Solution

Name:

Exam 1 - PHY 4604 - Fall 2002
Monday, September, 30, 2002

This exam is closed book and notes. You are not allowed (nor will you need) a calculator. Please use the space provided on the exam to do the problems. You may also use the backs of pages if additional space is needed.

1. Short answer section:

   (a) What is Planck's constant numerically?

   $\hbar \approx 1 \times 10^{-34}\ J \cdot s$

   (b) What is the continuity equation?

   $$\frac{\partial P}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

   (c) What is the probability current in one dimension?

   $$j = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x}\psi\right)$$

   (d) Given $\psi(p)$ how does one determine $\psi(x)$?

   $$\psi(x) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} e^{i\frac{p}{\hbar}x} \psi(p)$$

   (e) A set of wave functions, $u_n(x)$, for $n = 1, 2, 3 \ldots$ on the interval $[a, b]$ is said to be complete or satisfy the expansion function postulate if:

   any function $f(x)$ on $[a,b]$ can be written as $f(x) = \sum_{n=1}^{\infty} A_n u_n(x)$ for some A_n.

2. Consider the following wave functions on the interval $[-1, 1]$.

$$u_0(x) = \frac{1}{\sqrt{2}} \tag{1}$$

$$u_1(x) = \frac{\sqrt{3}}{\sqrt{2}} x \tag{2}$$

$$u_2(x) = ax^2 + b \tag{3}$$

These wave functions are orthonormal on the interval $[-1, 1]$.

(a) Determine values for the coefficients a and b which make u_2 orthogonal to u_1, u_0 and normalized.

Orthogonality of u_0 & u_2: $\int_{-1}^{1} dx\, u_0(x) u_2(x) = 0$

$$\rightarrow \int_{-1}^{1} dx \frac{1}{\sqrt{2}} (ax^2 + b) = 0$$

$$\rightarrow \frac{1}{\sqrt{2}} \left(\frac{2}{3} a + 2b\right) = 0 \rightarrow b = -\frac{1}{3} a$$

$$\rightarrow u_2(x) = a\left(x^2 - \frac{1}{3}\right)$$

Normalization of u_2: $\int_{-1}^{1} dx\, u_2(x) u_2(x) = 1$

$$= a^2 \int_{-1}^{1} dx \left(x^2 - \frac{1}{3}\right)^2$$

$$= a^2 \int_{-1}^{1} dx \left(x^4 - \frac{2}{3} x^2 + \frac{1}{9}\right)$$

$$= 2a^2 \left(\frac{1}{5} - \frac{2}{3}\frac{1}{3} + \frac{1}{9}\right)$$

$$= 2a^2 \left(\frac{9 - 10 + 5}{45}\right) = \frac{8}{45} a^2 \rightarrow \boxed{a = \sqrt{\frac{45}{8}} \quad b = -\sqrt{\frac{5}{8}}}$$

(b) What is the expectation value of x for u_0, u_1, and u_2?

Since $u_n(x)^2 = u_n(-x)^2$, $\langle x \rangle = \int_{-1}^{1} dx\, u_n(x)^2 x = 0$.

(c) What is Δx for u_1?

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\langle x^2 \rangle}$$

$$\langle x^2 \rangle = \int_{-1}^{1} dx\, \frac{3}{2} x^2 \cdot x^2 = \frac{3}{2} \cdot \frac{2}{5} = \frac{3}{5} \rightarrow \boxed{\Delta x = \sqrt{\frac{3}{5}}}$$

3. Consider the potential in one dimension $V(x) = 0$ for $x > 0$, $V(x) = V_o$ for $x < 0$, and $V(x) = \lambda(\hbar^2/2ma)\delta(x)$ at $x = 0$. Suppose $\underline{0 < E < V_o}$.

(a) What is the form of the solution to the time-independent Schrodinger equation for $x < 0$?

$$Ae^{\rho x} + Be^{-\rho x} \text{, where } \rho = \sqrt{\frac{2m(V_o - E)}{\hbar^2}}$$

For the wavefunction to be normalizable we must have $B = 0$.

(b) What is the form of the solution to the time-independent Schrodinger equation for $x > 0$?

$$Ce^{ikx} + De^{-ikx} \text{, where } k = \sqrt{\frac{2mE}{\hbar^2}}$$

(c) What are the boundary conditions at $x = 0$?

continuity of wavefunction: $\boxed{A = C + D}$

discontinuity of derivative at delta function:

$$\boxed{ik(C - D) - \rho A = \frac{\lambda}{a} A}$$

(d) Solve for the wave function for a wave coming **from the right**.

The second equation can be written as:

$$C - D = \frac{1}{ik}\left(\rho + \frac{\lambda}{a}\right)A$$

First eq: $C + D = A$.

$$\rightarrow \quad 2C = \left(1 + \frac{1}{ik}\left(\rho + \frac{\lambda}{a}\right)\right)A$$

$$2D = \left(1 - \frac{1}{ik}\left(\rho + \frac{\lambda}{a}\right)\right)A$$

For a wave incident from the right take $\boxed{D = 1}$

$$\rightarrow \quad \boxed{\begin{aligned} A &= \frac{2}{1 - \frac{1}{ik}\left(\rho + \frac{\lambda}{a}\right)} \\ C &= \frac{1 + \frac{1}{ik}\left(\rho + \frac{\lambda}{a}\right)}{1 - \frac{1}{ik}\left(\rho + \frac{\lambda}{a}\right)} \end{aligned}}$$

(e) Determine the transmission and reflection probabilities. Is probability conserved?

The reflection probability is $\left|\frac{C}{D}\right|^2 = 1$.

The transmission probability is zero ($E < V_0$), since $e^{\pm \rho x}$ does not carry any probability current (see l.c.).