Name: Solution

Final Exam - PHY 4604 - Fall 2002
Wednesday, December 18, 2002

This exam is closed book and notes. You are not allowed (nor will you need) a calculator. Please use the space provided on the exam to do the problems. You may also use the backs of pages if additional space is needed.

1. **Short answer section**

(a) What is Planck's constant numerically? Make sure to specify your units.

$\hbar \approx 1 \times 10^{-34}$ J·sec

(b) What is the time dependent Schrodinger equation?

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

(c) What is the definition of the Hermitian conjugate of an operator A?

The Hermitian conjugate of an operator A is denoted by $A^\dagger$ and satisfies

$$\langle \varphi | A\psi \rangle = \int dx\, \varphi^* A\psi = \int dx\, (A^\dagger \varphi)^* \psi = \langle A^\dagger \varphi | \psi \rangle.$$

(d) Express the time dependent Heisenberg operator, $A(t)$, in terms of the time independent Schrodinger operator, A.

$$A(t) = e^{iHt/\hbar} A e^{-iHt/\hbar}$$

(e) What are the bound state energies of the hydrogen atom?

$E = -E_I/n^2$, where $E_I \approx 13.6$ eV.

2. **One Dimensional Potentials**

Consider the one dimensional potential $V(x) = V_0$ for $x < 0$ and $V(x) = V_1$ for $x > 0$. Suppose that the energy, E, is greater than both V_0 and V_1.

(a) What is the form of the solution to the time independent Schrodinger equation for $x < 0$?

$$\psi(x<0) = Ae^{ik_0x} + Be^{-ik_0x},$$

where $k_0 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$.

(b) What is the form of the solution to the time independent Schrodinger equation for $x > 0$?

$$\psi(x>0) = Ce^{ik_1x} + De^{-ik_1x},$$

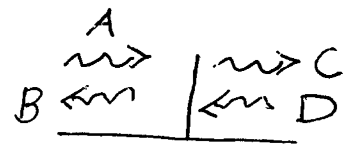
where $k_1 = \sqrt{\frac{2m(E-V_1)}{\hbar^2}}$

(c) What are the boundary conditions at $x = 0$?

$$\psi(0) = A + B = C + D$$

$$\psi'(0) = ik_0(A-B) = ik_1(C-D)$$

(d) Solve for the wave function for a wave coming from the left.



For a wavefunction coming from the left, $D = 0$.

$$\Rightarrow \quad A + B = C$$

$$i k_0 (A - B) = i k_1 C \quad \text{or} \quad A - B = \frac{k_1}{k_0} C$$

$$\Rightarrow 2A = \left(1 + \frac{k_1}{k_0}\right) C \quad \text{and} \quad 2B = \left(1 - \frac{k_1}{k_0}\right) C$$

$$\Rightarrow \boxed{\frac{C}{A} = \frac{2}{1 + k_1/k_0}} \quad \text{and} \quad \boxed{\frac{B}{A} = \frac{1 - k_1/k_0}{1 + k_1/k_0}}$$

(e) Determine the transmission and reflection probabilities. Is probability conserved?

Reflection probability $= R = \left|\frac{B}{A}\right|^2 = \left(\frac{1 - k_1/k_0}{1 + k_1/k_0}\right)^2 = \left(\frac{k_0 - k_1}{k_0 + k_1}\right)^2$

Transmission prob. $= T = \frac{k_1}{k_0}\left|\frac{C}{A}\right|^2 = \frac{k_1}{k_0}\left(\frac{2}{1 + k_1/k_0}\right)^2 = \frac{4 k_0 k_1}{(k_0 + k_1)^2}$

Since $\boxed{R = \left(\frac{k_0 - k_1}{k_0 + k_1}\right)^2 \text{ and } T = \frac{4 k_0 k_1}{(k_0 + k_1)^2}}$,

$$R + T = \frac{k_0^2 - 2 k_0 k_1 + k_1^2 + 4 k_0 k_1}{k_0^2 + 2 k_0 k_1 + k_1^2} = 1.$$

Probability is conserved.

3. **Simple Harmonic Oscillator & Time Dependence**

Consider the harmonic oscillator hamiltonian,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

The state of the system at $t = 0$ is

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\phi}}{\sqrt{2}}|1\rangle.$$

(a) What is the wave function at time t: $|\psi(t)\rangle$?

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}\frac{\hbar\omega}{2}t}|0\rangle + \frac{1}{\sqrt{2}} e^{i\varphi} e^{-\frac{i}{\hbar}\frac{3\hbar\omega}{2}t}|1\rangle$$

$$= \frac{1}{\sqrt{2}} e^{-i\frac{\omega}{2}t}|0\rangle + \frac{1}{\sqrt{2}} e^{i\varphi} e^{-i\frac{3\omega}{2}t}|1\rangle$$

(b) What is the expectation of the position operator at time t, $\langle x\rangle_t$? The position operator for the harmonic oscillator is $x = \sqrt{\hbar/2m\omega}(a + a^\dagger)$.

$$a|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{i\varphi} e^{-i\frac{3\omega}{2}t}|0\rangle$$

$$\langle\psi(t)|a|\psi(t)\rangle = \frac{1}{2} e^{i\varphi} e^{-i\omega t}$$

$$a^\dagger|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\frac{\omega}{2}t}|1\rangle + \frac{1}{\sqrt{2}} e^{i\varphi} e^{-i\frac{3\omega}{2}t}\sqrt{2}\,|2\rangle$$

$$\langle\psi(t)|a^\dagger|\psi(t)\rangle = \frac{1}{2} e^{-i\varphi} e^{i\omega t}$$ (as expected from $\langle\psi|a|\psi\rangle^* = \langle\psi|a^\dagger|\psi\rangle$)

$$\Rightarrow \langle x\rangle_t = \sqrt{\frac{\hbar}{2m\omega}}\,\frac{e^{i(\omega t-\varphi)} + e^{-i(\omega t-\varphi)}}{2}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t - \varphi)$$

(c) What is the expectation of the momentum at time t, $\langle p \rangle_t$? The momentum operator for the harmonic oscillator is $p = \sqrt{m\hbar\omega/2}i(a^\dagger - a)$.

$$\langle p \rangle_t = \sqrt{\frac{m\hbar\omega}{2}}\, i \left(\frac{e^{i(\omega t-\varphi)} - e^{-i(\omega t-\varphi)}}{2}\right)$$

$$= -\sqrt{\frac{m\hbar\omega}{2}} \sin(\omega t-\varphi)$$

(d) Test the uncertainty principle for $|\psi(t)\rangle$.

$$x^2 = \frac{\hbar}{2m\omega}(a+a^\dagger)^2 = \frac{\hbar}{2m\omega}(a^2 + a^\dagger a + a a^\dagger + a^{\dagger 2})$$

$$= \frac{\hbar}{2m\omega}(a^2 + 2a^\dagger a + 1 + a^{\dagger 2})$$

$$\rightarrow \frac{\langle \psi(t)|x^2|\psi(t)\rangle}{\hbar/2m\omega} = \frac{1}{2}\langle 0|2a^\dagger a + 1|0\rangle + \frac{1}{2}\langle 1|2a^\dagger a + 1|1\rangle$$

$$= \frac{1}{2} + \frac{3}{2} = 2$$

$$\rightarrow \boxed{(\Delta x)^2 = \frac{\hbar}{m\omega}\left(1 - \frac{1}{2}\cos^2(\omega t-\varphi)\right)}$$

$$p^2 = -\frac{m\hbar\omega}{2}(a^2 - 2a^\dagger a - 1 + a^{\dagger 2})$$

$$\rightarrow \langle \psi(t)|p^2|\psi(t)\rangle = m\hbar\omega$$

$$\rightarrow \boxed{(\Delta p)^2 = m\hbar\omega\left(1 - \frac{1}{2}\sin^2(\omega t-\varphi)\right)}$$

$$\Delta x \Delta p = \hbar\left\{\left(1-\frac{1}{2}\cos^2(\omega t-\varphi)\right)\left(1-\frac{1}{2}\sin^2(\omega t-\varphi)\right)\right\}^{1/2} \geq \frac{\hbar}{2}$$

4. **Spin**

Consider a spin 1/2 system. For your reference the Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) What are the eigenvalues and eigenvectors of the operator $(S_x + S_y)/\sqrt{2}$?

$$\frac{S_x + S_y}{\sqrt{2}} = \frac{\hbar}{2}\begin{pmatrix} 0 & \frac{1-i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\pi/4} \\ e^{i\pi/4} & 0 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & e^{-i\pi/4} \\ e^{i\pi/4} & -\lambda \end{vmatrix} = 0 = \lambda^2 - 1 \Rightarrow \lambda = \pm 1.$$

$$\lambda = 1, \text{ eigenvalue} = \frac{\hbar}{2}, \begin{pmatrix} -1 & e^{-i\pi/4} \\ e^{i\pi/4} & -1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

$$\rightarrow \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{+i\pi/4} \end{pmatrix} = \text{eigenvector}$$

$$\lambda = -1, \text{ eigenvalue} = -\frac{\hbar}{2}, \begin{pmatrix} 1 & e^{-i\pi/4} \\ e^{i\pi/4} & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

$$\rightarrow \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ e^{i\pi/4} \end{pmatrix} = \text{eigenvector}$$

(b) Suppose a measurement of this operator is made, and the system is found to be in the state corresponding to the larger eigenvalue. What is the probability that a subsequent measurement of S_z yields $\hbar/2$?

$$\text{Prob.}\{S_z = \tfrac{\hbar}{2}\} = \left|(1 \quad 0)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ e^{i\pi/4}\end{pmatrix}\right|^2 = \frac{1}{2}$$

(c) How would the result of part (b) change if we measure S_x instead of S_z?

$$\text{Prob.}\{S_x = \tfrac{\hbar}{2}\} = \left|\frac{1}{\sqrt{2}}(1 \quad 1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ e^{i\pi/4}\end{pmatrix}\right|^2$$

$$= \frac{1}{4}\left|1 + e^{i\pi/4}\right|^2$$

$$= \frac{1}{4}\left(2 + e^{i\pi/4} + e^{-i\pi/4}\right)$$

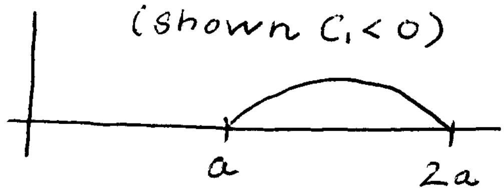
$$= \frac{1}{4}\left(2 + 2\cos\left(\frac{\pi}{4}\right)\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$

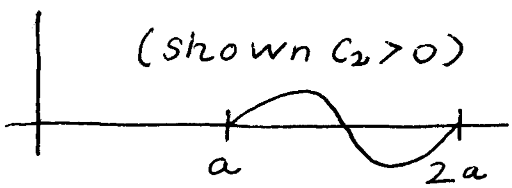
5. **Orthogonality & Completeness**

Consider a particle in a box between $x = a$ and $x = 2a$. In other words the potential is zero for $a < x < 2a$, and infinite elsewhere.

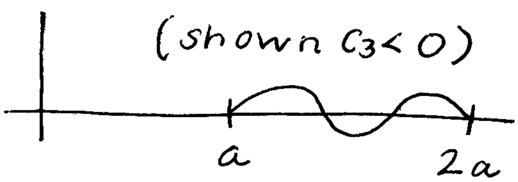
(a) Sketch the wave functions for the lowest three eigenstates.



$$\psi_1 = C_1 \sin\left(\frac{\pi x}{a}\right)$$



$$\psi_2 = C_2 \sin\left(\frac{2\pi x}{a}\right)$$



$$\psi_3 = C_3 \sin\left(\frac{3\pi x}{a}\right)$$

(b) What are the eigenstates and eigenvalues of the hamiltonian for this problem? Make sure to include the correct normalization constant.

Take $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$. Note that $\langle \psi_n | \psi_n \rangle = 1$.

$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2$, Here, $n = 1, 2, 3 \ldots$

(c) Show that the ground state and the first excited states are orthogonal.

$$\langle \psi_2 | \psi_1 \rangle = \frac{2}{a} \int_a^{2a} dx \, \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right)$$

$$= \frac{1}{a} \int_a^{2a} dx \left(\cos\left(\frac{\pi x}{a}\right) - \cos\left(\frac{3\pi x}{a}\right) \right)$$

$$= \frac{1}{a} \left(\frac{a}{\pi} \sin\left(\frac{\pi x}{a}\right) - \frac{a}{3\pi} \sin\left(\frac{3\pi x}{a}\right) \right) \Big|_a^{2a}$$

$$= 0$$

(d) Assuming that the eigenstates of the hamiltonian form a complete basis, express the function $f(x) = 1$ for $a < x < 1.5a$ and zero otherwise as a linear combination of the eigenstates. Make sure to determine the coefficients used in the linear combination.

$$\boxed{f(x) = \sum_{n=1}^{\infty} C_n \psi_n(x)}, \text{ where}$$

$$C_n = \langle \psi_n | f \rangle = \sqrt{\frac{2}{a}} \int_a^{2a} dx \, \sin\left(\frac{\pi n x}{a}\right) f(x)$$

$$= \sqrt{\frac{2}{a}} \int_a^{3a/2} dx \, \sin\left(\frac{n\pi x}{a}\right)$$

$$= \sqrt{\frac{2}{a}} \frac{a}{n\pi} \left(-\cos\left(\frac{n\pi x}{a}\right) \right) \Big|_a^{3a/2}$$

$$\boxed{C_n = \sqrt{\frac{2}{a}} \frac{a}{n\pi} \left(\cos(n\pi) - \cos(3n\pi/2) \right)}$$

6. **Angular Momentum**

(a) For $l = 1$ determine the matrix representation of the operator L_y.

$$\left.\begin{aligned} L_+ &= L_x + i L_y \\ L_- &= L_x - i L_y \end{aligned}\right\} \rightarrow L_y = \frac{L_+ - L_-}{2i} = \frac{L_+ - (L_+)^\dagger}{2i}$$

$$L_+ |1,0\rangle = \hbar\sqrt{1(1+1) - 0\cdot(1+0)}\, |1,1\rangle = \hbar\sqrt{2}\, |1,1\rangle$$

$$L_+ |1,-1\rangle = \hbar\sqrt{1(1+1) + 1\cdot(1-1}\, |1,0\rangle = \hbar\sqrt{2}\, |1,0\rangle$$

$$\rightarrow L_+ = \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \boxed{L_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}}.$$

(b) What is the eigenvector of L_y with zero eigenvalue?

$$\frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0 \rightarrow C_2 = 0 \text{ and } C_1 = C_3.$$

$$\rightarrow \text{eigenvector} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(c) Assuming one is in the zero eigenvalue state of part (b), what is the probability that a measurement of L_z will also yield zero?

$$\text{Prob.}\{L_z = 0\} = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right|^2 = 0$$

(d) What is the probability that a measurement of L_z will yield $\hbar$ instead of 0?

$$\text{Prob.}\{L_z = \hbar\} = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \frac{1}{2}.$$