Implications of the Schrodinger equation:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

a) Since this is first order in time, $|\psi(t)\rangle$ follows from $|\psi(t_0)\rangle$ at an initial time t_0.

b) Superposition principle: Since the Schrodinger equation is linear and homogeneous, if $|\psi_1(t)\rangle$ and $|\psi_2(t)\rangle$ are solutions, then $\lambda_1|\psi_1(t)\rangle + \lambda_2|\psi_2(t)\rangle$ is also a solution.

c) Conservation of probability:

$$\frac{d}{dt}\langle\psi(t)|\psi(t)\rangle = \underbrace{\left[\frac{d}{dt}\langle\psi(t)|\right]}_{-\frac{1}{i\hbar}\langle\psi(t)|H(t)}|\psi(t)\rangle + \langle\psi(t)|\underbrace{\left[\frac{d}{dt}|\psi(t)\rangle\right]}_{\frac{1}{i\hbar}H(t)|\psi(t)\rangle}$$

$$\rightarrow \frac{d}{dt}\langle\psi(t)|\psi(t)\rangle = -\frac{1}{i\hbar}\langle\psi(t)|H(t)|\psi(t)\rangle + \frac{1}{i\hbar}\langle\psi(t)|H(t)|\psi(t)\rangle$$

$$= 0$$

($H = \frac{p^2}{2m} + V(r)$ is Hermitian.)

(For probability current see Lecture 5a.)

d) Time evolution of mean value of an observable:

$$\langle A \rangle (t) = \langle \psi(t) | A | \psi(t) \rangle$$

$$\frac{d}{dt} \langle \psi(t) | A(t) | \psi(t) \rangle = \left[\frac{d}{dt} \langle \psi(t) | \right] A(t) | \psi(t) \rangle$$

$$+ \langle \psi(t) | A(t) \left[\frac{d}{dt} | \psi(t) \rangle \right]$$

$$+ \langle \psi(t) | \frac{\partial A}{\partial t} | \psi(t) \rangle$$

$$\Rightarrow \frac{d}{dt} \langle A \rangle = \frac{1}{i\hbar} \langle [A, H(t)] \rangle + \langle \frac{\partial A}{\partial t} \rangle$$

As an example, consider $H = \frac{P^2}{2m} + V(r)$.

$$\frac{d\langle R \rangle}{dt} = \frac{1}{i\hbar} \langle [R, H] \rangle = \frac{1}{i\hbar} \langle [R, \frac{P^2}{2m}] \rangle$$

$$\frac{d\langle P \rangle}{dt} = \frac{1}{i\hbar} \langle [P, H] \rangle = \frac{1}{i\hbar} \langle [P, V(R)] \rangle$$

$$[\vec{R}, \frac{P^2}{2m}] = \frac{1}{2m} (\vec{R} P^2 - P^2 \vec{R})$$

$$= \frac{1}{2m} (\vec{R} P^2 - \sum_\alpha P_\alpha \vec{R} P_\alpha + \sum_\alpha P_\alpha \vec{R} P_\alpha - P^2 \vec{R})$$

$$= \frac{1}{2m} \sum_\alpha [\vec{R}, P_\alpha] P_\alpha + \frac{1}{2m} \sum_\alpha P_\alpha [\vec{R}, P_\alpha]$$

Since $[R_\alpha, P_\beta] = i\hbar\,\delta_{\alpha\beta}$,

$$[\vec{R}, \frac{P^2}{2m}] = 2\,\frac{1}{2m}\,i\hbar\,\vec{P}, \text{ and}$$

$$\boxed{\frac{d\langle\vec{R}\rangle}{dt} = \frac{\langle\vec{P}\rangle}{m}}\,.$$

$$[P_x, V(r)]\psi = [\frac{\hbar}{i}\frac{\partial}{\partial x}, V(r)]\psi$$

$$= \frac{\hbar}{i}\left(\frac{\partial}{\partial x}(V\psi) - V\frac{\partial\psi}{\partial x}\right)$$

$$= \frac{\hbar}{i}\left(\frac{\partial V}{\partial x}\right)\psi$$

$$\Rightarrow [\vec{P}, V(r)] = \frac{\hbar}{i}\,\vec{\nabla}V$$

$$\Rightarrow \boxed{\frac{d\langle\vec{P}\rangle}{dt} = -\langle\vec{\nabla}V\rangle}$$

e) Solution of the Schrodinger equation:

Let the eigenstates of H be $|\psi_n\rangle$, $H|\psi_n\rangle = E_n|\psi_n\rangle$. Since the $|\psi_n\rangle$ are complete, a solution to the time dependent Schrodinger equation,

$$i\hbar \frac{d|\varphi(t)\rangle}{dt} = H\,|\varphi(t)\rangle,$$

can be written as

$$|\varphi(t)\rangle = \sum_n c_n(t)|\psi_n\rangle .$$

Substituting this into the Schrodinger equation,

$$\sum_n i\hbar \frac{dc_n(t)}{dt}|\psi_n\rangle = \sum_n H\,c_n(t)|\psi_n\rangle$$

$$= \sum_n E_n c_n(t)|\psi_n\rangle .$$

Take the dot product with $\langle\psi_m|$ $\Rightarrow$

$$i\hbar \frac{dc_m}{dt} = E_m c_m \Rightarrow \frac{dc_m}{dt} = -i\frac{E_m}{\hbar} c_m$$

$$\Rightarrow c_m(t) = c_m(0)\, e^{-i\frac{E_m}{\hbar}t}$$

$$\Rightarrow \boxed{|\varphi(t)\rangle = \sum_n c_n(0)\, e^{-i\frac{E_n}{\hbar}t}|\psi_n\rangle, \text{ where } c_n(0) = \langle\psi_n|\varphi(0)\rangle.}$$

f) Time evolution operator:

$$|\varphi(0)\rangle = \sum_n c_n |\psi_n\rangle$$

$$|\varphi(t)\rangle = \sum_n e^{-\frac{iE_n t}{\hbar}} c_n |\psi_n\rangle$$

$$\Rightarrow |\varphi(t)\rangle = \left(\sum_n e^{-\frac{iE_n t}{\hbar}} |\psi_n\rangle\langle\psi_n|\right) |\varphi(0)\rangle$$

$$= e^{-\frac{iHt}{\hbar}} |\varphi(0)$$

time evolution operator $U(t,0)$ for $H(\cancel{t})$

$H(\cancel{t})$ case: $|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$

$$\Rightarrow i\hbar \frac{\partial}{\partial t} U(t,t_0)|\psi(t_0)\rangle = H\, U(t,t_0)|\psi(t_0)\rangle$$

Since this is true for any $|\psi(t_0)\rangle$,

$$i\hbar \frac{\partial}{\partial t} U(t,t_0) = H\, U(t,t_0) \,,$$

$$U(t_0,t_0) = 1 \,.$$

Solution: $\boxed{U(t,t_0) = \exp\left(-\frac{iH}{\hbar}(t-t_0)\right)}$

g) Schrodinger & Heisenberg pictures: (for $H(\cancel{t})$)

(i)... $$\langle A\rangle(t) = \langle\psi(t)|A|\psi(t)\rangle \qquad \text{(Schrodinger)}$$

$$= \langle\psi(t_0)|e^{\frac{iH}{\hbar}(t-t_0)}A\,e^{-\frac{iH}{\hbar}(t-t_0)}|\psi(t_0)\rangle$$

Let $A_H(t) = e^{\frac{iH}{\hbar}(t-t_0)}A\,e^{-\frac{iH}{\hbar}(t-t_0)}$ (Heisenberg)

(ii)... $\Rightarrow$ $$\langle A\rangle(t) = \langle\psi(t_0)|A_H(t)|\psi(t_0)\rangle$$

Notation: $$\boxed{\begin{aligned}\langle A\rangle(t) &= \langle\psi_S(t)|A_S|\psi_S(t)\rangle\\ \langle A\rangle(t) &= \langle\psi_H|A_H(t)|\psi_H\rangle\end{aligned}}$$

$$i\hbar\frac{dA_H}{dt} = i\hbar\frac{d}{dt}\left(e^{\frac{iH}{\hbar}(t-t_0)}A_S\,e^{-\frac{iH}{\hbar}(t-t_0)}\right)$$

$$= -H\,A_H(t) + A_H(t)\,H + e^{\frac{iH}{\hbar}(t-t_0)}\,i\hbar\frac{dA_S}{dt}\,e^{-\frac{iH}{\hbar}(t-t_0)}$$

OR $$\boxed{i\hbar\frac{dA_H}{dt} = [A_H(t),H] + i\hbar\left(\frac{dA_S}{dt}\right)_H}$$