Exam 2

1. (a) $A^\dagger$ satisfies

$$\int dx\, \varphi^* A \psi = \int dx\, (A^\dagger \varphi)^* \psi$$

for any φ and ψ.

(b) $\delta(x - x_0)$

(c) $$\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m}$$

$$\frac{d\langle p\rangle}{dt} = \left\langle -\frac{dV}{dx}\right\rangle$$

(d) $A(t) = e^{\frac{iHt}{\hbar}} A\, e^{-\frac{iHt}{\hbar}}$

(e) $\sum_n |u_n\rangle\langle u_n| = \mathbb{1}$

2. (a) $1 = \langle\psi(t=0)|\psi(t=0)\rangle$

$$= \left(\sum_{n=0}^{\infty} c_n^* \langle n|\right)\left(\sum_{m=0}^{\infty} c_m |m\rangle\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n^* c_m \delta_{nm}$$

$$\boxed{1 = \sum_{n=0}^{\infty} |c_n|^2}$$

(b) $|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-\frac{iE_n t}{\hbar}} |n\rangle$

$$\boxed{|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-i\omega(n+\frac{1}{2})t} |n\rangle}$$

(c) $$\langle x\rangle_t = \left(\sum_{n=0}^{\infty} c_n^* e^{i\omega(n+\frac{1}{2})t}\langle n|\right)\sqrt{\frac{\hbar}{2m\omega}}\,(a+a^\dagger)\left(\sum_{m=0}^{\infty} c_m e^{-i\omega(m+\frac{1}{2})t}|m\rangle\right)$$

$$= \left(\sum_{n=0}^{\infty} c_n^* e^{i\omega(n+\frac{1}{2})t}\langle n|\right)\times \sqrt{\frac{\hbar}{2m\omega}}\left(\sum_{m=0}^{\infty} c_m e^{-i\omega(m+\frac{1}{2})t}\left(\sqrt{m}\,|m-1\rangle + \sqrt{m+1}\,|m+1\rangle\right)\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n^* c_m e^{i\omega(n-m)t}\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{m}\,\delta_{n,m-1} + \sqrt{m+1}\,\delta_{n,m+1}\right)$$

$$\boxed{\langle x\rangle_t = \sqrt{\frac{\hbar}{2m\omega}}\sum_{m=0}^{\infty} c_{m-1}^* c_m e^{-i\omega t}\sqrt{m} + c_{m+1}^* c_m e^{i\omega t}\sqrt{m+1}}$$

(d) Suppose c_N and $c_{N+1} \neq 0$ and all other terms are zero.

$$\langle x\rangle_t = \sqrt{\frac{\hbar}{2m\omega}}\left(c_N^* c_{N+1} e^{-i\omega t}\sqrt{N+1} + c_{N+1}^* c_N e^{i\omega t}\sqrt{N+1}\right)$$

Let $c_N = \frac{1}{\sqrt{2}}$ and $c_{N+1} = \frac{i}{\sqrt{2}}$.

$$\rightarrow \langle x\rangle_t = \sqrt{\frac{\hbar}{2m\omega}}\,\frac{1}{2}\left(i e^{-i\omega t} - i e^{i\omega t}\right)\sqrt{N+1}$$

$$= \sqrt{\frac{N+1}{2}}\sqrt{\frac{\hbar}{2m\omega}}\,\sin(\omega t)$$

$$\rightarrow 5 = \sqrt{\frac{N+1}{2}} \rightarrow N+1 = 50 \rightarrow N = 49$$

One soln. is: $|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega(49+\frac{1}{2})t}|49\rangle + \frac{i}{\sqrt{2}} e^{-i\omega(50+\frac{1}{2})t}|50\rangle.$

(e) $[a^\dagger, a^3] = a^\dagger a^3 - a^3 a^\dagger$

$$\begin{aligned} &= a^\dagger a a^2 - a a^\dagger a^2 && = [a^\dagger, a]\,a^2 \\ &+ a a^\dagger a^2 - a^2 a^\dagger a && + a[a^\dagger, a]a \\ &+ a^2 a^\dagger a - a^3 a^\dagger && + a^2[a^\dagger, a] \end{aligned}$$

$$= -a^2 - a^2 - a^2 = -3a^2.$$

3. Let $E_n = \frac{\hbar^2}{2m}\frac{\pi^2 n^2}{a^2}$ and $\varphi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right)$.

(a) $E = E_2 + E_3$

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\varphi_2(x_1)\varphi_3(x_2) - \varphi_3(x_1)\varphi_2(x_2)\right)$$

(b) $|\psi(x_1, x_2)|^2 dx_1 dx_2$ is the probability to find an electron between x_1 & $x_1 + dx_1$ and an electron between x_2 & $x_2 + dx_2$.

(c) Probability of either electron at $x = x_0$ is

$$\int_0^a dx_1 |\psi(x_1, x_0)|^2 + \int_0^a dx_2 |\psi(x_0, x_2)|^2$$

$$= \frac{2}{2}\int_0^a dx_1 \left|\varphi_2(x_1)\varphi_3(x_0) - \varphi_3(x_1)\varphi_2(x_0)\right|^2$$

$$= |\varphi_3(x_0)|^2 + |\varphi_2(x_0)|^2$$

(d) $E = E_2 + E_2$ since

groundstate $E = E_1 + E_1$
1^{st} excited state $E = E_1 + E_2$
and $E_2 + E_2 < E_1 + E_3$.

$$\psi(x_1, x_2) = \varphi_2(x_1)\varphi_2(x_2)$$