General Two-Level Systems:

no coupling: $H_0|\varphi_1\rangle = E_1|\varphi_1\rangle$

$H_0|\varphi_2\rangle = E_2|\varphi_2\rangle$

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

with coupling: $H = H_0 + W$, $W = \begin{pmatrix} 0 & W_{12} \\ W_{12}^* & 0 \end{pmatrix}$

$$\rightarrow \quad H = \begin{pmatrix} E_1 & W_{12} \\ W_{12}^* & E_2 \end{pmatrix}$$

Eigenvalues & eigenvectors:

$$0 = \begin{vmatrix} E_1 - \lambda & W_{12} \\ W_{12}^* & E_2 - \lambda \end{vmatrix} = (E_1 - \lambda)(E_2 - \lambda) - |W_{12}|^2$$

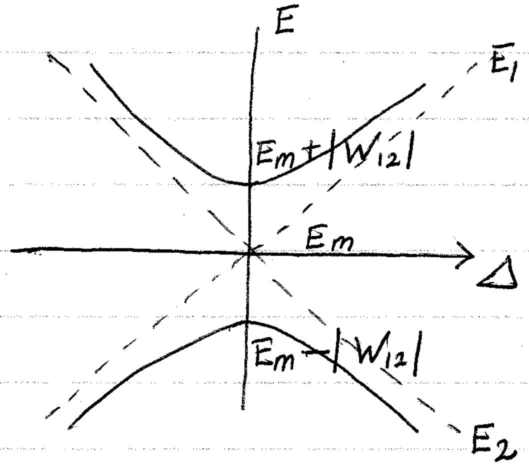
$$\lambda^2 - (E_1 + E_2)\lambda + E_1 E_2 - |W_{12}|^2 = 0$$

$$\rightarrow \lambda = \frac{E_1 + E_2}{2} \pm \frac{\sqrt{(E_1 + E_2)^2 - 4E_1E_2 + 4|W_{12}|^2}}{2}$$

$$= \frac{E_1 + E_2}{2} \pm \frac{1}{2}\sqrt{(E_1 - E_2)^2 + 4|W_{12}|^2}$$

Let $E_m = \frac{1}{2}(E_1 + E_2)$, $\Delta = \frac{1}{2}(E_1 - E_2)$

$$\rightarrow \lambda = E_m \pm \sqrt{\Delta^2 + |W_{12}|^2} \equiv E_\pm$$



$$E_+ = E_m + \sqrt{\Delta^2 + |W_{12}|^2}$$

$$E_- = E_m - \sqrt{\Delta^2 + |W_{12}|^2}$$

For $\Delta \gg |W_{12}|$, $E_+ \approx \frac{E_1 + E_2}{2} + \frac{E_1 - E_2}{2} = E_1$

$$E_- \approx \underbrace{\frac{E_1 + E_2}{2}}_{E_m} - \underbrace{\frac{E_1 - E_2}{2}}_{|\Delta|} = E_2$$

For $\Delta \ll -|W_{12}|$, $E_+ \approx \frac{E_1 + E_2}{2} + \frac{E_2 - E_1}{2} = E_2$

$$E_1 \approx \frac{E_1 + E_2}{2} - \frac{E_2 - E_1}{2} = E_1$$

$|\psi_\pm\rangle$: $\lambda = \dfrac{E_1+E}{2} \pm \sqrt{\Delta^2 + |W_{12}|^2}$

$$0 = \begin{pmatrix} E_1-\lambda & W_{12} \\ W_{12}^* & E_2-\lambda \end{pmatrix} |\psi_\pm\rangle$$

$$= \begin{pmatrix} \Delta \mp \sqrt{\Delta^2+|W_{12}|^2} & W_{12} \\ W_{12}^* & -\Delta \mp \sqrt{\Delta^2+|W_{12}|^2} \end{pmatrix} |\psi_\pm\rangle$$

Limiting cases:

(i) $\Delta = 0$, $\begin{pmatrix} \mp|W_{12}| & W_{12} \\ W_{12}^* & \mp|W_{12}| \end{pmatrix} |\psi_\pm\rangle = 0$
(degenerate)

$$\rightarrow |\psi_\pm\rangle \propto \begin{pmatrix} W_{12} \\ \pm|W_{12}| \end{pmatrix}$$

$$\rightarrow |\psi_\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\varphi} \\ \pm 1 \end{pmatrix}, \text{ where } \frac{W_{12}}{|W_{12}|} = e^{-i\varphi}.$$

For $W_{12} \in \mathbb{R}$ and $W_{12} > 0$, $(\varphi = 0)$, and

$$|\psi_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{(antibonding)}$$

$$|\psi_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{(bonding)}$$

(ii) $\Delta \gg |W_{12}|$, $\begin{pmatrix} \Delta \mp |\Delta| & 0 \\ 0 & -\Delta \mp |\Delta| \end{pmatrix} |\psi_\pm\rangle = 0$

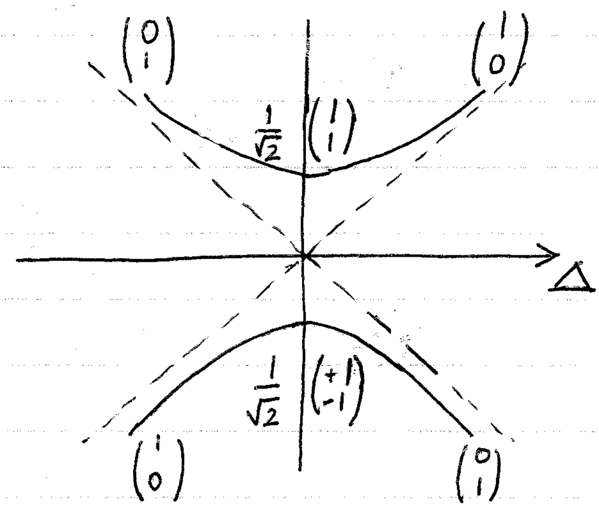
$$\begin{pmatrix} 0 & 0 \\ 0 & -2\Delta \end{pmatrix} |\psi_+\rangle = 0 \rightarrow |\psi_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2\Delta & 0 \\ 0 & 0 \end{pmatrix} |\psi_-\rangle = 0 \rightarrow |\psi_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(iii) $\Delta \ll -|W_{12}|$, $\begin{pmatrix} \Delta \mp |\Delta| & 0 \\ 0 & -\Delta \mp |\Delta| \end{pmatrix} |\psi_\pm\rangle = 0$

$$\begin{pmatrix} 2\Delta & 0 \\ 0 & 0 \end{pmatrix} |\psi_+\rangle = 0 \rightarrow |\psi_+\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -2\Delta \end{pmatrix} |\psi_-\rangle = 0 \rightarrow |\psi_-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



(For $W_{12} > 0$)

General case:

$$H = \begin{pmatrix} E_1 & W_{12} \\ W_{12}^* & E_2 \end{pmatrix} = \left(\frac{E_1+E_2}{2}\right)\mathbb{1}$$

$$+\left(\frac{E_1-E_2}{2}\right)\sigma_z$$

$$+ Re\{W_{12}\}\sigma_x \leftarrow Re\{W_{21}\}\sigma_x$$

$$- Im\{W_{12}\}\sigma_y \leftarrow + Im\{W_{21}\}\sigma_y$$

Let $W_{21} = |W_{21}|e^{i\varphi}$

$$\Rightarrow H = E_m\mathbb{1} + \sqrt{\Delta^2 + |W_{21}|^2}\;\hat{n}\cdot\vec{\sigma}$$

$$\hat{n}_x = \frac{|W_{21}|\cos\varphi}{\sqrt{\Delta^2+|W_{21}|^2}} = \sin\theta\cos\varphi \;;\; \sin\theta = \frac{|W_{21}|}{\sqrt{\Delta^2+|W_{21}|^2}}$$

$$\hat{n}_y = \frac{|W_{21}|\sin\varphi}{\sqrt{\Delta^2+|W_{21}|^2}} = \sin\theta\sin\varphi$$

$$\hat{n}_z = \frac{\Delta}{\sqrt{\Delta^2+|W_{21}|^2}} = \cos\theta \;;\; \cos\theta = \frac{\Delta}{\sqrt{\Delta^2+|W_{21}|^2}}$$

We know the eigenvectors of this matrix:

$$|\psi_+\rangle = \cos\left(\frac{\theta}{2}\right)e^{-i\varphi/2}|\varphi_1\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi/2}|\varphi_2\rangle, \; E_+ = E_m + \sqrt{\Delta^2+|W_{21}|^2}$$

$$|\psi_-\rangle = -\sin\left(\frac{\theta}{2}\right)e^{-i\varphi/2}|\varphi_1\rangle + \cos\left(\frac{\theta}{2}\right)e^{i\varphi/2}|\varphi_2\rangle, \; E_- = E_m - \sqrt{\Delta^2+|W_{21}|^2}$$

Rabi oscillations:

Suppose at $t=0$, $|\psi(t)\rangle = |\varphi_1\rangle$.

$$|\varphi_1\rangle = e^{i\varphi/2}\left(\cos\left(\tfrac{\theta}{2}\right)|\psi_+\rangle - \sin\left(\tfrac{\theta}{2}\right)|\psi_-\rangle\right)$$

$$|\psi(t)\rangle = e^{i\varphi/2}\left(\cos\left(\tfrac{\theta}{2}\right)e^{-i\frac{E_1 t}{\hbar}}|\psi_+\rangle - \sin\left(\tfrac{\theta}{2}\right)e^{-i\frac{E_2 t}{\hbar}}|\psi_-\rangle\right)$$

What is the probability we are in $|\varphi_2\rangle$ at a time t?

$$|\varphi_2\rangle = e^{-i\varphi/2}\left(\sin\left(\tfrac{\theta}{2}\right)|\psi_+\rangle + \cos\left(\tfrac{\theta}{2}\right)|\psi_1\rangle\right)$$

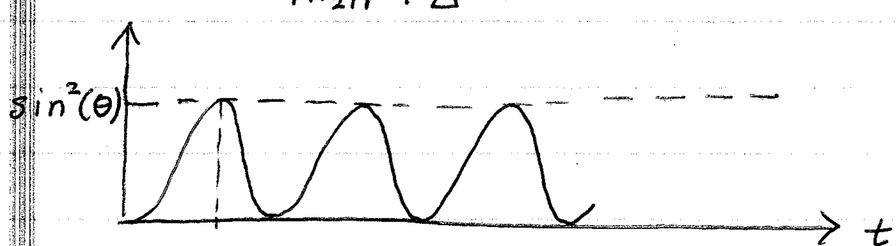
$$\rightarrow \langle\varphi_2|\psi(t)\rangle = e^{i\varphi}\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right)\left[e^{-iE_+t/\hbar} - e^{-iE_-t/\hbar}\right]$$

$$\rightarrow P_{12} = |\langle\varphi_2|\psi(t)\rangle|^2 = \sin^2\left(\tfrac{\theta}{2}\right)\cos^2\left(\tfrac{\theta}{2}\right)\left[2 - 2\cos\left(\tfrac{E_+ - E}{\hbar}t\right)\right]$$

$$= \frac{2}{4}\sin^2(\theta)\left[1 - \cos\left(\frac{E_+ - E_-}{\hbar}\right)t\right]$$

$$= \sin^2(\theta)\sin^2\left(\frac{E_+ - E_-}{2\hbar}t\right)$$

or
$$= \frac{|W_{21}|^2}{|W_{21}|^2 + \Delta^2}\sin^2\left(\frac{\sqrt{\Delta^2 + |W_{21}|^2}}{\hbar}t\right)$$



$$\frac{E_+ - E_1}{2\hbar}t = \frac{\pi}{2}$$

$$\rightarrow t = \pi\hbar/(E_+ - E_-)$$