}} = A_2 e^{-ik\frac{a}{2}} + A_2' e^{ik\frac{a}{2}}$$

$$\varphi'\left(\tfrac{-a}{2}\right) = \rho B_1' e^{-\rho\frac{a}{2}} = ikA_2 e^{-ik\frac{a}{2}} - ikA_2' e^{+ik\frac{a}{2}}$$

$$\Rightarrow \frac{\varphi'\left(\frac{-a}{2}\right)}{\varphi\left(\frac{a}{2}\right)} = \rho = ik \frac{A_2 e^{-ik\frac{a}{2}} - A_2' e^{ik\frac{a}{2}}}{A_2 e^{-ik\frac{a}{2}} + A_2' e^{ik\frac{a}{2}}} \qquad (ii)$$

Multiply out (i) & (ii):

$$(i) \quad (-\rho - ik)e^{ik\frac{a}{2}} A_2 + (-\rho + ik)e^{-ik\frac{a}{2}} A_2' = 0$$

$$(ii) \quad (\rho - ik)e^{-ik\frac{a}{2}} A_2 + (\rho + ik)e^{ik\frac{a}{2}} A_2' = 0$$

$$\Rightarrow \frac{A_2}{A_{2'}} = \frac{(-\rho+ik)e^{-ik\frac{a}{2}}}{(\rho+ik)e^{ik\frac{a}{2}}} = \frac{(\rho+ik)e^{ik\frac{a}{2}}}{(-\rho+ik)e^{-ik\frac{a}{2}}}$$

$$\Rightarrow (-\rho+ik)^2 e^{-ika} = (\rho+ik)^2 e^{ika}$$

$$\Rightarrow \frac{(-\rho+ik)^2}{(\rho+ik)^2} = e^{2ika}$$

$$\Rightarrow \frac{\rho-ik}{\rho+ik} = \pm e^{ika}$$

Let $\cos\theta = \frac{\rho}{\sqrt{\rho^2+k^2}} > 0$ and $\sin\theta = \frac{k}{\sqrt{\rho^2+k^2}} > 0$.

$\rho = \sqrt{\frac{2m(0-E)}{\hbar^2}}$ and $k = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$

$\Rightarrow \rho^2 + k^2 = \frac{2mV_0}{\hbar^2}$, Let $= k_0 = \sqrt{\frac{2mV_0}{\hbar^2}}$. $\sin\theta = \frac{k}{k_0}$

$$\frac{e^{-i\theta}}{e^{i\theta}} = \pm e^{ika} \Rightarrow e^{-2i\theta} = \pm e^{ika}$$

$$\Rightarrow e^{-i\theta} = (1, -1, i, -i) \times e^{i\frac{ka}{2}}$$

$$\Rightarrow -\theta = \frac{ka}{2} + \varphi \text{ , where } \varphi = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$$-k/k_0 = -\sin(\theta) = \sin(-\theta) = \sin\left(\frac{ka}{2} + \varphi\right)$$

$$\rho/k_0 = \cos(\theta) = \cos(-\theta) = \cos\left(\frac{ka}{2} + \varphi\right)$$

$$-k/\rho = -\tan(\theta) = \tan(-\theta) = \tan\left(\frac{ka}{2} + \varphi\right)$$

$\varphi = 0 \qquad \sin\left(\frac{ka}{2}\right) = -k/k_0 \;\rightarrow\; k/k_0 = -\sin\left(\frac{ka}{2}\right), \quad \frac{3\pi}{2} < \frac{ka}{2} < 2\pi$

$\qquad\qquad \cos\left(\frac{ka}{2}\right) = \rho/k_0$

$\varphi = \frac{\pi}{2} \qquad \cos\left(\frac{ka}{2}\right) = -k/k_0 \;\rightarrow\; k/k_0 = -\cos\left(\frac{ka}{2}\right), \quad \pi < \frac{ka}{2} < \frac{3\pi}{2}$

$\qquad\qquad -\sin\left(\frac{ka}{2}\right) = \rho/k_0$

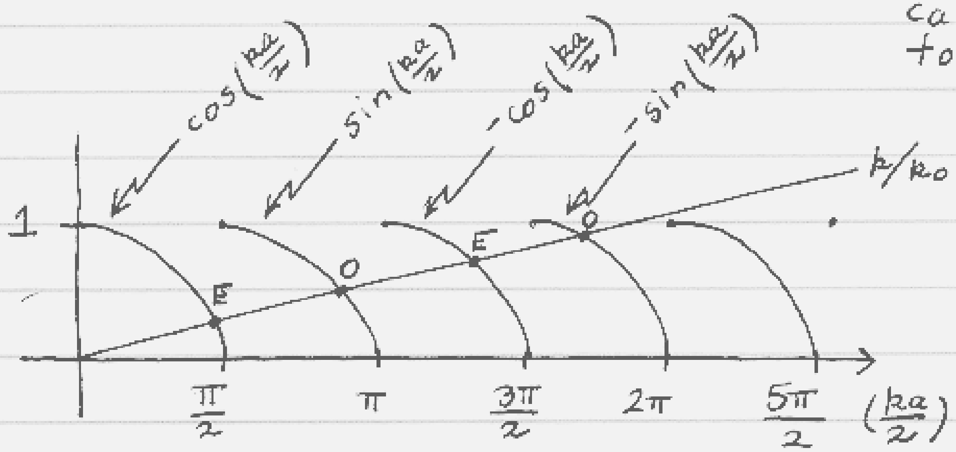
$\varphi = \pi \qquad -\sin\left(\frac{ka}{2}\right) = -k/k_0 \;\rightarrow\; k/k_0 = \sin\left(\frac{ka}{2}\right), \quad \frac{\pi}{2} < \frac{ka}{2} < \pi$

$\qquad\qquad -\cos\left(\frac{ka}{2}\right) = \rho/k_0$

$\varphi = \frac{3\pi}{2} \qquad -\cos\left(\frac{ka}{2}\right) = -k/k_0 \;\rightarrow\; k/k_0 = \cos\left(\frac{ka}{2}\right), \quad 0 < \frac{ka}{2} < \frac{\pi}{2}$

$\qquad\qquad \sin\left(\frac{ka}{2}\right) = \rho/k_0$

can add $2\pi n$ to each of these



There is always at least one solution.
From the graph above, the number N of solutions is

$$N = 1 + \mathrm{int}\left(\frac{k_0 a/2}{\pi/2}\right)$$

$$= 1 + \mathrm{int}\left(\frac{k_0 a}{\pi}\right)$$

$$\frac{A_2}{A_2'} = -\frac{(\rho - ik)}{(\rho + ik)}\frac{e^{-ik\frac{a}{2}}}{e^{ik\frac{a}{2}}} = -\frac{e^{-i\theta}}{e^{i\theta}}\,e^{-ika} = -e^{-2i\theta}e^{-ika}$$

Since $-\theta = \frac{ka}{2} + \varphi$, $+2(-\theta) = -2\theta = ka + 2\varphi$,

$$\frac{A_2}{A_2'} = -e^{i2\varphi} \rightarrow$$

$\varphi = 0$	$A_2/A_2' = -1$	$\frac{3\pi}{2} < \frac{ka}{2} < 2\pi$
$\varphi = \pi/2$	$A_2/A_2' = 1$	$\pi < \frac{ka}{2} < \frac{3\pi}{2}$
$\varphi = \pi$	$A_2/A_2' = -1$	$\frac{\pi}{2} < \frac{ka}{2} < \pi$
$\varphi = \frac{3\pi}{2}$	$A_2/A_2 = 1$	$0 < \frac{ka}{2} < \frac{\pi}{2}$

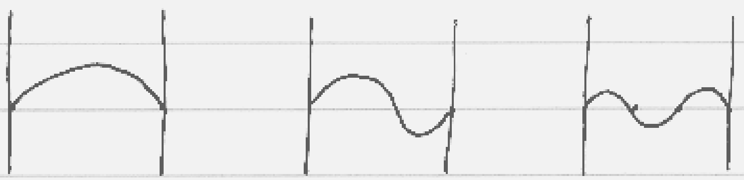
Thus, the solutions alternate between even $\left(\frac{A_2}{A_2'} = 1\right)$ and odd $\left(\frac{A_2}{A_2'} = -1\right)$. See previous page.

For a very deep well, V_0 is large and k_0 is large, $k_0 a \gg 1$. The solutions are roughly

$$\frac{ka}{2} \approx \frac{\pi}{2}, \pi, \frac{3\pi}{2},$$

$$\rightarrow ka \approx \pi, 2\pi, 3\pi, \ldots \rightarrow k = \frac{\pi}{a}, \frac{2\pi}{a}, \ldots$$

$$k = \frac{\pi n}{a} = \frac{2\pi}{\lambda}, \text{ or } \lambda = \frac{2a}{n}.$$



$n = 1$ $\quad$ $n = 2$ $\quad$ $n = 3$ $\quad \ldots$ (infinite square well)