Homework 11:

2.

(a) $H = \hbar\omega \begin{pmatrix} \frac{1}{2} & & & \\ & \frac{3}{2} & & \\ & & \frac{5}{2} & \\ & & & \ddots \end{pmatrix}$; $H|\psi\rangle = \frac{\hbar\omega}{\sqrt{6}} \begin{pmatrix} \frac{1}{2} \\ 3 \\ \frac{5}{2} \\ 0 \\ \vdots \end{pmatrix}$;

$$\langle\psi|H|\psi\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 2 & 1 & 0 & \cdots \end{pmatrix} \frac{\hbar\omega}{\sqrt{6}} \begin{pmatrix} \frac{1}{2} \\ 3 \\ \frac{5}{2} \\ \vdots \end{pmatrix} = \frac{\hbar\omega}{6}\left(\frac{1}{2} + 6 + \frac{5}{2}\right) = \hbar\omega\frac{9}{6}$$

$$\Rightarrow \boxed{\langle H\rangle = \frac{3}{2}\hbar\omega}$$

(b) $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) = \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & \sqrt{1} & & & \\ \sqrt{1} & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & \sqrt{3} & \\ & & \sqrt{3} & 0 & \\ & & & & \ddots \end{pmatrix}$

$P = \sqrt{\frac{m\omega\hbar}{2}}\, i(a^\dagger - a) = \sqrt{\frac{m\omega\hbar}{2}}\, i \begin{pmatrix} 0 & -\sqrt{1} & & \\ \sqrt{1} & 0 & -\sqrt{2} & \\ & \sqrt{2} & 0 & -\sqrt{3} \\ & & \sqrt{3} & 0 \end{pmatrix}$

$x|\psi\rangle = \sqrt{\frac{\hbar}{2m\omega}} \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1+\sqrt{2} \\ 2\sqrt{2} \\ \sqrt{3} \\ 0 \\ \vdots \end{pmatrix}$; $\langle\psi|x|\psi\rangle = \sqrt{\frac{\hbar}{2m\omega}} \frac{1}{6}(2 + 2 + 2\sqrt{2} + 2\sqrt{2})$

$$\boxed{\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\, \frac{2}{3}(1 + \sqrt{2})}$$

$x^2|\psi\rangle = \left(\frac{\hbar}{2m\omega}\right)\frac{1}{\sqrt{6}} \begin{pmatrix} 1+\sqrt{2} \\ 2+4 \\ 2+\sqrt{2}+3 \\ \\ \end{pmatrix}$; $\langle\psi|x^2|\psi\rangle = \left(\frac{\hbar}{2m\omega}\right)\frac{1}{6}(1 + \sqrt{2} + 12 + 2 + \sqrt{2} + 3)$

$$\boxed{\langle x^2\rangle = \left(\frac{\hbar}{2m\omega}\right)\frac{1}{3}(9 + \sqrt{2})}$$

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2 = \left(\frac{\hbar}{2m\omega}\right)\left(\frac{9+\sqrt{2}}{3} - \frac{4(1+\sqrt{2})^2}{9}\right)$$

$$= \left(\frac{\hbar}{2m\omega}\right)\left(3 + \frac{\sqrt{2}}{3} - \frac{4}{3} - \frac{8\sqrt{2}}{9}\right)$$

$$\boxed{(\Delta x)^2 = \left(\frac{\hbar}{2m\omega}\right)\left(\frac{5}{3} - \frac{5\sqrt{2}}{9}\right)}$$

$$p|\psi\rangle = \sqrt{\frac{m\omega\hbar}{2}}\, i \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1-\sqrt{2} \\ 2\sqrt{2} \\ \sqrt{3} \\ 0 \\ \vdots \end{pmatrix} ; \quad \langle\psi|p|\psi\rangle = \sqrt{\frac{m\omega\hbar}{2}}\, i \frac{1}{6}(-2+2-2\sqrt{2}+2\sqrt{2})$$

$$\boxed{\langle p\rangle = 0}$$

$$p^2|\psi\rangle = \frac{m\omega\hbar}{2} \frac{-1}{\sqrt{6}} \begin{pmatrix} \sqrt{2}-1 \\ -2-4 \\ \sqrt{2}-5 \end{pmatrix} ; \quad \langle\psi|p^2|\psi\rangle = \frac{m\omega\hbar}{2} \frac{-1}{6}(\sqrt{2}-1-12+\sqrt{2}-5)$$

$$= \frac{m\omega\hbar}{2} \frac{1}{6}(18-2\sqrt{2})$$

$$\boxed{\langle p^2\rangle = \frac{m\omega\hbar}{2} \frac{1}{3}(9-\sqrt{2})}$$

$$\boxed{(\Delta p)^2 = \langle p^2\rangle = \frac{m\omega\hbar}{2} \frac{9-\sqrt{2}}{3}}$$

(c) $\Delta x \Delta p = \hbar \left\{ \frac{5}{3}\left(1-\frac{\sqrt{2}}{3}\right)\left(\frac{9-\sqrt{2}}{3}\right) \right\}^{1/2} \approx 1.5\hbar \geqslant \frac{\hbar}{2}$.

4. The non-zero elements of L_+ are:

$$\langle \tfrac{3}{2}, \tfrac{3}{2} | L_+ | \tfrac{3}{2}, \tfrac{1}{2} \rangle = \hbar \sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2} - \tfrac{1}{2}\cdot\tfrac{3}{2}} = \sqrt{3}\,\hbar$$

$$\langle \tfrac{3}{2}, \tfrac{1}{2} | L_+ | \tfrac{3}{2}, -\tfrac{1}{2} \rangle = \hbar \sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2} - (-\tfrac{1}{2})\tfrac{1}{2}} = 2\hbar$$

$$\langle \tfrac{3}{2}, -\tfrac{1}{2} | L_+ | \tfrac{3}{2}, -\tfrac{3}{2} \rangle = \hbar \sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2} - (-\tfrac{3}{2})(-\tfrac{1}{2})} = \sqrt{3}\,\hbar$$

$$\Rightarrow L_+ = \hbar \begin{pmatrix} 0 & \sqrt{3} & & \\ & 0 & 2 & \\ & & 0 & \sqrt{3} \\ & & & 0 \end{pmatrix} \Rightarrow L_- = (L_+)^\dagger = \hbar \begin{pmatrix} 0 & & & \\ \sqrt{3} & 0 & & \\ & 2 & 0 & \\ & & \sqrt{3} & 0 \end{pmatrix}$$

$$\Rightarrow L_x = \frac{L_+ + L_-}{2} = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & & \\ \sqrt{3} & 0 & 2 & \\ & 2 & 0 & \sqrt{3} \\ & & \sqrt{3} & 0 \end{pmatrix}$$

$$L_y = \frac{L_+ - L_-}{2i} = \frac{\hbar}{2i} \begin{pmatrix} 0 & \sqrt{3} & & \\ -\sqrt{3} & 0 & 2 & \\ & -2 & 0 & \sqrt{3} \\ & & -\sqrt{3} & 0 \end{pmatrix}$$

Check the commutator:

$$L_x L_y = \frac{\hbar^2}{4i} \begin{pmatrix} -3 & 0 & 2\sqrt{3} & 0 \\ 0 & -1 & 0 & 2\sqrt{3} \\ -2\sqrt{3} & 0 & +1 & 0 \\ 0 & -2\sqrt{3} & -3 & 3 \end{pmatrix}$$

$$L_y L_x = \frac{\hbar^2}{4i} \begin{pmatrix} 3 & 0 & 2\sqrt{3} & 0 \\ 0 & 1 & 0 & 2\sqrt{3} \\ -2\sqrt{3} & 0 & -1 & 0 \\ 0 & -2\sqrt{3} & 0 & -3 \end{pmatrix}$$

$$\rightarrow L_x L_y - L_y L_x = \frac{\hbar^2}{4i} \begin{pmatrix} -6 & & & \\ & -2 & & \\ & & 2 & \\ & & & 6 \end{pmatrix} = i\hbar \frac{\hbar}{2} \begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & 3 \end{pmatrix} = i\hbar L_z. //$$

7. The eigenstate of S_y with eigen value of $-\hbar/2$ is

$$|\downarrow,\hat{y}\rangle = \begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

The probability of being in this state is:

$$|\langle\downarrow,\hat{y}|\psi\rangle|^2 = \left| \left(\tfrac{-i}{\sqrt{2}} \ \tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} \cos\alpha \\ \sin\alpha\, e^{i\beta} \end{pmatrix} \right|^2$$

$$= \left| \frac{-i\cos\alpha}{\sqrt{2}} + \frac{\sin\alpha}{\sqrt{2}}(\cos\beta + i\sin\beta) \right|^2$$

$$= \frac{1}{2}\left((\sin\alpha\cos\beta)^2 + (\sin\alpha\sin\beta - \cos\alpha)^2 \right)$$

$$= \frac{1}{2}\left(\sin^2\alpha\cos^2\beta + \sin^2\alpha\sin^2\beta + \cos^2\alpha - 2\sin\alpha\sin\beta\cos\alpha \right)$$

$$= \frac{1}{2}\left(\sin^2\alpha + \cos^2\alpha - (2\sin\alpha\cos\alpha)\sin\beta \right)$$

$$\boxed{|\langle\downarrow,\hat{y}|\psi\rangle|^2 = \frac{1}{2}(1 - \sin(2\alpha)\sin\beta)}$$

10. $L_x = \dfrac{L_+ + L_-}{2} = \dfrac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$

The eigenvalues & eigenvectors of L_x are:

$$\lambda = \hbar: \ |\psi\rangle = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}; \quad \lambda = 0: \ |\psi\rangle = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix};$$

$$\lambda = -\hbar: \ |\psi\rangle = \begin{pmatrix} -1/2 \\ 1/\sqrt{2} \\ -1/2 \end{pmatrix}.$$

The probability of being in the $\lambda = 0$ state is:

$$\left| \left(-\tfrac{1}{\sqrt{2}} \ 0 \ \tfrac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{26}}\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \right|^2 = \frac{1}{2 \cdot 26}(-1-3)^2 = \frac{4}{13}$$

12. $S_x + S_y = \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}$. Find the eigenvalues:

$$\begin{vmatrix} -\lambda & 1-i \\ 1+i & -\lambda \end{vmatrix}^2 = \lambda^2 - |1+i|^2 = \lambda^2 - 2 = 0 \rightarrow \lambda = \pm\sqrt{2}.$$

For the larger eigenvalue, $\lambda = \sqrt{2}$,

$$\begin{pmatrix} -\sqrt{2} & 1-i \\ 1+i & -\sqrt{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \rightarrow c_1 = \frac{1-i}{\sqrt{2}} c_2 = e^{-i\pi/4} c_2$$

$$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\pi/4} \\ 1 \end{pmatrix}.$$

The probability that S_z yields $\hbar/2$ is the probability that one is in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$$\text{Prob.} = \left| \frac{e^{-i\pi/4}}{\sqrt{2}} \right|^2 = \frac{1}{2}.$$

14. The eigenvectors & eigenvalues of the spin $\frac{1}{2}$ operators, $S_\alpha = \frac{\hbar}{2}\sigma_\alpha$, are

operator	eigenvalue	eigenvector
S_z	$\hbar/2$	$\|\uparrow, \hat{z}\rangle = \begin{pmatrix}1\\0\end{pmatrix}$
S_z	$-\hbar/2$	$\|\downarrow, \hat{z}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$
S_x	$\hbar/2$	$\|\uparrow, \hat{x}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$
S_x	$-\hbar/2$	$\|\downarrow, \hat{x}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$
S_y	$\hbar/2$	$\|\uparrow, \hat{y}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}-i\\1\end{pmatrix}$
S_y	$-\hbar/2$	$\|\downarrow, \hat{y}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}i\\1\end{pmatrix}$

The problem specifies that initially the state is an eigenstate of S_x with eigenvalue $+\hbar/2$.

$$\rightarrow |\psi(t=0)\rangle = |\uparrow, \hat{x}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.$$

With a magnetic field in the $\hat{z}$ direction the hamiltonian is

$$H = \hbar\omega\sigma_z, \text{ where } \omega = \frac{eg}{4mc}.$$

This has eigenstates & eigenvalues:

$|\uparrow, \hat{z}\rangle$: $\quad E = \hbar\omega$
$|\downarrow, \hat{z}\rangle$: $\quad E = -\hbar\omega$.

$$\Rightarrow |\psi(T)\rangle = e^{-i\omega T}|\uparrow, \hat{z}\rangle\langle\uparrow, \hat{z}|\psi(0)\rangle$$
$$+ e^{i\omega T}|\downarrow, \hat{z}\rangle\langle\downarrow, \hat{z}|\psi(0)\rangle$$

In matrix notation this is simply

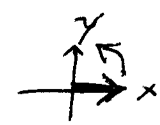
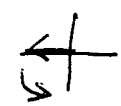
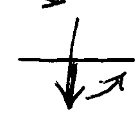
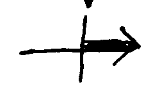
$$|\psi(T)\rangle = e^{-i\omega T}\begin{pmatrix}1\\0\end{pmatrix}(1\ 0)|\psi(0)\rangle + e^{i\omega T}(0\ 1)\begin{pmatrix}0\\1\end{pmatrix}|\psi(0)\rangle$$

$$=\left\{e^{-i\omega T}\begin{pmatrix}1&0\\0&0\end{pmatrix} + e^{i\omega T}\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}|\psi(0)\rangle$$

$$=\begin{pmatrix}e^{-i\omega T}&0\\0&e^{i\omega T}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$=\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega T}\\e^{i\omega T}\end{pmatrix}.$$

Does this make sense? (extra)

time	wavefunction		
$T=0$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$	$=\|\uparrow,\hat{x}\rangle$	
$\omega T=\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}(1-i)/\sqrt{2}\\(1+i)/\sqrt{2}\end{pmatrix}$	$=\frac{1+i}{\sqrt{2}}\|\uparrow,\hat{y}\rangle$	
$\omega T=\frac{\pi}{2}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}-i\\i\end{pmatrix}$	$=(-i)\|\downarrow,\hat{x}\rangle$	
$\omega T=\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}(-1-i)/\sqrt{2}\\(-1+i)/\sqrt{2}\end{pmatrix}$	$=\frac{-1+i}{\sqrt{2}}\|\downarrow,\hat{y}\rangle$	
$\omega T=\pi$	$\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\-1\end{pmatrix}$	$=-\|\uparrow,\hat{x}\rangle$	
$\omega T=\frac{5\pi}{4}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}(-1+i)/\sqrt{2}\\(-1-i)/\sqrt{2}\end{pmatrix}$	$=\frac{-1-i}{\sqrt{2}}\|\uparrow,\hat{y}\rangle$	
$\omega T=\frac{3\pi}{2}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}i\\-i\end{pmatrix}$	$=(i)\|\downarrow,\hat{x}\rangle$	
$\omega T=\frac{7\pi}{4}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}(1+i)/\sqrt{2}\\(1-i)/\sqrt{2}\end{pmatrix}$	$=\frac{1-i}{\sqrt{2}}\|\downarrow,\hat{y}\rangle$	
$\omega T=2\pi$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$	$=\|\uparrow,\hat{x}\rangle$	

Yes, the spin is rotating about the z-axis.

$$|\psi(2T)\rangle = e^{-i\omega T}|\uparrow,\hat{y}\rangle\langle\uparrow,\hat{y}|\psi(T)\rangle$$
$$+ e^{i\omega T}|\downarrow,\hat{y}\rangle\langle\downarrow,\hat{y}|\psi(T)\rangle$$
$$= \left\{\frac{e^{-i\omega T}}{2}\begin{pmatrix}-i\\1\end{pmatrix}(i\ \ 1) + \frac{e^{i\omega T}}{2}\begin{pmatrix}i\\1\end{pmatrix}(-i\ \ 1)\right\}|\psi(T)\rangle$$
$$= \left\{\frac{e^{-i\omega T}}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} + \frac{e^{i\omega T}}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\right\}|\psi(T)\rangle$$
$$= \begin{pmatrix}\cos(\omega T) & -\sin(\omega T)\\ \sin(\omega T) & \cos(\omega T)\end{pmatrix}|\psi(T)\rangle$$
$$= \begin{pmatrix}\cos(\omega T) & -\sin(\omega T)\\ \sin(\omega T) & \cos(\omega T)\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega T}\\ e^{i\omega T}\end{pmatrix}$$

The probability of being in the $|\uparrow,\hat{x}\rangle$ state is

Prob. $= |\langle\uparrow,\hat{x}|\psi(2T)\rangle$, where

$$\langle\uparrow,\hat{x}|\psi(2T)\rangle = \frac{1}{\sqrt{2}}(1\ \ 1)\begin{pmatrix}\cos\omega T & -\sin\omega T\\ \sin\omega T & \cos\omega T\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega T}\\ e^{i\omega T}\end{pmatrix}$$
$$= \frac{1}{2}\left((\cos\omega T+\sin\omega T)\ \ (\cos\omega T - \sin\omega T)\right)\begin{pmatrix}e^{-i\omega T}\\ e^{i\omega T}\end{pmatrix}$$
$$= \frac{1}{2}\left(2\cos^2\omega T - 2i\sin^2\omega T\right)$$
$$= \cos^2\omega T - i\sin^2\omega T$$

$$\Rightarrow \boxed{\text{Prob.} = |\langle\uparrow,\hat{x}|\psi(2T)\rangle|^2 = \cos^4\omega T + \sin^4\omega T}$$

As a check,

$$\langle\downarrow,\hat{x}|\psi(2T)\rangle = \frac{1}{\sqrt{2}}(1\ \ -1)\begin{pmatrix}\cos\omega T & -\sin\omega T\\ \sin\omega T & \cos\omega T\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega T}\\ e^{i\omega T}\end{pmatrix}$$
$$= \frac{1}{2}\left((\cos\omega T-\sin\omega T)\ \ -(\sin\omega T+\cos\omega T)\right)\begin{pmatrix}e^{-i\omega T}\\ e^{i\omega T}\end{pmatrix}$$
$$= \frac{1}{2}\left(-2\cos\omega T\sin\omega T - 2i\sin\omega T\cos\omega T\right)$$

$|\langle\downarrow,\hat{x}|\psi(2T)\rangle|^2 = 2\cos^2\omega T\sin^2\omega T$ so the sum of the two probabilities is one. ✓