## Linear Operators:

Definition:

$$A(\lambda_1 |\psi_1\rangle + \lambda_2 |\psi_2\rangle) = \lambda_1 A|\lambda_1\rangle + \lambda_2 A|\psi_2\rangle$$

Notation:

$$(AB)|\psi\rangle = A(B|\psi\rangle)$$

$$[A,B] = AB - BA$$

An example:

$$\left[\frac{d}{dx}, x\right]\psi(x) = \frac{d}{dx} x\,\psi(x) - x\frac{d}{dx}\psi(x)$$

$$= \psi(x) + x\frac{d\psi}{dx} - x\frac{d\psi}{dx}$$

$$= \psi(x)$$

$$\Rightarrow \left[\frac{d}{dx}, x\right] = 1$$

Adjoint operator:

$$|\psi'\rangle = A|\psi\rangle$$

$$\langle\psi|A^{+} = \langle\psi'|$$

$$\begin{tikzcd}
|\psi\rangle \arrow[r, "A"] \arrow[d, leftrightarrow] & |\psi'\rangle = A|\psi\rangle \arrow[d, leftrightarrow] \\
\langle\psi| \arrow[r, "A^{+}"] & \langle\psi'| = \langle\psi|A^{+}
\end{tikzcd}$$

$\leftrightarrow$ is the conjugate or adjoint relation.

Notation: $|A\psi\rangle = A|\psi\rangle$

$\langle A\psi| = \langle\psi|A^{+}$

Properties of adjoint operator:

(i) $\langle \psi | A^\dagger | \varphi \rangle = \langle A\psi | \varphi \rangle = \langle \varphi | A\psi \rangle^* = \langle \varphi | A | \psi \rangle^*$

(ii) $(A^\dagger)^\dagger = A$

(iii) $(\lambda A)^\dagger = \lambda^* A^\dagger$

(iv) $(A + B)^\dagger = A^\dagger + B^\dagger$

(v) $(AB)^\dagger = B^\dagger A^\dagger$

Proof of (v): Let $|\varphi\rangle = AB|\psi\rangle = A|\chi\rangle$, ie. $|\chi\rangle = B|\psi\rangle$

$$|\varphi\rangle = AB|\psi\rangle \Rightarrow \langle\varphi| = \langle\psi|(AB)^\dagger$$

$$\left.\begin{array}{ll} |\varphi\rangle = A|\chi\rangle & \Rightarrow \langle\varphi| = \langle\chi|A^\dagger \\ |\chi\rangle = B|\psi\rangle & \Rightarrow \langle\chi| = \langle\psi|B^\dagger \end{array}\right\} \Rightarrow \langle\varphi| = \langle\psi|B^\dagger A^\dagger$$

Since $\langle\varphi| = \langle\psi|(AB)^\dagger = \langle\psi|B^\dagger A^\dagger$ for any $\langle\psi|$, $(AB)^\dagger = B^\dagger A^\dagger$.

Hermitian operators: $A = A^\dagger$

$$\Rightarrow \langle\psi|A^\dagger|\varphi\rangle = \langle\psi|A|\varphi\rangle = \langle\varphi|A|\psi\rangle^*$$