Homework 5:

1. We showed in class that $(AB)^\dagger = B^\dagger A^\dagger$. If A and B are hermitian, then $= BA$.

$$(A+B)^\dagger = (A^\dagger + B^\dagger) = A+B$$

$$((A+B)(A+B))^\dagger = (A+B)^\dagger (A+B)^\dagger = (A+B)(A+B)$$

This shows that $(A+B)^n$ is hermitian for $n=1$ and 2. Next we show that if $(A+B)^{n-1}$ is hermitian, then $(A+B)^n$ is hermitian. This will complete a proof by induction.

$$\begin{aligned}((A+B)^n)^\dagger &= ((A+B)(A+B)^{n-1})^\dagger = ((A+B)^{n-1})^\dagger (A+B)^\dagger \\ &= (A+B)^{n-1}(A+B) \\ &= (A+B)^n \checkmark\end{aligned}$$

2. We know that $(A+B)^\dagger = A^\dagger + B^\dagger$, $(A^\dagger)^\dagger = A$, and $(cA)^\dagger = c^* A^\dagger$.

$$\rightarrow (A+A^\dagger)^\dagger = A^\dagger + (A^\dagger)^\dagger = A^\dagger + A \checkmark$$

$$(i(A-A^\dagger))^\dagger = -i(A-A^\dagger)^\dagger = -i(A^\dagger - A) = i(A-A^\dagger)$$

3. If H is hermitian, then from 1. above H^n is also hermitian. $i^* = -i$ and $(i)^{n*} = (-i)^n$.

$$\begin{aligned}(e^{iH})^\dagger &= \left(\sum_{n=0}^{\infty} \frac{i^n H^n}{n!}\right)^\dagger = \sum_{n=0}^{\infty} \frac{(i^n)^*(H^n)^\dagger}{n!} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} H^n \\ &= e^{-iH}\end{aligned}$$

4. Since $\langle\psi|\psi\rangle \geq 0$ for any ψ, $\langle\psi+\lambda\phi|\psi+\lambda\phi\rangle \geq 0$. Here ψ & ϕ are arbitrary wave functions and $\lambda \in \mathbb{C}$.

$$\langle\psi+\lambda\phi|\psi+\lambda\phi\rangle = \int dx\,(\psi^*+\lambda^*\phi^*)(\psi+\lambda\phi)$$

$$= \int dx\, \phi^*\phi$$

$$+ |\lambda|^2 \int dx\, \psi^*\psi$$

$$+ \lambda \int dx\, \psi^*\phi$$

$$+ \lambda^* \int dx\, \phi^*\psi$$

$$\rightarrow \langle\phi|\phi\rangle + |\lambda|^2\langle\psi|\psi\rangle + \lambda\langle\psi|\phi\rangle + \lambda^*\langle\phi|\psi\rangle \geq 0$$

$$\rightarrow \langle\phi|\phi\rangle + (\mathrm{Re}\lambda)^2\langle\psi|\psi\rangle + 2(\mathrm{Re}\lambda)(\mathrm{Re}\{\langle\psi|\phi\rangle\})$$
$$+ (\mathrm{Im}\lambda)^2\langle\psi|\psi\rangle - 2(\mathrm{Im}\lambda)(\mathrm{Im}\{\langle\psi|\phi\rangle\}) \geq 0$$

Minimize this with respect to $\mathrm{Re}\lambda$ and $\mathrm{Im}\lambda$.

$$\frac{\partial}{\partial \mathrm{Re}\lambda} = 0 \rightarrow 2(\mathrm{Re}\lambda)\langle\psi|\psi\rangle + 2\,\mathrm{Re}\{\langle\psi|\phi\rangle\} = 0$$
$$\frac{\partial}{\partial \mathrm{Im}\lambda} = 0 \rightarrow 2(\mathrm{Im}\lambda)\langle\psi|\psi\rangle - 2\,\mathrm{Im}\{\langle\psi|\phi\rangle\} = 0$$

$$\rightarrow \mathrm{Re}\lambda = \frac{-\mathrm{Re}\{\langle\psi|\phi\rangle\}}{\langle\psi|\psi\rangle} \qquad \mathrm{Im}\lambda = \frac{\mathrm{Im}\{\langle\psi|\phi\rangle\}}{\langle\psi|\psi\rangle}$$

$$\rightarrow \lambda = -\frac{\langle\psi|\phi\rangle^*}{\langle\psi|\psi\rangle} = -\frac{\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle}$$

$$\& \ \lambda^* = -\frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}$$

Substituting into the inequality for either λ & λ^* or $Re\{\lambda\}$ & $Im\{\lambda\}$,

$$\langle\phi|\phi\rangle + \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle} - \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle} - \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle} \geq 0$$

$$\langle\phi|\phi\rangle - \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle} \geq 0$$

$$\boxed{\langle\phi|\phi\rangle\langle\psi|\psi\rangle \geq |\langle\psi|\phi\rangle|^2}$$

6. $\langle A^2\rangle = \int dx\, \psi^* A A \psi = \int dx\, (A^\dagger\psi)^*(A\psi)$

$= \int dx\, (A\psi)^*(A\psi) \geq 0$

7. Suppose u_n is an eigenfunction of H with eigenvalue $\mathcal{E}_n$.

$\langle u_n|H|u_n\rangle = \mathcal{E}_n\langle u_n|u_n\rangle$ is real $\rightarrow$ $\mathcal{E}_n$ is real.

$H^4|u_n\rangle = (\mathcal{E}_n)^4|u_n\rangle = 1|u_n\rangle$

$\rightarrow (\mathcal{E}_n)^4 = 1. \rightarrow \mathcal{E}_n = \pm 1, \pm i$

Since H is hermitian, $\mathcal{E}_n = \pm 1$. If H was not hermitian, then $\mathcal{E}_n$ could be ± 1 or $\pm i$.

8. $\langle U\psi|U\psi\rangle = \int dx\, (U\psi)^* U\psi = \int dx\, \psi^* U^\dagger U \psi$

$= \int dx\, \psi^* 1 \psi = \langle\psi|\psi\rangle = 1.$

9. From problem 3, $(e^{iA})^{+} = e^{-iA}$.

For operator $e^{A}e^{B}$ does not in general equal e^{A+B}; however, if $[A,B]=0$, then $e^{A}e^{B}=e^{A+B}$. Since $[iA,-iA]=0$,

$$(e^{iA})(e^{-iA}) = 1,$$

$$(e^{-iA})(e^{iA}) = 1. \rightarrow e^{iA} \text{ is unitary.}$$

10. $\langle v_b | v_a \rangle = \langle U u_b | U u_a \rangle$

$$= \int dx\, (U u_b)^* (U u_a)$$

$$= \int dx\, u_b^* U^{+} U u_a$$

$$= \int dx\, u_b^* u_a$$

$$= \langle u_b | u_a \rangle = \delta_{ab}$$