Dirac Notation:

In 3D, $\vec{v} = (\hat{x}\cdot\vec{v})\,\hat{x} + (\hat{y}\cdot\vec{v})\,\hat{y} + (\hat{z}\cdot\vec{v})\,\hat{z}$.

$$|v\rangle = \langle\hat{x}|v\rangle\,|\hat{x}\rangle + \langle\hat{y}|v\rangle\,|\hat{y}\rangle + \langle\hat{z}|v\rangle\,|\hat{z}\rangle$$

dot product

vector pointing in $\hat{x}$ direction.

In function spaces we have

$$\varphi(x) = \sum_{n=1}^{\infty} \langle\psi_n|\varphi\rangle\,\psi_n(x)$$

$$|\varphi\rangle = \sum_{n=1}^{\infty} \langle\psi_n|\varphi\rangle\,|\psi_n\rangle.$$

vector in function space

dot product

unit vector in function space

Let $|x_0\rangle$ be the delta function peaked at x_0.

$$\langle x_0 | \varphi \rangle = \int dx\, \delta(x - x_0)\, \varphi(x) = \varphi(x_0)$$

$$\langle \varphi | x_0 \rangle = \int dx\, \varphi^*(x)\, \delta(x - x_0) = \varphi^*(x_0)$$

$$\langle x_0 | x_0' \rangle = \int dx\, \delta(x - x_0)\, \delta(x - x_0') = \delta(x_0 - x_0')$$

We can also expand $|\varphi\rangle$ in $|x\rangle$ instead of $|\Psi_n\rangle$:

$$|\varphi\rangle = \int dx\, \varphi(x)\, |x\rangle$$

Check: $\langle x_0 | \varphi \rangle = \varphi(x_0) = \int dx\, \varphi(x) \langle x_0 | x \rangle$

$$= \int dx\, \varphi(x)\, \delta(x - x_0)$$

$$= \varphi(x_0) \checkmark$$

Operators: vector $\longrightarrow$ vector

$$|\varphi_1\rangle \longrightarrow |\varphi_2\rangle$$

Identity operator: $|\varphi\rangle \longrightarrow |\varphi\rangle$ (general case next lecture)

In 3D, $\vec{V} = (\hat{x}\cdot\vec{V})\hat{x} + (\hat{y}\cdot\vec{V})\hat{y} + (\hat{z}\cdot\vec{V})\hat{z}$

$$= \hat{x}(\hat{x}\cdot\vec{V}) + \hat{y}(\hat{y}\cdot\vec{V}) + \hat{z}(\hat{z}\cdot\vec{V})$$

$$= (\hat{x}\,\hat{x}\cdot + \hat{y}\,\hat{y}\cdot + \hat{z}\,\hat{z}\cdot)\vec{V}$$

identity operator

$$\mathbb{1} = (\hat{x}\hat{x}\cdot + \hat{y}\hat{y}\cdot + \hat{z}\hat{z}\cdot)$$

Function space:

$$|\varphi\rangle = \sum_{n=1}^{\infty} \langle\psi_n|\varphi\rangle\ |\psi_n\rangle$$

$$= \sum_{n=1}^{\infty} |\psi_n\rangle\langle\psi_n|\varphi\rangle \quad \Rightarrow \mathbb{1} = \sum_{n=1}^{\infty} |\psi_n\rangle\langle\psi_n|$$

$$|\varphi\rangle = \int dx\ \varphi(x)\,|x\rangle$$

$$= \int dx\ \langle x|\varphi\rangle\,|x\rangle$$

$$= \int dx\ |x\rangle\langle x|\varphi\rangle \quad \Rightarrow \mathbb{1} = \int dx\ |x\rangle\langle x|$$

Completeness condition:

$$\sum_{n=1}^{\infty} \psi_n^*(x)\,\psi_n(x') = \delta(x-x')$$

$$= \sum_{n=1}^{\infty} \langle \psi_n | x \rangle \langle x' | \psi_n \rangle = \langle x' | x \rangle$$

$$= \sum_{n=1}^{\infty} \langle x' | \psi_n \rangle \langle \psi_n | x \rangle = \langle x' | x \rangle$$

Thus, the completeness condition is equivalent to the identity operator being written as

$$\sum_{n=1}^{\infty} | \psi_n \rangle \langle \psi_n | = \mathbb{1} \, .$$

If $|\varphi\rangle$ is a vector, what is $\langle\varphi|$?

$|\varphi\rangle \in$ function space (ket)

$\langle\varphi| \in$ conjugate or dual space (bra)

$\varphi(x) = \langle x|\varphi\rangle$; $\varphi^*(x) = \langle\varphi|x\rangle$

Remember, $\langle\varphi_1|\varphi_2\rangle = \int dx\, \varphi_1^*(x)\,\varphi_2(x)$.

$\rightarrow \langle\varphi_1|\varphi_2\rangle^* = \langle\varphi_2|\varphi_1\rangle$

$$\langle\varphi|\lambda_1\psi_1 + \lambda_2\psi_2\rangle = \lambda_1\langle\varphi|\psi_1\rangle + \lambda_2\langle\varphi|\psi_2\rangle$$

$$\langle\lambda_1\varphi_1 + \lambda_2\varphi_2|\psi\rangle = \lambda_1^*\langle\varphi_1|\psi\rangle + \lambda_2^*\langle\varphi_2|\psi\rangle$$

ket: $|\lambda_1\psi_1 + \lambda_2\psi_2\rangle = \lambda_1|\psi_1\rangle + \lambda_2|\psi_2\rangle$

bra: $\langle\lambda_1\varphi_1 + \lambda_2\varphi_2| = \lambda_1^*\langle\varphi_1| + \lambda_2^*\langle\varphi_2|$