Homework 3

1. a. Linear since $\mathcal{O}_1(\psi_1+\psi_2) = \mathcal{O}_1\psi_1 + \mathcal{O}_1\psi_2$
$\mathcal{O}_1(c\,\psi_1) = c\,\mathcal{O}_1\psi_1$

b. Linear for same reasons.

c. Not linear since $\mathcal{O}_3(c\psi) = c^*\mathcal{O}_3(\psi)$ not $c\,\mathcal{O}_3(\psi)$

d. Not linear since $e^{\psi_1+\psi_2} \neq e^{\psi_1} + e^{\psi_2}$

e. Not linear since $\frac{d(\psi_1+\psi_2)}{dx} + a \neq \frac{d\psi_1}{dx} + a + \frac{d\psi_2}{dx} + a$

f. Linear for same reasons as a.

2. a. $\mathcal{O}_2\mathcal{O}_6\psi = x\frac{d}{dx}\int_{-\infty}^{x} dx'\,\psi(x')\,x' = x\,\psi(x)\,x = x^2\psi(x)$

$$\mathcal{O}_6\mathcal{O}_2\psi = \int_{-\infty}^{x} dx'\left(x'\frac{d\psi(x')}{dx'}\right)x' = \int_{-\infty}^{x} dx'\,(x')^2\frac{d\psi}{dx'}(x')$$

$$= (x')^2\,\psi(x')\Big|_{-\infty}^{x} - \int_{-\infty}^{x} dx'\,2x'\,\psi(x')$$

$$= x^2\frac{d\psi}{dx} - 2\mathcal{O}_6\psi$$

$\Rightarrow [\mathcal{O}_2,\mathcal{O}_6]\psi = 2\mathcal{O}_6\psi$, i.e. $[\mathcal{O}_2,\mathcal{O}_6] = 2\mathcal{O}_6$

b. $\mathcal{O}_1\mathcal{O}_2\psi = x^3\,x\frac{d\psi}{dx} = x^4\frac{d\psi}{dx}$

$\mathcal{O}_2\mathcal{O}_1\psi = x\frac{d}{dx}(x^3\psi) = x^4\frac{d\psi}{dx} + 3x^3\psi$

$\Rightarrow [\mathcal{O}_1,\mathcal{O}_2]\psi = -3x^3\psi$, i.e. $[\mathcal{O}_1,\mathcal{O}_2] = -3x^3$

3. $A_n^{(+)} = \int_{-a}^{a} dx\, u_n^{(+)}(x) f(x)$, where $u_n^{(+)}(x) = \frac{1}{\sqrt{a}} \cos\left(\frac{\pi[n-\frac{1}{2}]}{a} x\right)$

$A_n^{(-)} = \int_{-a}^{a} dx\, u_n^{(-)}(x) f(x)$, where $u_n^{(-)}(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi n}{a} x\right)$

for $n = 1, 2, 3, \ldots$

a. $f(x) = 1$ for $-\frac{a}{2} < x < a/2$, $= 0$ otherwise.

$$A_n^{(+)} = \int_{-a/2}^{a/2} dx \frac{1}{\sqrt{a}} \cos\left(\frac{\pi[n-\frac{1}{2}]}{a} x\right)$$

$$= \frac{1}{\sqrt{a}} \frac{+\sin\left(\frac{\pi[n-\frac{1}{2}]x}{a}\right)}{\pi[n-\frac{1}{2}]/a}\Bigg|_{-a/2}^{a/2}$$

$$= \frac{1}{\sqrt{a}} \frac{a}{\pi[n-\frac{1}{2}]} 2\sin\left(\frac{\pi}{2}[n-\frac{1}{2}]\right)$$

$A_n^{(-)} = 0$ because $u_n^-(x)$ is odd and $f(x)$ is even – and the range of integration is $[-\frac{a}{2}, \frac{a}{2}]$.

b. $f(x) = \sin(3\pi x/a)$.

$A_n^{(+)} = 0$ since $u_n^+(x)$ is even, $f(x)$ is odd, and the integration range is $(-a, a)$.

$$A_n^{(-)} = \int_{-a}^{a} dx \frac{1}{\sqrt{a}} \sin\left(\frac{\pi n}{a} x\right) \sin\left(\frac{3\pi}{a} x\right)$$

$$= \frac{1}{\sqrt{a}} \int_{-a}^{a} dx \frac{1}{2}\left\{\cos\left(\frac{\pi}{a}(n-3)x\right) - \cos\left(\frac{\pi}{a}(n+3)x\right)\right\}$$

$$= \frac{1}{\sqrt{a}} \frac{1}{2}\left\{ \frac{\sin\left(\frac{\pi}{a}(n-3)x\right)}{\frac{\pi}{a}(n-3)}\Bigg|_{-a}^{a} \mp \frac{\sin\left(\frac{\pi}{a}(n+3)x\right)}{\frac{\pi}{a}(n+3)}\Bigg|_{-a}^{a} \right\} \quad \text{for } n \neq 3$$

$= 0$ for $n \neq 3$

$$= \frac{1}{\sqrt{a}} \int_{-a}^{a} dx \, \frac{1}{2} \left\{ 1 - \cos\left(\frac{\pi}{a}(n+3)x\right) \right\}$$

$$= \frac{1}{\sqrt{a}} \, \frac{1}{2} \, 2a = \sqrt{a} \quad \text{for } n=3$$

c. $A_n^{(+)}$

$$= \int_{-a}^{a} dx \, \frac{1}{\sqrt{a}} \cos\left(\frac{\pi}{a}[n-\tfrac{1}{2}]x\right)(a^2-x^2)$$

$$= \int_{-a}^{a} dx \, \frac{1}{\sqrt{a}} (a^2-x^2) \frac{d}{dx} \frac{\sin\left(\frac{\pi}{a}[n-\frac{1}{2}]x\right)}{\frac{\pi}{a}[n-\frac{1}{2}]}$$

$$= \frac{1}{\sqrt{a}} (a^2-x^2) \frac{\sin\left(\frac{\pi}{a}[n-\frac{1}{2}]x\right)}{\frac{\pi}{a}[n-\frac{1}{2}]} \Bigg|_{-a}^{a}$$

$$- \int_{-a}^{a} dx \, \frac{1}{\sqrt{a}} \frac{d}{dx}(a^2-x^2) \frac{\sin\left(\frac{\pi}{a}[n-\frac{1}{2}]x\right)}{\frac{\pi}{a}[n-\frac{1}{2}]}$$

$$= \int_{-a}^{a} dx \, \frac{1}{\sqrt{a}} \, 2x \, \frac{\sin\left(\frac{\pi}{a}[n-\frac{1}{2}]x\right)}{\frac{\pi}{a}[n-\frac{1}{2}]}$$

$$= \int_{-a}^{a} dx \, \frac{1}{\sqrt{a}} \, 2x \, \frac{d}{dx} \, \frac{-\cos\left(\frac{\pi}{a}[n-\frac{1}{2}]x\right)}{\left(\frac{\pi}{a}[n-\frac{1}{2}]\right)^2}$$

$$= \frac{1}{\sqrt{a}} \, 2x \, \frac{-\cos\left(\frac{\pi}{a}[n-\frac{1}{2}]x\right)}{\left(\frac{\pi}{a}[n-\frac{1}{2}]\right)^2} \Bigg|_{-a}^{a}$$

$$- \int_{-a}^{a} dx \, \frac{1}{\sqrt{a}} \frac{d}{dx}(2x) \, \frac{-\cos\left(\frac{\pi}{a}[n-\frac{1}{2}]x\right)}{\left(\frac{\pi}{a}[n-\frac{1}{2}]\right)^2}$$

$$= \frac{2}{\sqrt{a}} \int_{-a}^{a} dx \, \frac{\cos\left(\frac{\pi}{a}[n-\frac{1}{2}]x\right)}{\left(\frac{\pi}{a}[n-\frac{1}{2}]\right)^2}$$

$$= \frac{2}{\sqrt{a}} \, \frac{+\sin\left(\frac{\pi}{a}[n-\frac{1}{2}]x\right)}{\left(\frac{\pi}{a}[n-\frac{1}{2}]\right)^3} \Bigg|_{-a}^{a} = \frac{4}{\sqrt{a}} \, \frac{\sin\left(\pi[n-\frac{1}{2}]\right)}{\left(\frac{\pi}{a}[n-\frac{1}{2}]\right)^3}$$

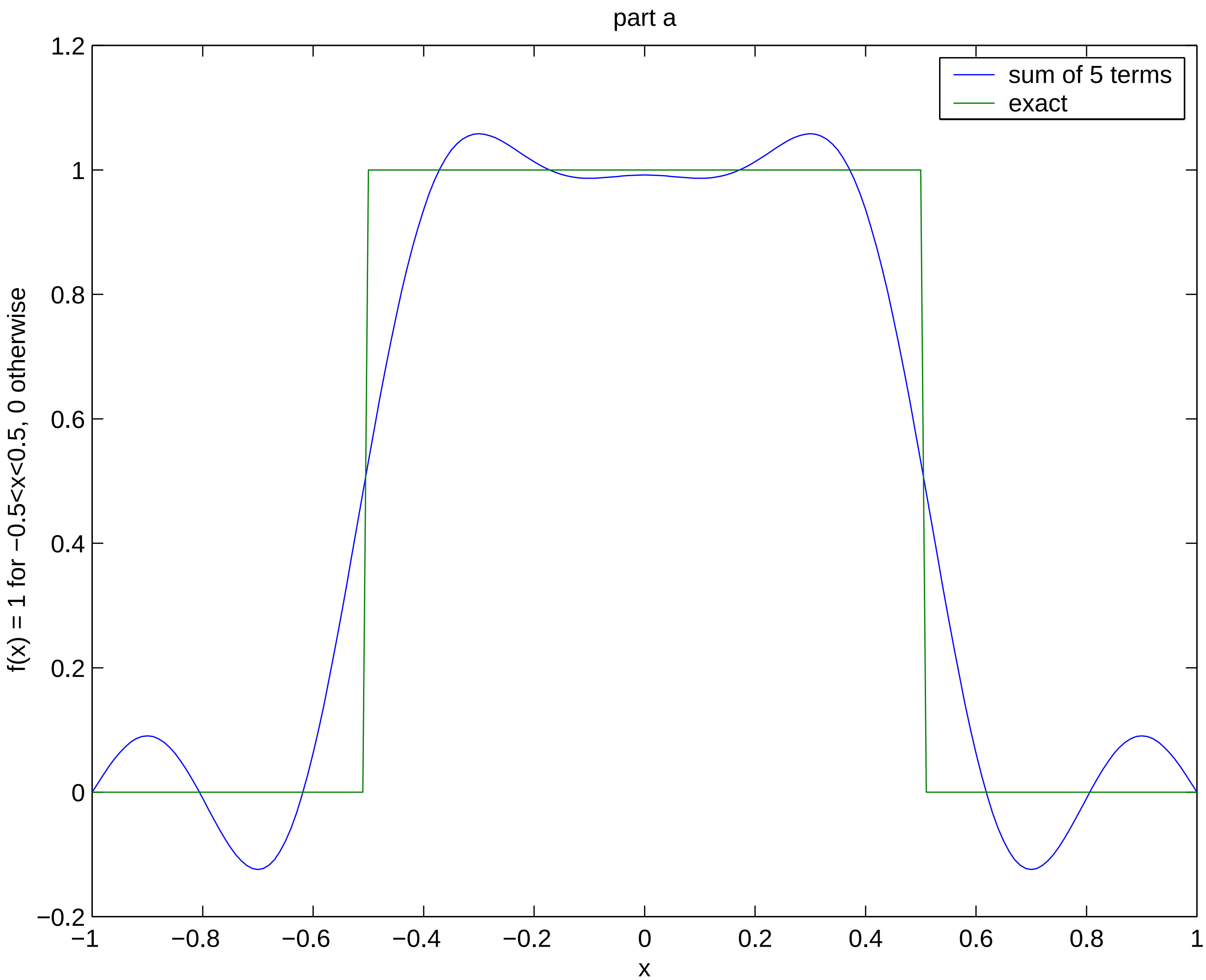
$A_n^{(-)} = 0$ because $f(x)$ is even.

4. From part 3 we have:

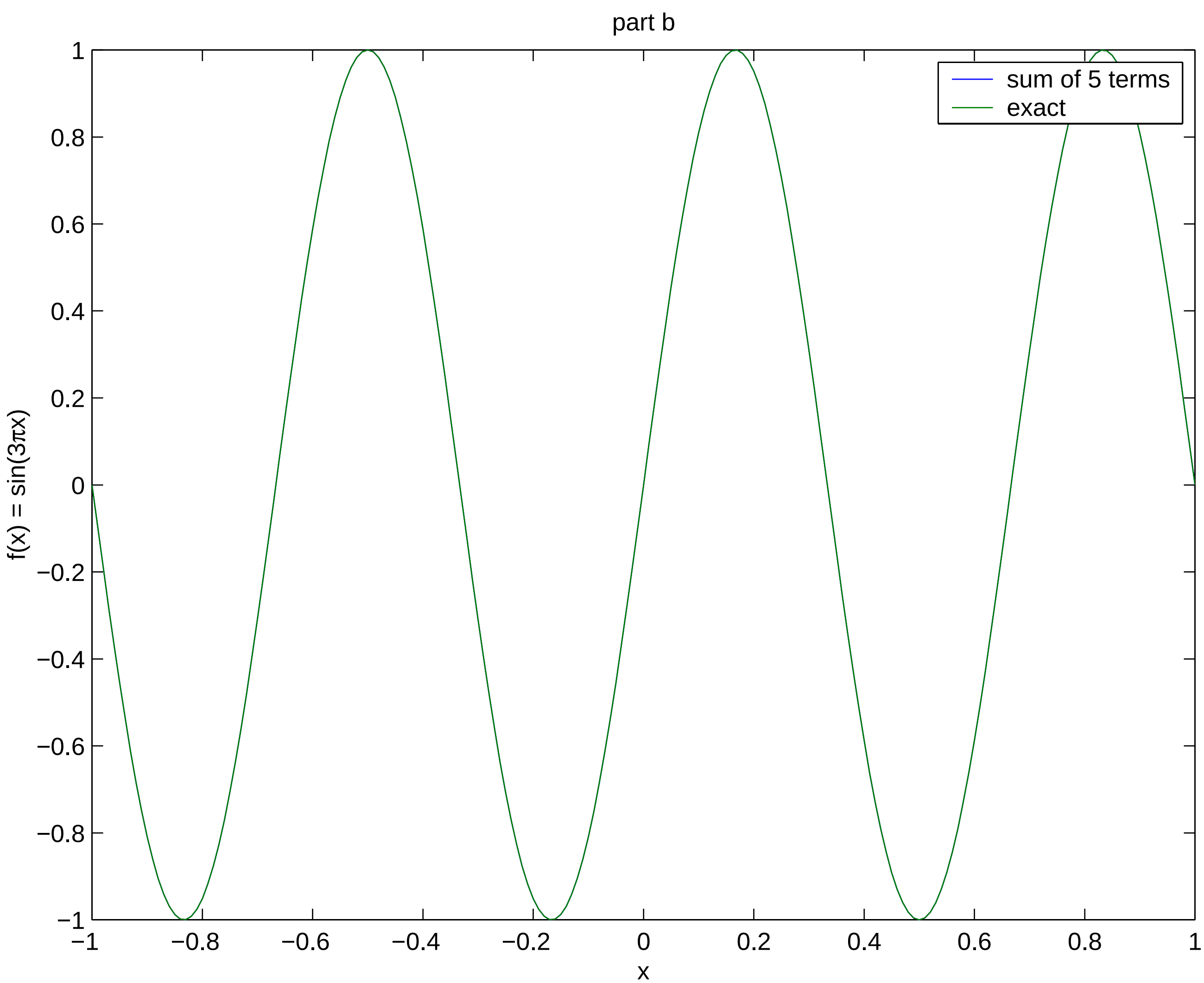
| | $A_n^{(+)}$ | $A_n^{(-)}$ |
|---|---|---|
| a. | $\sqrt{a}\,\dfrac{\sin\left(\frac{\pi}{2}[n-\frac{1}{2}]\right)}{\frac{\pi}{2}[n-\frac{1}{2}]}$ | $0$ |
| b. | $0$ | $\sqrt{a}\,\delta_{n,3}$ |
| c. | $\sqrt{a}\,4\,\dfrac{\sin\left(\pi[n-\frac{1}{2}]\right)}{\left(\pi[n-\frac{1}{2}]\right)^3}$ | $0$ |

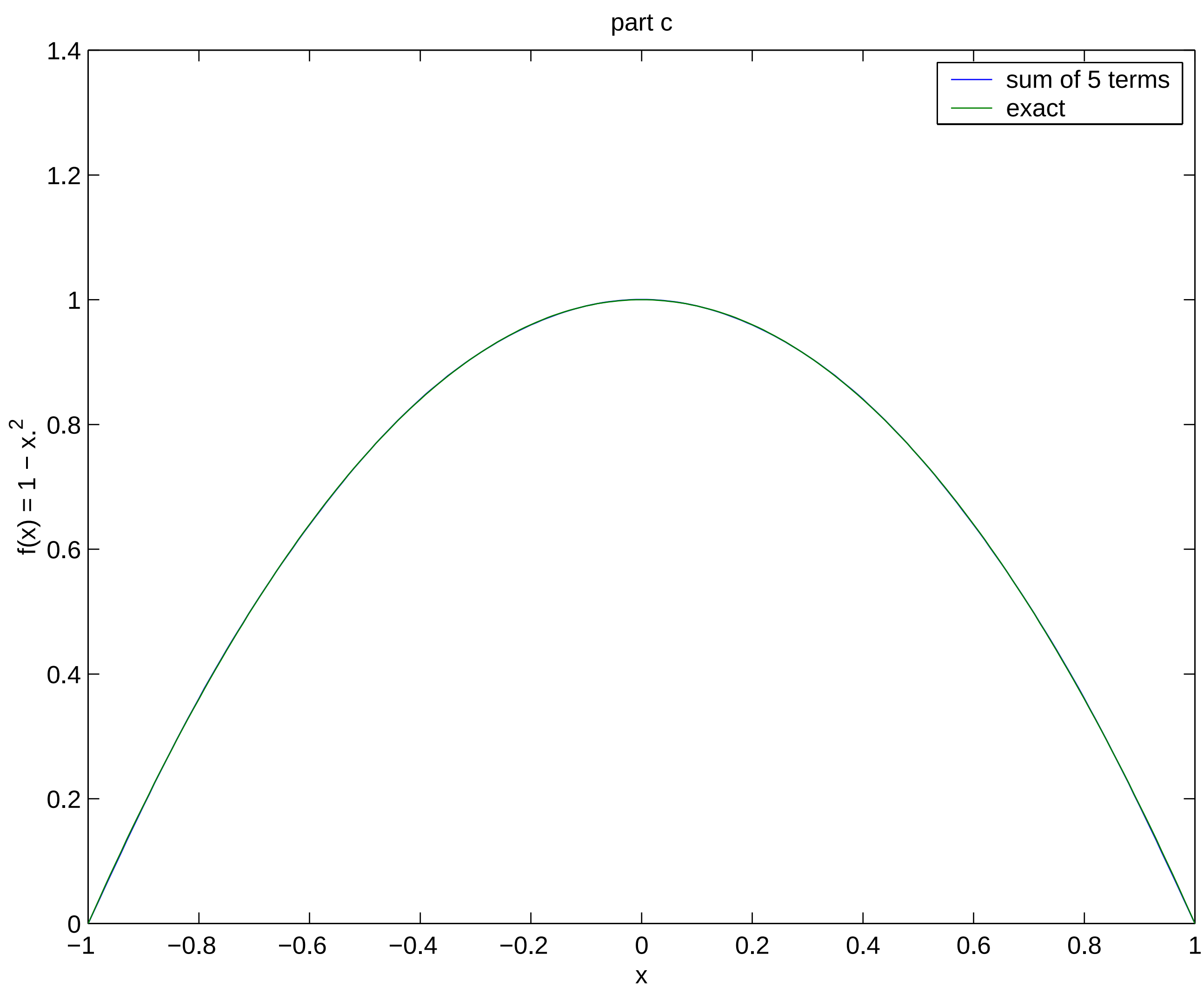
The sums of Eq. (2) are compared to the exact results in the following plots. Note that the agreement is perfect in case b and nearly so in case c.

```
% Homework assignment 3 - Fall 2002
% Note a = 1.

x = -1:0.01:1;

% There are more elegant ways to do this in matlab, but
% this method should be easy for you to understand.

parta = (sin(pi/4)/(pi/4))*cos(pi*x/2);
parta = parta + (sin(3*pi/4)/(3*pi/4))*cos(3*pi*x/2);
parta = parta + (sin(5*pi/4)/(5*pi/4))*cos(5*pi*x/2);
parta = parta + (sin(7*pi/4)/(7*pi/4))*cos(7*pi*x/2);
parta = parta + (sin(9*pi/4)/(9*pi/4))*cos(9*pi*x/2);

partaexact = (x >= -0.5).*(x <= 0.5);

figure(1)
plot(x,parta,x,partaexact)
legend('sum of 5 terms','exact')
xlabel('x')
ylabel('f(x) = 1 for -0.5<x<0.5, 0 otherwise')
title('part a')

partb = sin(3*pi*x);
partbexact = sin(3*pi*x);

figure(2)
plot(x,partb,x,partbexact)
legend('sum of 5 terms','exact')
xlabel('x')
ylabel('f(x) = sin(3\pix)')
title('part c')

partc = 4*sin(pi/2)*cos(pi*x/2)/(pi/2).^3;
partc = partc + 4*sin(3*pi/2)*cos(3*pi*x/2)/(3*pi/2).^3;
partc = partc + 4*sin(5*pi/2)*cos(5*pi*x/2)/(5*pi/2).^3;
partc = partc + 4*sin(7*pi/2)*cos(7*pi*x/2)/(7*pi/2).^3;
partc = partc + 4*sin(9*pi/2)*cos(9*pi*x/2)/(9*pi/2).^3;

partcexact = 1 - x.^2;

figure(3)
plot(x,partc,x,partcexact)
legend('sum of 5 terms','exact')
xlabel('x')
ylabel('f(x) = 1 - x.^2')
title('part c')
```



part a
sum of 5 terms
exact
f(x) = 1 for −0.5<x<0.5, 0 otherwise
x
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


part b
sum of 5 terms
exact
f(x) = sin(3πx)
x
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


part c
sum of 5 terms
exact
f(x) = 1 − x.^2
x
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1