Homework 6:

1. a) $Hu_n(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}u_n(x) = \frac{\hbar^2}{2m}\left(\frac{\pi n}{a}\right)^2 u_n(x) = E_n u_n(x)$

$$\rightarrow \boxed{E_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{a}\right)^2}$$

b) $\psi(x,t) = \frac{1}{\sqrt{2}}u_1(x)e^{-\frac{iE_1 t}{\hbar}} + \frac{i}{\sqrt{2}}u_2(x)e^{-\frac{iE_2 t}{\hbar}}$

c)
$$\begin{aligned}|\psi(x,t)|^2 &= \frac{1}{2}|u_1(x)|^2 + \frac{1}{2}|u_2(x)|^2 \\ &+ \frac{1}{2}\, i u_1(x)u_2(x)\, e^{-i\frac{(E_2-E_1)t}{\hbar}} \\ &- \frac{1}{2}\, i u_1(x)u_2(x)\, e^{+i\frac{(E_2-E_1)t}{\hbar}} \\ &= \frac{1}{2}\left(u_1(x)^2 + u_2(x)^2 + 2u_1(x)u_2(x)\sin\left(\frac{(E_2-E_1)t}{\hbar}\right)\right)\end{aligned}$$

This has period $\frac{(E_2-E_1)T}{\hbar} = 2\pi \rightarrow T = \frac{2\pi\hbar}{E_2-E_1} = \frac{h}{E_2-E_1}$.

d) At $t=0$, $|\psi(x,t)|^2 = \frac{1}{2}(u_1(x)^2 + u_2(x)^2)$.

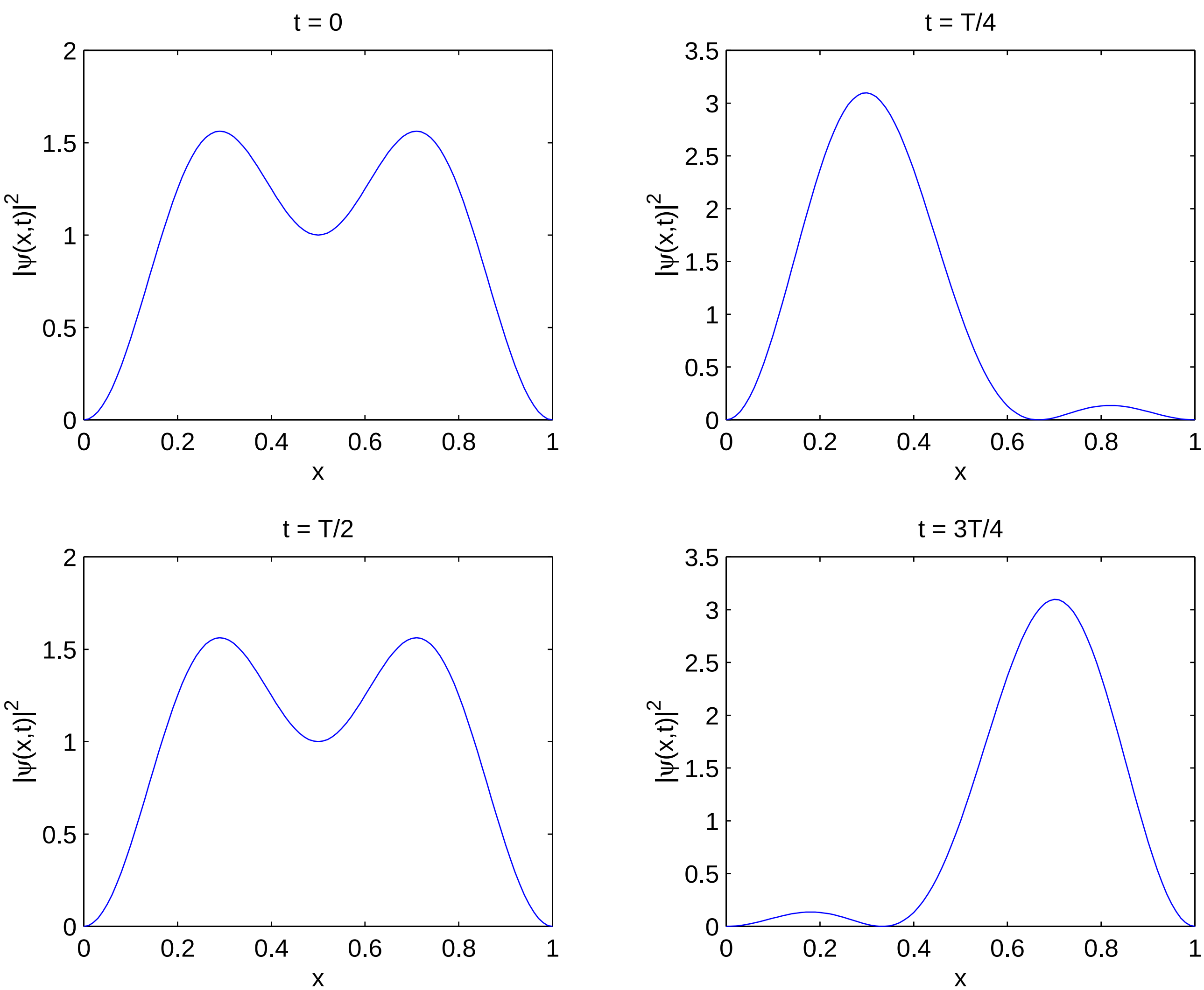
At $t = \frac{T}{4}$, $|\psi(x,t)|^2 = \frac{1}{2}(u_1(x)^2 + u_2(x)^2 + 2u_1(x)u_2(x))$
$= \frac{1}{2}(u_1(x) + u_2(x))^2$.

At $t = \frac{T}{2}$, $|\psi(x,t)|^2 = \frac{1}{2}(u_1(x)^2 + u_2(x)^2)$.

At $t = \frac{3T}{4}$, $|\psi(x,t)|^2 = \frac{1}{2}(u_1(x)^2 + u_2(x)^2 - 2u_1(x)u_2(x))$
$= \frac{1}{2}(u_1(x) - u_2(x))^2$.

These are plotted on the following page.

```
x = 0:0.01:1;
u1 = sqrt(2)*sin(pi*x);
u2 = sqrt(2)*sin(2*pi*x);

P0 = 0.5*(u1.^2 + u2.^2);
P1 = 0.5*(u1 + u2).^2;
P2 = P0;
P3 = 0.5*(u1 - u2).^2;

subplot(2,2,1)
plot(x,P0)
title('t = 0')
xlabel('x')
ylabel('|\psi(x,t)|^2')

subplot(2,2,2)
plot(x,P1)
title('t = T/4')
xlabel('x')
ylabel('|\psi(x,t)|^2')

subplot(2,2,3)
plot(x,P2)
title('t = T/2')
xlabel('x')
ylabel('|\psi(x,t)|^2')

subplot(2,2,4)
plot(x,P3)
title('t = 3T/4')
xlabel('x')
ylabel('|\psi(x,t)|^2')
```


t = 0
t = T/4
t = T/2
t = 3T/4
|ψ(x,t)|2
x
0
0.2
0.4
0.6
0.8
1
0.5
1.5
2
2.5
3
3.5

e) $\langle m|x|n\rangle = \int_0^a dx\, \frac{2}{a} \sin\left(\frac{\pi m x}{a}\right) x \sin\left(\frac{\pi n x}{a}\right)$

$$= \frac{1}{a}\int_0^a dx\; x\left(\cos\left(\frac{\pi(m-n)x}{a}\right) - \cos\left(\frac{\pi(m+n)x}{a}\right)\right)$$

If $m \neq n$, $= \frac{1}{a}\int_0^a dx\; x \frac{a}{\pi(m-n)}\frac{d}{dx}\sin\left(\frac{\pi(m-n)x}{a}\right)$

$$- x\frac{a}{\pi(m+n)}\frac{d}{dx}\sin\left(\frac{\pi(m+n)x}{a}\right)$$

$$= \frac{1}{a}\int_0^a dx\; \frac{-a}{\pi(m-n)} \sin\left(\frac{\pi(m-n)x}{a}\right)$$

$$\frac{+a}{\pi(m+n)}\sin\left(\frac{\pi(m+n)x}{a}\right)$$

$$= \frac{1}{a}\left(\frac{a^2}{\pi^2(m-n)^2}\cos\left(\frac{\pi(m-n)x}{a}\right)\Big|_0^a\right.$$

$$\left.\frac{-a^2}{\pi^2(m+n)^2}\cos\left(\frac{\pi(m+n)x}{a}\right)\Big|_0^a\right)$$

$$= \frac{1}{a}\frac{a^2}{\pi^2(m-n)^2} \times \begin{cases} 0 & \text{if } m-n = 0, \pm 2, \pm 4, \ldots \\ -2 & \text{if } m-n = \pm 1, \pm 3, \ldots \end{cases}$$

$$- \frac{1}{a}\frac{a^2}{\pi^2(m+n)^2} \times \begin{cases} 0 & \text{if } m+n = 0, \pm 2, \pm 4, \ldots \\ -2 & \text{if } m+n = \pm 1, \pm 3, \ldots \end{cases}$$

If $m = n$, $= \frac{a}{2}$.

$$\boxed{\begin{aligned} &\langle 1|x|1\rangle = \langle 2|x|2\rangle = \frac{a}{2} \\ &\langle 1|x|2\rangle = \langle 2|x|1\rangle = \left(\frac{a}{\pi^2}(-2) - \frac{a}{\pi^2}\frac{(-2)}{9}\right) = \frac{-16}{9\pi^2}a. \end{aligned}}$$

$\langle m|p|n\rangle = \int_0^a dx\, \frac{2}{a}\sin\left(\frac{\pi m x}{a}\right)\frac{\hbar}{i}\frac{d}{dx}\sin\left(\frac{\pi n x}{a}\right)$

$$= \frac{2}{a}\frac{\pi n}{a}\frac{\hbar}{i}\int_0^a dx\; \sin\left(\frac{\pi m x}{a}\right)\cos\left(\frac{\pi n x}{a}\right)$$

$$= \frac{\pi n}{a^2}\frac{\hbar}{i}\int_0^a dx\left(\sin\left(\frac{\pi(m-n)x}{a}\right) + \sin\left(\frac{\pi(m+n)x}{a}\right)\right)$$

If $m \neq n$, $= \frac{\pi n}{a^2}\frac{\hbar}{i}\left(\frac{a}{\pi(m-n)} - \cos\left(\frac{\pi(m-n)x}{a}\right) + \frac{a}{\pi(m+n)} - \cos\left(\frac{\pi(m+n)x}{a}\right)\right)\Big|_0^a$

$$= \frac{\pi n}{a^2}\frac{\hbar}{i}\left(\frac{a}{\pi(m-n)}\begin{Bmatrix} +2 \text{ if } m-n \text{ odd} \\ 0 \text{ if } m-n \text{ even}\end{Bmatrix} + \frac{a}{\pi(m+n)}\begin{Bmatrix} +2 \text{ if } m+n \\ \text{is odd,} \\ 0 \text{ if even}\end{Bmatrix}\right)$$

If $m=n$, $= \frac{\pi n}{a^2}\frac{\hbar}{i}\int_0^a dx\, 2\sin\left(\frac{\pi n x}{a}\right)\cos\left(\frac{\pi n x}{a}\right)$

$$= \frac{\pi n}{a^2}\frac{\hbar}{i}\int_0^a dx\,\sin\left(\frac{2\pi n x}{a}\right)$$

$$= \frac{\pi n}{a^2}\frac{\hbar}{i}\frac{a}{2\pi n}-\cos\left(\frac{2\pi n x}{a}\right)\Big|_0^a$$

$$= 0.$$

$$\rightarrow \boxed{\langle 1|p|1\rangle = \langle 2|p|2\rangle = 0}$$

$$\langle 1|p|2\rangle = \frac{\pi n}{a^2}\frac{\hbar}{i}\left(\frac{a}{\pi(m-n)}(+2) + \frac{a}{\pi(m+n)}(+2)\right)$$

$$= \frac{\hbar}{i}\frac{2}{a}n\left(\frac{1}{m-n} + \frac{1}{m+n}\right)$$

$$= \frac{\hbar}{i}\frac{2}{a}n\frac{2m}{m^2-n^2} = \frac{4mn}{m^2-n^2}\frac{\hbar}{i}\frac{1}{a}$$

$$\boxed{\langle 1|p|2\rangle = -\frac{8}{3}\frac{\hbar}{i}\frac{1}{a} \quad \text{and } \langle 2|p|1\rangle = \frac{8}{3}\frac{\hbar}{i}\frac{1}{a}}$$

f) $\langle x\rangle_t = \left(\langle 1|\frac{1}{\sqrt{2}}e^{i\frac{E_1 t}{\hbar}} - \langle 2|\frac{i}{\sqrt{2}}e^{i\frac{E_2 t}{\hbar}}\right) x$

$$\left(\frac{1}{\sqrt{2}}e^{-i\frac{E_1 t}{\hbar}}|1\rangle + \frac{i}{\sqrt{2}}e^{-i\frac{E_2 t}{\hbar}}|2\rangle\right)$$

$$= \frac{1}{2}\langle 1|x|1\rangle + \frac{1}{2}\langle 2|x|2\rangle$$

$$+ \frac{i}{2}e^{i\left(\frac{E_1-E_2}{\hbar}\right)t}\langle 1|x|2\rangle - \frac{i}{2}e^{i\left(\frac{E_2-E_1}{\hbar}\right)t}\langle 2|x|1\rangle$$

$$\boxed{\langle x\rangle_t = \frac{a}{2} - \frac{16}{9\pi^2}a\sin\left(\left(\frac{E_2-E_1}{\hbar}\right)t\right)}$$

$$\langle p\rangle_t = \frac{1}{2}\langle 1|p|1\rangle + \frac{1}{2}\langle 2|p|2\rangle$$

$$+ \frac{i}{2}e^{i\left(\frac{E_1-E_2}{\hbar}\right)t}\langle 1|p|2\rangle - \frac{i}{2}e^{i\left(\frac{E_2-E_1}{\hbar}\right)t}\langle 2|p|1\rangle$$

$$= \frac{1}{2}\left(-\frac{8}{3}\right)\frac{\hbar}{a}e^{i\left(\frac{E_1-E_2}{\hbar}\right)t} - \frac{1}{2}\frac{8}{3}\frac{\hbar}{a}e^{i\left(\frac{E_2-E_1}{\hbar}\right)t}$$

$$\boxed{\langle p\rangle_t = -\frac{8}{3}\frac{\hbar}{a}\cos\left(\left(\frac{E_2-E_1}{\hbar}\right)t\right)}$$

g) $$\frac{d\langle x\rangle_t}{dt} = -\frac{16}{9\pi^2}\, a\, \frac{E_2-E_1}{\hbar}\cos\left(\left(\frac{E_2-E_1}{\hbar}\right)t\right)$$

$$= -\frac{16}{9\pi^2}\, a\, \frac{1}{\hbar}\frac{\hbar^2}{2m}\frac{\pi^2(4-1)}{a^2}\cos\left(\left(\frac{E_2-E_1}{\hbar}\right)t\right)$$

$$= \frac{1}{m}\left(-\frac{8}{3}\right)\frac{\hbar}{a}\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)$$

$$= \frac{1}{m}\langle p\rangle_t \quad \checkmark$$

2. a) $$H\psi = -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x)$$

$$BH\psi = -\frac{\hbar^2}{2m}\left.\frac{d^2\psi}{dx^2}\right|_{a-x} + V(a-x)\psi(a-x)$$

$$= -\frac{\hbar^2}{2m}\frac{d^2\psi(a-x)}{dx^2} + V(x)\psi(a-x)$$

$$B\psi = \psi(a-x)$$

$$HB\psi = -\frac{\hbar^2}{2m}\frac{d^2\psi(a-x)}{dx^2} + V(x)\psi(a-x)$$

$\rightarrow HB\psi = BH\psi$, i.e. $[H,B]=0$ //

b) $$\int_0^a dx\, \varphi^*(x)\, B\, \psi(x) = \int_0^a dx\, \varphi^*(x)\psi(a-x)$$

Let $x' = a-x$, $\rightarrow x = a - x'$

$$= \int_a^0 -dx'\, \varphi^*(a-x')\psi(x')$$

$$= \int_0^a dx'\, \varphi^*(a-x')\psi(x')$$

$$= \int_0^a dx' (B\varphi)^*\psi$$

$\Rightarrow B = B^\dagger$, i.e. B is hermitian.

c) $$\left.\begin{aligned} Bf(x) &= f(a-x) \\ Bf(a-x) &= f(a-(a-x)) = f(x) \end{aligned}\right\} \rightarrow B^2 f(x) = f(x),$$

i.e. $B^2 = \mathbb{1}$.

d) If $B\psi = \lambda\psi$, then $B^2\psi = \lambda^2\psi = \psi$. $\Rightarrow \lambda = \pm 1$.

e) Since $[B, H] = 0$, the eigenstates of H may be also taken to be eigenstates of B. Since the u_n are not degenerate, the eigenstates u_n are also eigenstates of B.

$$B|u_n\rangle = \pm |u_n\rangle.$$

$$\begin{aligned}
B u_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi(a-x)}{a}\right) &= \sqrt{\frac{2}{a}} \sin\left(\pi - \frac{\pi x}{a}\right) \\
&= -\sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a} - \pi\right) \\
&= \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \\
&= u_1(x)
\end{aligned}$$

$$\begin{aligned}
B u_2(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi(a-x)}{a}\right) &= \sqrt{\frac{2}{a}} \sin\left(2\pi - \frac{2\pi x}{a}\right) \\
&= -\sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right) = -u_2(x)
\end{aligned}$$

3. a)
$$\begin{aligned}
\frac{d x(t)}{dt} &= \frac{i}{\hbar}[H, x(t)] \\
&= \frac{i}{\hbar}[H, e^{iHt} x e^{-iHt}] \\
&= \frac{i}{\hbar} e^{iHt}[H, x] e^{-iHt}
\end{aligned}$$

$$\begin{aligned}
[H, x] &= \left[\frac{p^2}{2m} + \frac{1}{2} m\omega^2 x, x\right] \\
&= \left[\frac{p^2}{2m}, x\right] \\
&= \frac{1}{2m}\left(p^2 x - pxp + pxp - xp^2\right) \\
&= \frac{1}{2m}\left(p[p, x] + [p, x]p\right) \\
&= -i\hbar \frac{p}{m}
\end{aligned}$$

$$\Rightarrow \boxed{\frac{dx(t)}{dt} = \frac{p(t)}{m}}$$

$$\frac{dp(t)}{dt} = \frac{i}{\hbar}[H, p(t)]$$
$$= \frac{i}{\hbar}[H, e^{iHt} p e^{-iHt}]$$
$$= \frac{i}{\hbar} e^{iHt} [H, p] e^{-iHt}$$

$$[H, p] = [\frac{p^2}{2m} + V(x), p] = [V(x), \frac{\hbar}{i}\frac{d}{dx}]$$
$$= \frac{\hbar}{i}\left(V(x)\frac{d}{dx} - \frac{d}{dx}V(x)\right)$$
$$= \frac{\hbar}{i}\left(-\frac{dV}{dx}\right) = \frac{\hbar}{i}(-m\omega^2 x)$$

$$\Rightarrow \boxed{\frac{dp(t)}{dt} = -m\omega^2 x(t)}$$

b) $\dfrac{d^2x(t)}{dt^2} = \dfrac{1}{m}\dfrac{dp}{dt} = -\omega^2 x(t)$

$\Rightarrow x(t) = \cos(\omega t)\,\mathcal{O}_1 + \sin(\omega t)\,\mathcal{O}_2$, where $\mathcal{O}_1$ & $\mathcal{O}_2$ are operators.

$$p(t) = m\frac{dx(t)}{dt} = -m\omega \sin(\omega t)\,\mathcal{O}_1 + m\omega\cos(\omega t)\,\mathcal{O}_2$$

$$x(0) = \mathcal{O}_1$$
$$p(0) = m\omega\,\mathcal{O}_2$$

$$\Rightarrow \boxed{\begin{aligned} x(t) &= \cos(\omega t)\,x(0) + \frac{p(0)}{m\omega}\sin(\omega t) \\ p(t) &= \cos(\omega t)\,p(0) - m\omega\, x(0)\sin(\omega t) \end{aligned}}$$

c) Since $\langle x\rangle_t = \langle\psi(0)|\,x(t)\,|\psi(0)\rangle$
$\langle p\rangle_t = \langle\psi(0)|\,p(t)\,|\psi(0)\rangle$,

$$\langle x\rangle_t = \cos(\omega t)\langle x\rangle_0 + \sin(\omega t)\frac{\langle p\rangle_0}{m}$$
$$\langle p\rangle_t = \cos(\omega t)\langle p\rangle_0 - \sin(\omega t)\, m\omega\,\langle x\rangle_0 ,$$

which is the classical solution.