Homework 4:

1. The flux or probability current is

$$\frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x}\psi\right) =$$

$$= \frac{\hbar}{2mi}\Big((A^* e^{-ikx} + B^* e^{ikx})\, ik\, (Ae^{ikx} - Be^{-ikx})$$
$$-(-ik)(A^* e^{-ikx} - B^* e^{ikx})(Ae^{ikx} + Be^{-ikx})\Big)$$

$$= \frac{\hbar k}{2m}\Big((|A|^2 - |B|^2 - A^* B e^{-2ikx} + AB^* e^{2ikx})$$
$$+ (|A|^2 - |B|^2 + A^* B e^{-2ikx} - AB^* e^{2ikx})\Big)$$

$$= \frac{\hbar k}{m}(|A|^2 - |B|^2).$$

→ Conservation of probability implies

$$|A|^2 - |B|^2 = |C|^2 - |D|^2$$

$$\rightarrow \underbrace{|A|^2 + |D|^2}_{\text{incoming}} = \underbrace{|B|^2 + |C|^2}_{\text{outgoing}}$$

$$= |S_{21}A + S_{22}D|^2 + |S_{11}A + S_{12}D|^2$$
$$= (|S_{21}|^2 + |S_{11}|^2)|A|^2 + (|S_{22}|^2 + |S_{12}|^2)|D|^2$$
$$+ (S_{21}S_{22}^* + S_{11}S_{12}^*)AD^*$$
$$+ (S_{22}S_{21}^* + S_{12}S_{11}^*)DA^*$$

The only way this can be true for arbitrary A and D is if

$$1 = |S_{21}|^2 + |S_{11}|^2 = |S_{22}|^2 + |S_{12}|^2$$

$$0 = S_{21}S_{22}^* + S_{11}S_{12}^* = S_{22}S_{21}^* + S_{12}S_{11}^*.$$

The unitarity condition is that

$$S S^{\dagger} = 1 \quad \text{or equivently that } S^{\dagger} S = 1,$$

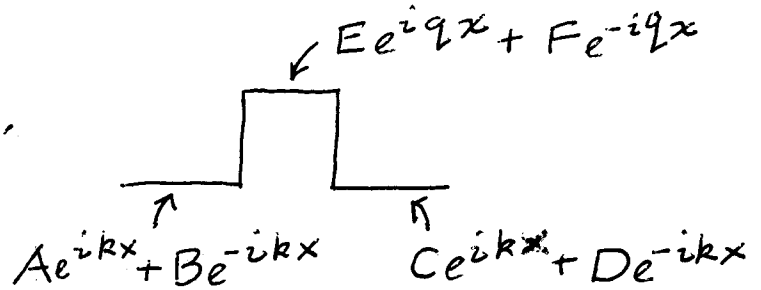
where $S^{\dagger} = \begin{pmatrix} S_{11}^{*} & S_{21}^{*} \\ S_{12}^{*} & S_{22}^{*} \end{pmatrix}$.

$$S^{\dagger} S = \begin{pmatrix} S_{11}^{*} & S_{21}^{*} \\ S_{12}^{*} & S_{22}^{*} \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

$$= \begin{pmatrix} |S_{11}|^2 + |S_{21}|^2 & S_{11}^{*} S_{12} + S_{21}^{*} S_{22} \\ S_{11} S_{12}^{*} + S_{21} S_{22}^{*} & |S_{12}|^2 + |S_{22}|^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

2.



$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$q = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$$\psi(-a) = Ae^{-ika} + Be^{ika} = Ee^{-iqa} + Fe^{iqa}$$

$$\psi'(-a) = ik(Ae^{-ika} - Be^{ika}) = iq(Ee^{-iqa} - Fe^{iqa})$$

$$\psi(a) = Ce^{ika} + De^{-ika} = Ee^{iqa} + Fe^{-iqa}$$

$$\psi'(a) = ik(Ce^{ika} - De^{-ika}) = iq(Ee^{iqa} - Fe^{-iqa})$$

Let $\tilde{A} = Ae^{-ika}$, $\tilde{B} = Be^{ika}$, $\tilde{C} = Ce^{ika}$, $\tilde{D} = De^{-ika}$.

$$\Rightarrow \tilde{A} + \tilde{B} = Ee^{-iqa} + Fe^{iqa} \quad \ldots (1)$$

$$\frac{k}{q}(\tilde{A} - \tilde{B}) = Ee^{-iqa} - Fe^{iqa} \quad \ldots (2)$$

$$\tilde{C} + \tilde{D} = Ee^{iqa} + Fe^{-iqa} \quad \ldots (3)$$

$$\frac{k}{q}(\tilde{C} - \tilde{D}) = Ee^{iqa} - Fe^{-iqa} \quad \ldots (4)$$

$$\left.\begin{aligned} (1)+(2) &\Rightarrow 2E = \left(1+\frac{k}{q}\right)e^{iqa}\tilde{A} + \left(1-\frac{k}{q}\right)e^{iqa}\tilde{B} \\ (3)+(4) &\Rightarrow 2E = \left(1+\frac{k}{q}\right)e^{-iqa}\tilde{C} + \left(1-\frac{k}{q}\right)e^{-iqa}\tilde{D} \end{aligned}\right\} (5)$$

$$\left.\begin{aligned} (1)-(2) &\Rightarrow 2F = \left(1-\frac{k}{q}\right)e^{-iqa}\tilde{A} + \left(1+\frac{k}{q}\right)e^{-iqa}\tilde{B} \\ (3)-(4) &\Rightarrow 2F = \left(1-\frac{k}{q}\right)e^{iqa}\tilde{C} + \left(1+\frac{k}{q}\right)e^{+iqa}\tilde{D} \end{aligned}\right\} (6)$$

$$(5)\ldots\quad \left(\frac{k}{q}+1\right)e^{iqa}\tilde{A} + \left(\frac{k}{q}-1\right)e^{-iqa}\tilde{D} = \left(\frac{k}{q}-1\right)e^{iqa}\tilde{B} + \left(\frac{k}{q}+1\right)e^{-iqa}\tilde{C}$$

$$(6)\ldots\quad \left(\frac{k}{q}-1\right)e^{-iqa}\tilde{A} + \left(\frac{k}{q}+1\right)e^{iqa}\tilde{D} = \left(\frac{k}{q}+1\right)e^{-iqa}\tilde{B} + \left(\frac{k}{q}-1\right)e^{iqa}\tilde{C}$$

Write this as a matrix equation:

$$\begin{pmatrix} \left(\frac{k}{q}+1\right)e^{iqa} & \left(\frac{k}{q}-1\right)e^{-iqa} \\ \left(\frac{k}{q}-1\right)e^{-iqa} & \left(\frac{k}{q}+1\right)e^{iqa} \end{pmatrix}\begin{pmatrix} \tilde{A} \\ \tilde{D} \end{pmatrix} =$$

$$= \begin{pmatrix} \left(\frac{k}{q}+1\right)e^{-iqa} & \left(\frac{k}{q}-1\right)e^{iqa} \\ \left(\frac{k}{q}-1\right)e^{iqa} & \left(\frac{k}{q}+1\right)e^{-iqa} \end{pmatrix}\begin{pmatrix} \tilde{C} \\ \tilde{B} \end{pmatrix}$$

$$\begin{pmatrix} \tilde{C} \\ \tilde{B} \end{pmatrix} = \frac{1}{\left(\frac{k}{q}+1\right)^2 e^{-i2qa} - \left(\frac{k}{q}-1\right)^2 e^{i2qa}} \times$$

$$\times \begin{pmatrix} \left(\frac{k}{q}+1\right)e^{-iqa} & -\left(\frac{k}{q}-1\right)e^{iqa} \\ -\left(\frac{k}{q}-1\right)e^{iqa} & \left(\frac{k}{q}+1\right)e^{-iqa} \end{pmatrix}$$

$$\times \begin{pmatrix} \left(\frac{k}{q}+1\right)e^{iqa} & \left(\frac{k}{q}-1\right)e^{-iqa} \\ \left(\frac{k}{q}-1\right)e^{-iqa} & \left(\frac{k}{q}+1\right)e^{iqa} \end{pmatrix}\begin{pmatrix} \tilde{A} \\ \tilde{D} \end{pmatrix}$$

$$= \frac{1}{\left(\frac{k}{q}+1\right)^2 e^{-i2qa} - \left(\frac{k}{q}-1\right)^2 e^{i2qa}} \times$$

$$\begin{pmatrix} \left(\frac{k}{q}+1\right)^2 - \left(\frac{k}{q}-1\right)^2 & \left(\frac{k}{q}+1\right)\left(\frac{k}{q}-1\right)\left(e^{-2iqa} - e^{2iqa}\right) \\ \left(\frac{k}{q}+1\right)\left(\frac{k}{q}-1\right)\left(e^{-2iqa} - e^{2iqa}\right) & \left(\frac{k}{q}+1\right)^2 - \left(\frac{k}{q}-1\right)^2 \end{pmatrix}\begin{pmatrix} \tilde{A} \\ \tilde{D} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} Ce^{ika} \\ Be^{ika} \end{pmatrix} = \frac{1}{4\frac{k}{q}\cos(2qa) - 2i\left(\left(\frac{k}{q}\right)^2+1\right)\sin(2qa)} \times$$

$$\times \begin{pmatrix} 4\frac{k}{q} & -2i\left(\left(\frac{k}{q}\right)^2-1\right)\sin(2qa) \\ -2i\left(\left(\frac{k}{q}\right)^2-1\right)\sin(2qa) & 4\frac{k}{q} \end{pmatrix} \begin{pmatrix} Ae^{-ika} \\ De^{-ika} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} C \\ B \end{pmatrix} = \frac{e^{-i2ka}}{2\frac{k}{q}\cos(2qa) - i\left(\left(\frac{k}{q}\right)^2+1\right)\sin(2qa)} \times$$

$$\times \begin{pmatrix} 2\frac{k}{q} & i\left(1-\left(\frac{k}{q}\right)^2\right)\sin(2qa) \\ i\left(1-\left(\frac{k}{q}\right)^2\right)\sin(2qa) & 2\frac{k}{q} \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} S_{11} = S_{22} &= \frac{2\frac{k}{q}e^{-i2ka}}{2\frac{k}{q}\cos(2qa) - i\left(\left(\frac{k}{q}\right)^2+1\right)\sin(2qa)} \\ S_{12} = S_{21} &= \frac{i\left(1-\left(\frac{k}{q}\right)^2\right)\sin(2qa)e^{-i2ka}}{2\frac{k}{q}\cos(2qa) - i\left(\left(\frac{k}{q}\right)^2+1\right)\sin(2qa)} \end{aligned}}$$

<u>Checks:</u> $$|S_{11}|^2 = \frac{4\left(\frac{k}{q}\right)^2}{4\left(\frac{k}{q}\right)^2\cos^2(2qa) + \left(\left(\frac{k}{q}\right)^2+1\right)^2\sin^2(2qa)} = |S_{22}|^2$$

$$|S_{21}|^2 = \frac{\left(1-\left(\frac{k}{q}\right)^2\right)^2\sin^2(2qa)}{4\left(\frac{k}{q}\right)^2\cos^2(2qa) + \left(\left(\frac{k}{q}\right)^2+1\right)^2\sin^2(2qa)} = |S_{12}|^2$$

Using $\cos^2(2qa) = 1 - \sin^2(2qa)$, the denominator may be written as

$$4\left(\frac{k}{q}\right)^2 + \left(\left(\frac{k}{q}\right)^2 - 1\right)^2\sin^2(2qa),$$

and $|S_{11}|^2 + |S_{21}|^2 = |S_{22}|^2 + |S_{12}|^2 = 1.$ ✓

The other relation is $S_{11}S_{12}^* + S_{21}S_{22}^* = 0$:

$$S_{11}S_{12}^* = \frac{2\frac{k}{q}(-i)(1-(\frac{k}{q})^2)\sin(2qa)}{4(\frac{k}{q})^2 + ((\frac{k}{q})^2-1)^2\sin^2(2qa)}$$

$$S_{21}S_{22}^* = \frac{2\frac{k}{q}(i)(1-(\frac{k}{q})^2)\sin(2qa)}{4(\frac{k}{q})^2 + ((\frac{k}{q})^2-1)^2\sin^2(2qa)}$$

These clearly sum to zero. ✓

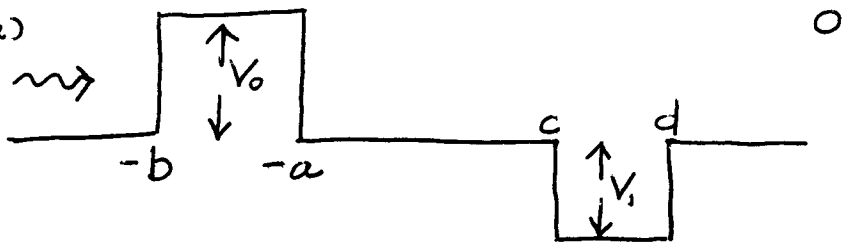
We will do one more check which is not required for this problem. For a wave coming from the left, $A=1$, $B=R$, $C=T$, $D=0$.

$$\Rightarrow \begin{pmatrix} T \\ R \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow T = S_{11} \text{ and } R = S_{21}.$$

Our expressions for S_{11} and S_{21} agree with those in (5-26). ✓ Note that their q was different because they had a potential well; however, the form of the solution (5-23) is the same because $E > V$.

5. (a)



$0 < E < V_0$

Define the following wavevectors:

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$q = \sqrt{\frac{2m(E+V_1)}{\hbar^2}}$$

$$\rho = \sqrt{\frac{2m(V_0-E)}{\hbar^2}} \quad ,$$

$$\begin{aligned}
\psi(x<-b) &= e^{ikx} + Re^{-ikx} \\
\psi(-b<x<-a) &= Ae^{\rho x} + Be^{-\rho x} \\
\psi(-a<x<c) &= Ce^{ikx} + De^{-ikx} \\
\psi(c<x<d) &= Ee^{iqx} + Fe^{-iqx} \\
\psi(d<x) &= Te^{ikx}
\end{aligned}$$

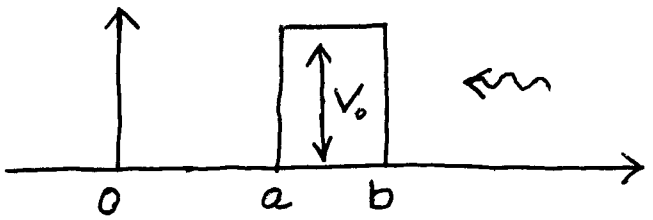
Continuity of $\psi(x)$ boundary conditions:

$$\begin{aligned}
\psi(-b) &= e^{-ikb} + Re^{ikb} = Ae^{-\rho b} + Be^{\rho b} \\
\psi(-a) &= Ae^{-\rho a} + Be^{\rho a} = Ce^{-ika} + De^{ika} \\
\psi(c) &= Ce^{ikc} + De^{-ikc} = Ee^{iqc} + Fe^{-iqc} \\
\psi(d) &= Ee^{iqd} + Fe^{-iqd} = Te^{ikd} \quad ,
\end{aligned}$$

Continuity of $\psi'(x)$ boundary conditions:

$$\begin{aligned}
\psi'(-b) &= ik(e^{-ikb} - Re^{ikb}) = \rho(Ae^{-\rho b} - Be^{\rho b}) \\
\psi'(-a) &= \rho(Ae^{-\rho a} - Be^{\rho a}) = ik(Ce^{-ika} - De^{ika}) \\
\psi'(c) &= ik(Ce^{ikc} - De^{-ikc}) = iq(Ee^{iqc} - Fe^{-iqc}) \\
\psi'(d) &= iq(Ee^{iqd} - Fe^{-iqd}) = ik(Te^{ikd}) \quad .
\end{aligned}$$

(b)



$0 < E < V_0$

Define the wavevectors:

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\rho = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \; .$$

$$\begin{aligned}
\psi(x<0) &= Te^{-ikx} \\
\psi(0<x<a) &= Ae^{-ikx} + Be^{ikx} \\
\psi(a<x<b) &= Ce^{-\rho x} + De^{\rho x} \\
\psi(b<x) &= e^{-ikx} + Re^{ikx}
\end{aligned}$$

Continuity of $\psi(x)$ boundary conditions:

$$\begin{aligned}
\psi(0) &= T = A + B \\
\psi(a) &= Ae^{-ika} + Be^{ika} = Ce^{-\rho a} + De^{\rho a} \\
\psi(b) &= Ce^{-\rho b} + De^{\rho b} = e^{-ikb} + Re^{ikb} \; .
\end{aligned}$$

$\psi'(x)$ boundary conditions:

$$\begin{aligned}
-ik(A-B) + ikT &= \frac{\lambda}{a}\psi(0) = \frac{\lambda}{a}T \\
\psi'(a) &= -ik(Ae^{-ika} - Be^{ika}) = -\rho(Ce^{-\rho a} - De^{\rho a}) \\
\psi'(b) &= -\rho(Ce^{-\rho b} - De^{\rho b}) = -ik(1e^{-ikb} - Re^{ikb}) \; .
\end{aligned}$$