II. Wave Function Vector Space:

Consider the following basis on $[-L/2, L/2]$:

$$\psi_n(x) = C_n e^{i2\pi n x/L}.$$

$$\langle\psi_n|\psi_n\rangle = \int_{-L/2}^{L/2} dx\, \psi_n^*(x)\psi_n(x)$$

$$= C_n^2 \int_{-L/2}^{L/2} dx = C_n^2 \cdot L = 1$$

$$\rightarrow C_n = \frac{1}{\sqrt{L}}$$

For $n \neq n'$, $$\langle\psi_n|\psi_{n'}\rangle = \int_{-L/2}^{L/2} dx \frac{e^{-i2\pi n x/L}}{\sqrt{L}} \frac{e^{i2\pi n' x/L}}{\sqrt{L}}$$

$$= \int_{-L/2}^{L/2} dx \frac{1}{L} \exp(i2\pi(n'-n)\frac{x}{L})$$

$$= \frac{1}{L} \frac{\exp(i2\pi(n'-n)x/L)}{i2\pi(n'-n)/L}\Big|_0^L$$

$$= \frac{1}{L} \frac{\exp(i2\pi(n'-n)) - 1}{i2\pi(n'-n)/L} = 0$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{\sqrt{L}} e^{-i2\pi n x/L} \frac{e^{i2\pi n x'/L}}{\sqrt{L}} = \frac{1}{L}\sum_{n=-\infty}^{+\infty} e^{i2\pi n(x'-x)/L}$$

$$= \lim_{N\to\infty} \frac{1}{L} \sum_{n=-N}^{N} e^{i2\pi n(x-x')/L}$$

$$= \lim_{N\to\infty} \frac{1}{L} \frac{\sin\left(2\pi(N+\frac{1}{2})(x-x')/L\right)}{\sin\left(2\pi(x-x')/2L\right)}$$

For $x, x' \in [-L/2, L/2]$,

$$= \lim_{N\to\infty} \frac{1}{L} \frac{\sin(2\pi(N+\frac{1}{2})(x-x')/L)}{2\pi(x-x')/2L}$$

$$= \lim_{N\to\infty} \frac{1}{\pi} \frac{\sin(2\pi(N+\frac{1}{2})(x-x')/L)}{(x-x')} = \delta(x-x') \checkmark$$

Thus, any function $\psi(x)$ on $[-L/2, L/2]$ can be written

$$\psi(x) = \sum_{n=-\infty}^{+\infty} \frac{e^{i(2\pi n/L)x}}{\sqrt{L}} \langle \psi_n | \psi \rangle$$

$$\langle \psi_n | \psi \rangle = \int_{-L/2}^{L/2} dx' \frac{e^{-i(2\pi n/L)x'}}{\sqrt{L}} \psi(x')$$

What happens as $L \to \infty$? First rewrite this as:

$$\psi(x) = \sum_{n=-\infty}^{\infty} \frac{e^{ik_n x}}{L} \sqrt{L} \langle \psi_n | \psi \rangle$$

$$\sqrt{L} \langle \psi_n | \psi \rangle = \int_{-L/2}^{L/2} dx' e^{-ik_n x'} \psi(x')$$

where $k_n = 2\pi n/L$. Note that $k_{n+1} - k_n = \Delta k = \frac{2\pi}{L}$.
Thus, this can also be written as

$$\psi(x) = \frac{1}{2\pi} \sum_n \Delta k \, e^{ik_n x} \sqrt{L} \langle \psi_n | \psi \rangle$$

$$\sqrt{L} \langle \psi_n | \psi \rangle = \int_{-L/2}^{L/2} dx' e^{-ik_n x'} \psi(x')$$

Define $g(k) = \sqrt{L} \langle \psi_n | \psi \rangle$.

$$\rightarrow \boxed{\begin{aligned} \psi(x) &= \int \frac{dk}{2\pi} e^{ikx} g(k) \\ g(k) &= \int dx' e^{-ikx'} \psi(x') \end{aligned}}$$

Orthonormality becomes:

$$\langle \psi_n | \psi_{n'} \rangle = \delta_{nn'}$$

$$\int_{-L/2}^{L/2} dx \frac{e^{-ik_n x}}{\sqrt{L}} \frac{e^{ik_{n'} x}}{\sqrt{L}} = \delta_{k_n k_{n'}}$$

$$\int_{-L/2}^{L/2} dx\, e^{-ik_n x} e^{ik_{n'} x} = L \delta_{k_n k_{n'}} = 2\pi \frac{\delta_{k_n k_{n'}}}{\Delta k} \quad \text{since } \Delta k = \frac{2\pi}{L}$$

In the limit as $L \to \infty$ this becomes:

$$\boxed{\int_{-\infty}^{+\infty} dx\, e^{-ikx} e^{ik'x} = 2\pi \delta(k - k')}$$

The closure condition becomes:

$$\sum_n \psi_n^*(x)\,\psi_n(x') = \delta(x-x')$$

$$\sum_{n=-\infty}^{+\infty} \frac{e^{-ik_n x}}{\sqrt{L}}\,\frac{e^{ik_n x'}}{\sqrt{L}} = \delta(x-x')$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{2\pi}\underbrace{\left(\frac{2\pi}{L}\right)}_{\Delta k} e^{-ik_n x}\, e^{ik_n x'} = \delta(x-x')$$

As $L \to \infty$, the sum becomes an integral

$$\boxed{\int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{-ikx}\, e^{ikx'} = \delta(x-x')}\,.$$

Note: These are the same mathematical equations:

$$\int_{-\infty}^{+\infty} dx\, e^{i(k'-k)x} = 2\pi\,\delta(k-k')$$

$$\int_{-\infty}^{+\infty} dk\, e^{i(x'-x)k} = 2\pi\,\delta(x-x')\,.$$