Solution

Name:

**Exam 3 - PHY 4604 - Fall 2002**
Monday, December 9, 2002

This exam is closed book and notes. You are not allowed (nor will you need) a calculator. Please use the space provided on the exam to do the problems. You may also use the backs of pages if additional space is needed.

1. **Short answer section**

   (a) What is the orthogonality condition for the spherical harmonics?

$$\langle \ell m | \ell' m' \rangle = \delta_{\ell\ell'} \delta_{mm'} = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi \, Y_{\ell m}(\theta,\varphi)^* Y_{\ell' m'}(\theta,\varphi)$$

   (b) What is the form of the solution to the Schrodinger equation for a central potential?

$$\psi(r,\theta,\varphi) = Y_{\ell m}(\theta,\varphi) R(r)$$

   (c) What is the radial Schrodinger equation for $u(r)$? How is $u(r)$ related to $R(r)$? What is the normalization condition for $u(r)$?

$$\left( -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} + V(r) \right) u(r) = E\, u(r)$$

$$\int_0^{\infty} dr \, |u(r)|^2 = 1$$

   (d) What is the Bohr radius (numerically)?

$$a_0 \approx 0.52 \, \text{Å}$$

   (e) What are the bound state energies of the hydrogen atom?

$$E_n = -E_I/n^2 \quad , \text{where } E_I = 13.6 \, eV \text{ (Rydberg)}$$

2. **Angular momentum**

(a) Compute the commutator $[L_x^2, L_y^2]$.

$$[L_x^2, L_y^2] = L_x^2 L_y^2 - L_y^2 L_x^2 - L_x L_y^2 L_x + L_x L_y^2 L_x$$

$$= L_x [L_x, L_y^2] + [L_x, L_y^2] L_x$$

$$[L_x, L_y^2] = L_x L_y^2 - L_y^2 L_x - L_y L_x L_y + L_y L_x L_y$$

$$= [L_x, L_y] L_y + L_y [L_x, L_y]$$

$$= i\hbar (L_z L_y + L_y L_z)$$

$$\rightarrow [L_x^2, L_y^2] = i\hbar (L_x L_z L_y + L_x L_y L_z + L_z L_y L_x + L_y L_z L_x).$$

(b) For $l = 2$ compute the matrix representation of the operator $L_y$.

$$\left.\begin{array}{l} L_+ = L_x + iL_y \\ L_- = L_x - iL_y \end{array}\right\} \longrightarrow L_y = \frac{1}{2i}(L_+ - L_-)$$

The non-zero matrix elements of $L_+$ for $\ell = 2$ are:

$$L_+|2,-2\rangle = \hbar\sqrt{2\cdot 3 - -2(-1)}\;|2,-1\rangle = 2\hbar|2,-1\rangle$$
$$L_+|2,-1\rangle = \hbar\sqrt{2\cdot 3 - 0}\;|2,0\rangle = \sqrt{6}\,\hbar|2,0\rangle$$
$$L_+|2,0\rangle = \hbar\sqrt{2\cdot 3 - 0}\;|2,1\rangle = \sqrt{6}\,\hbar|2,1\rangle$$
$$L_+|2,1\rangle = \hbar\sqrt{2\cdot 3 - 1\cdot 2}\;|2,2\rangle = 2\hbar|2,2\rangle$$

$$\longrightarrow L_+ = \hbar\begin{pmatrix} 0 & 2 & & & \\ & 0 & \sqrt{6} & & \\ & & 0 & \sqrt{6} & \\ & & & 0 & 2 \\ & & & & 0 \end{pmatrix} \quad \text{and } L_- = \hbar\begin{pmatrix} 0 & & & & \\ 2 & 0 & & & \\ & \sqrt{6} & 0 & & \\ & & \sqrt{6} & 0 & \\ & & & 2 & 0 \end{pmatrix}$$

$$\longrightarrow L_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & 2 & & & \\ -2 & 0 & \sqrt{6} & & \\ & -\sqrt{6} & 0 & \sqrt{6} & \\ & & -\sqrt{6} & 0 & 2 \\ & & & -2 & 0 \end{pmatrix}$$

3. **Central potentials**

Consider the the potential in three dimensions: $V(r) = 0$ for $r < a$ and $V(r) = \infty$ for $r \geq a$.

(a) Using the spherical Bessel functions what is the general form of the radial wave function, $R(r)$, for angular momentum $l$ and energy $E > 0$?

$R(r) = A j_l(kr) + B n_l(kr)$, where $E = \frac{\hbar^2 k^2}{2m}$.

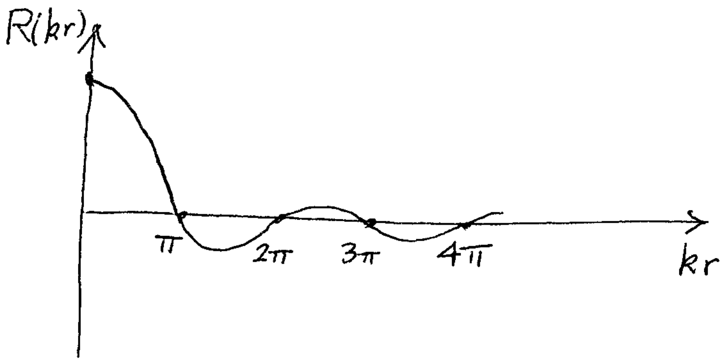
(b) What is the boundary condition at $r = a$? What is the behavior of $R(r)$ as $r \to 0$?

$R(a) = 0$ and $R(r) \sim r^{l+1}$ as $r \to 0$.

(c) What is the solution for $R(r)$ for $l = 0$ consistent with (b) and (c) above? Specify $R(r)$ explicitly up to an overall normalization constant.

$R(r) = A j_0(kr) = A \frac{\sin(kr)}{kr}$

(d) Plot the radial wave function, $R(r)$.




(e) Determine the allowed energies.

$ka = n\pi$ for $n = 1, 2, 3, \ldots$

$\rightarrow k = \frac{n\pi}{a}$ and $E = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2$

4. **Spin**

(a) Consider a system of spin 1/2. What are the eigenvalues and eigenvectors of the operator $(S_x + S_z)/\sqrt{2}$? This operator is the spin operator in the $(\mathbf{x} + \mathbf{z})/\sqrt{2}$ direction.

$$\frac{S_x + S_z}{\sqrt{2}} = \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad \text{Diagonalize } \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}:$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 1 - 1 = 0 \longrightarrow \lambda = \pm\sqrt{2}$$

$$\lambda = \sqrt{2}: \begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0 \rightarrow C_1 = (+1+\sqrt{2})C_2 \ \& \ C_2 = (\sqrt{2}-1)C_1$$

$$(1+\sqrt{2})^2 + 1 = 1 + 2 + 2\sqrt{2} + 1 = 4 + 2\sqrt{2}$$

$$\lambda = -\sqrt{2}: \begin{pmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0 \longrightarrow C_1 = (1-\sqrt{2})C_2 \ \& \ C_2 = -(1+\sqrt{2})C_1$$

$$(1-\sqrt{2})^2 + 1 = 1 + 2 - 2\sqrt{2} + 1 = 4 - 2\sqrt{2}$$

$\Longrightarrow$

| eigenvalue | eigenvector |
|---|---|
| $+\hbar/2$ | $\frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$ |
| $-\hbar/2$ | $\frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} -1 \\ +(1+\sqrt{2}) \end{pmatrix}$ |

(b) Suppose a measurement of this operator is made, and the system is found to be in the state corresponding to the smaller eigenvalue. What is the probability that a subsequent measurement of $S_z$ yields $-\hbar/2$?

$$\text{Prob.}\left(S_z = -\frac{\hbar}{2}\right) = \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} \right|^2$$

$$= \frac{1}{4-2\sqrt{2}}$$

$$= \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} -1 \\ 1+\sqrt{2} \end{pmatrix} \right|^2$$

$$= \frac{1+2\sqrt{2}+2}{4+2\sqrt{2}} = \frac{3+2\sqrt{2}}{4+2\sqrt{2}}$$

(c) How would the results of part (b) change if we measured $S_x$ instead of $S_z$?

$$\text{Prob.}\left(S_x = -\frac{\hbar}{2}\right) = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} \right|^2$$

$$= \frac{1}{2} \frac{1}{4-2\sqrt{2}} \left(1-\sqrt{2}-1\right)^2$$

$$= \frac{1}{4-2\sqrt{2}}$$

There is no change in the probability.