Eigenvalues / eigenvectors:

$$A|\psi\rangle = \lambda|\psi\rangle$$

linear operator eigenvector eigenvalue

Using $\mathbb{1} = \sum_n |\psi_n\rangle\langle\psi_n|$,

$$A\mathbb{1}|\psi\rangle = A\sum_n |\psi_n\rangle\langle\psi_n|\psi\rangle = \lambda|\psi\rangle$$

$$\rightarrow \sum_n \langle\psi_m|A|\psi_n\rangle\langle\psi_n|\psi\rangle = \lambda\langle\psi_m|\psi\rangle$$

This is a matrix equation

$$\begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & A_{22} & \\ \vdots & & \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} = \lambda\begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}, \text{ where } \begin{array}{l} c_n = \langle\psi_n|\psi\rangle \\ A_{mn} = \langle\psi_m|A|\psi_n\rangle \end{array}$$

$$\Rightarrow (A - \lambda\mathbb{1})\begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} = 0 \Rightarrow \det(A - \lambda\mathbb{1}) = 0.$$

The eigen values of a Hermitian operator are real:

(i) Hermitian: $A^{\dagger} = A$

$\rightarrow \langle\psi|A|\psi\rangle^{*} = \langle\psi|A^{\dagger}|\psi\rangle = \langle\psi|A|\psi\rangle$

$\rightarrow \langle\psi|A|\psi\rangle$ is real.

$$A|\psi\rangle = \lambda|\psi\rangle \;\rightarrow\; \langle\psi|A|\psi\rangle = \lambda\langle\psi|\psi\rangle$$

$$\rightarrow \lambda = \frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle} \text{ is real.} \;//$$

(ii) Two eigenvectors of a Hermitian operator corresponding to two different eigenvalues are orthogonal:

$$A|\psi\rangle = \lambda|\psi\rangle$$
$$A|\varphi\rangle = \mu|\varphi\rangle \;\rightarrow\; \langle\varphi|A^{\dagger} = \langle\varphi|A = \mu\langle\varphi|$$

$$\langle\varphi|A|\psi\rangle = \lambda\langle\varphi|\psi\rangle$$
$$\langle\varphi|A|\psi\rangle = \mu\langle\varphi|\psi\rangle$$

$$\rightarrow 0 = (\lambda-\mu)\langle\varphi|\psi\rangle \;//$$

(iii) For any finite dimensional vectorspace, it is always possible to form an orthonormal basis with eigenvectors of a Hermitian operator.

→ Thus, let $|\psi_n\rangle$ be eigenvectors of A with eigenvalues λ_n.

$$A|\psi_n\rangle = \lambda_n|\psi_n\rangle$$

$$\langle\psi_m|A|\psi_n\rangle = \lambda_n\langle\psi_m|\psi_n\rangle = \lambda_n\delta_{mn}$$

$$\rightarrow A = \mathbb{1}A\mathbb{1} = \left(\sum_m |\psi_m\rangle\langle\psi_m|\right)A\left(\sum_n|\psi_n\rangle\langle\psi_n|\right)$$

$$= \sum_{m,n}|\psi_m\rangle\langle\psi_m|A|\psi_n\rangle\langle\psi_n|$$

$$= \sum_n \lambda_n |\psi_n\rangle\langle\psi_n|$$

(iv) For an infinite dimensional vector space, it is <u>not</u> always possible to form an orthonormal basis with eigenvectors of a Hermitian operator. Define those operators for which it is possible as observables.

(v) Suppose we have two Hermitian operators, A & B. Can one find a basis (orthonormal) with eigenvectors of both A and B?

YES if $[A,B]=0$
No if $[A,B]\neq 0$

Proof of necessary condition:

Suppose $A|\psi_n\rangle = a_n|\psi_n\rangle \rightarrow A = \sum_n a_n |\psi_n\rangle\langle\psi_n|$
$B|\psi_n\rangle = b_n|\psi_n\rangle \qquad B = \sum_n b_n |\psi_n\rangle\langle\psi_n|$

$$AB = \sum_n a_n b_n |\psi_n\rangle\langle\psi_n| \rightarrow AB = BA$$
$$BA = \sum_n b_n a_n |\psi_n\rangle\langle\psi_n| \rightarrow [A,B]=0$$

Proof of sufficient condition:

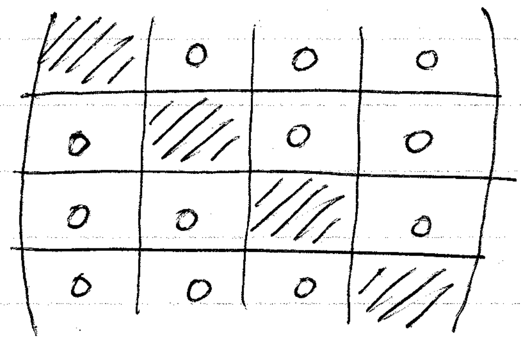
Suppose $[A,B]=0$.

If $|\psi_1\rangle$ and $|\psi_2\rangle$ are eigenvectors of A with <u>different</u> eigenvalues, then

$$0 = \langle\psi_1|[A,B]|\psi_2\rangle = \langle\psi_1|AB|\psi_2\rangle - \langle\psi|BA|\psi_2\rangle$$

$$= a_1\langle\psi_1|B|\psi_2\rangle - a_2\langle\psi_1|B|\psi_2\rangle$$
$\Rightarrow \langle\psi_1|B|\psi_2\rangle = 0$ since $a_1 \neq a_2$.

This means that B does not link different eigenspaces of A. In a basis of A, B is block diagonal.

$B =$ 

Each eigenspace of A may be diagonalized with respect to B creating a basis of eigenvectors of both A and B. //