Homework 7

1. (a) $\langle \phi | \psi \rangle = \int dx \, \phi^*(x) \psi(x)$

(b) $\langle \phi | p | \psi \rangle = \int dx \, \phi^*(x) \frac{\hbar}{i} \frac{d}{dx} \psi(x)$

(c) $\langle u_n | u_m \rangle = \delta_{nm}$

(d) $\sum_n |u_n\rangle\langle u_n| = \mathbb{1}$

(e) $\delta(x - x_0)$

(f) $A^\dagger$ is the operator which satisfies

$$\int dx \, \phi^*(x) A \psi(x) = \int dx \, (A^\dagger \phi)^* \psi(x),$$

for any ϕ and ψ.

(g) (i) $A = A^\dagger$

(ii) $\langle \psi | A | \psi \rangle$ is real for any ψ.

(h) $(AB)^\dagger = B^\dagger A^\dagger$

(i) $(cA)^\dagger = c^* A^\dagger$

(j) One can find a complete set of simultaneous eigenvectors of A & B, i.e., $|u_n\rangle$ so that

$$A|u_n\rangle = a_n |u_n\rangle \text{ and}$$
$$B|u_n\rangle = b_n |u_n\rangle.$$

(k) $\Delta A = \langle A^2 \rangle - \langle A \rangle^2$
$\Delta B = \langle B^2 \rangle - \langle B \rangle^2$

$$\Delta A \Delta B \geq \pm \frac{i}{2} \langle [A, B] \rangle$$

(l) $\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m}$ and $\frac{d\langle p\rangle}{dt} = -\left\langle\frac{dV}{dx}\right\rangle$

(m) Let $|\psi(t=0)\rangle = \sum_n c_n |u_n\rangle$.

Then $|\psi(t)\rangle = \sum_n c_n e^{\frac{-iE_nt}{\hbar}} |u_n\rangle$, where $H|u_n\rangle = E_n|u_n\rangle$.

(n) $e^{-iHt/\hbar}$

(o) In the Schrodinger picture the wavefunction is time dependent and the operators are time independent. In the Heisenberg picture the wavefunction is time independent and Operators are time dependent.

(p) $A(t) = e^{iHt/\hbar} A e^{-iHt/\hbar}$

(q) $E_n = \hbar\omega(n+\frac{1}{2})$

(r) $a^\dagger |n\rangle = \sqrt{n+1}\,|n\rangle$ and $a|n\rangle = \sqrt{n}\,|n\rangle$

(s) $[a, a^\dagger] = 1$

2. (a) $[AB, C] = ABC - CAB - ACB + ACB$

$$= A[B,C] + [A,C]B$$

(b) $\int dx\, \varphi^* \frac{d}{dx}\psi = -\int dx \frac{d\varphi^*}{dx}\psi$ ... assuming boundary term vanishes

$$= \int dx \left(-\frac{d\varphi}{dx}\right)^* \psi$$

$$\Rightarrow \left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}$$

3, Principles of quantum mechanics

The wave functions

$$\phi_n(x) = \frac{e^{ik_n x}}{\sqrt{L}},$$

$k_n = 2\pi n/L$ and $n = 0, \pm 1, \pm 2, \ldots$, form a complete orthonormal basis on the interval $[0, L]$. They are also eigenvectors of the hamiltonian $H = \frac{p^2}{2m}$. Since the wave functions are periodic, $\phi_n(0) = \phi_n(L)$, these wave functions may be thought of as being on a ring with circumference L.

(a) Suppose the wave function at $t = 0$ is

$$\psi(x, 0) = \frac{e^{2\pi i x/L}}{\sqrt{2L}} + i\frac{e^{4\pi i x/L}}{\sqrt{2L}}.$$

Determine the wave function, $\psi(x, t)$, at an arbitrary time.

$$H\phi_n = \frac{-\hbar^2}{2m}\frac{d^2\phi_n}{dx^2} = \frac{\hbar^2 k_n^2}{2m}\phi_n \rightarrow E_n = \frac{\hbar^2 k_n^2}{2m}$$

Since $\phi_1(x) = \frac{e^{2\pi i x/L}}{\sqrt{L}}$ and $\phi_2(x) = \frac{e^{4\pi i x/L}}{\sqrt{L}}$,

$$\psi(x,t) = \frac{e^{2\pi i x/L}}{\sqrt{2L}} e^{-iE_1 t/\hbar} + i\frac{e^{4\pi i x/L}}{\sqrt{2L}} e^{-iE_2 t/\hbar}.$$

(b) Compute the probability density at time t.

$$|\psi(x,t)|^2 = \psi(x,t)\,\psi^*(x,t)$$

$$= \frac{1}{2L} + \frac{1}{2L} + i\frac{e^{2\pi i x/L}}{2L} e^{-i(E_2-E_1)t/\hbar} \; \frac{e^{-2\pi i x/L}}{2L} e^{i(E_2-E_1)t/\hbar}$$

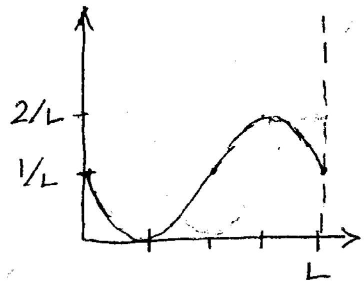
$$= \frac{1}{L}(1 - \sin(kx - \omega t)), \quad k = 2\pi/L, \; \omega = (E_2 - E_1)/\hbar$$

(c) The probability density is periodic in time. What is the period, τ, of the oscillations?

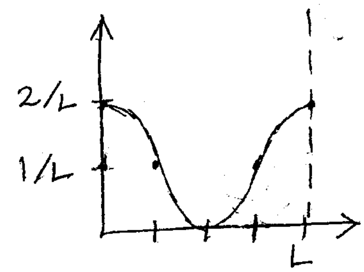
$$\tau = \frac{2\pi}{\omega} = \frac{2\pi\hbar}{E_2 - E_1} = \frac{2\pi\hbar}{\frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2(2^2 - 1^2)} = \frac{1}{6\pi}\frac{2mL^2}{\hbar}$$

(d) Sketch the probability density at $\tau = 0$, $\tau/4$, $\tau/2$, and $3\tau/4$.

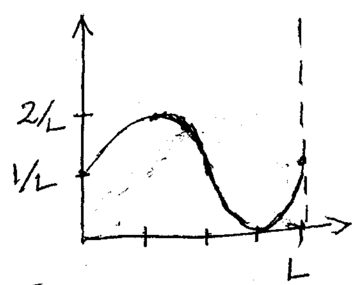
$|\psi(x,0)|^2 = \frac{1}{L}(1-\sin(kx))$



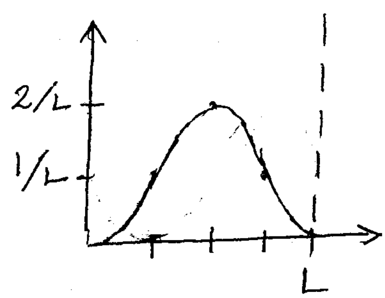
$|\psi(x,\frac{\tau}{4})|^2 = \frac{1}{L}(1-\sin(kx-\frac{\pi}{2}))$



$|\psi(x,\frac{\tau}{2})|^2 = \frac{1}{L}(1-\sin(kx-\pi))$



$|\psi(x,\frac{3\tau}{4})|^2 = \frac{1}{L}(1-\sin(kx-\frac{3\pi}{2}))$



(e) If the position of the particle is measured at $t = \frac{\tau}{4}$, what is the probability that the particle is between 0 and $L/2$, i.e., $0 \le x \le L/2$?

$$\text{Probability} = \int_0^{L/2} |\psi(x,\tfrac{\tau}{4})|^2 dx$$
$$= \int_0^{L/2} \frac{1}{L}\left(1+\cos\left(\frac{2\pi x}{L}\right)\right) dx$$
$$= \frac{1}{L}\left(\frac{L}{2} + \frac{L}{2\pi}\sin\left(\frac{2\pi x}{L}\right)\Big|_0^{L/2}\right)$$
$$= \frac{1}{2}$$

4. Harmonic oscillator

(a) At $t=0$ a harmonic oscillator is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle.$$

What is the wave function at an arbitrary time t?

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = \frac{1}{\sqrt{2}} e^{-iE_0 t/\hbar}|0\rangle + \frac{i}{\sqrt{2}} e^{-iE_1 t/\hbar}|1\rangle$$

(b) For the harmonic oscillator the position and momentum operators may be written as

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$$
$$p = \sqrt{\frac{m\hbar\omega}{2}}\, i\,(a^\dagger - a).$$

Compute the expectation value of the position and momentum as a function of time for the above wave function. Do the results for $\langle x\rangle$ and $\langle p\rangle$ make sense when compared to the classical equations of motion?

$$\langle x\rangle = \langle\psi(t)|x|\psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\Big(\frac{1}{2}\langle 0|(a^\dagger+a)|0\rangle + \frac{1}{2}\langle 1|(a^\dagger+a)|1\rangle + \frac{ie^{-i(E_1-E_0)t/\hbar}}{2}\langle 0|(a^\dagger+a)|1\rangle - \frac{ie^{i(E_1-E_0)t/\hbar}}{2}\langle 1|(a^\dagger+a)|0\rangle\Big)$$

$$\boxed{\langle x\rangle(t) = \sqrt{\frac{\hbar}{2m\omega}}\sin(\omega t)}\text{, where } E_n = \hbar\omega\left(n+\frac{1}{2}\right)$$

$$\langle p\rangle = \langle\psi(t)|x|\psi(t)\rangle = \sqrt{\frac{m\hbar\omega}{2}}\Big(\frac{1}{2}\langle 0|i(a^\dagger-a)|0\rangle + \frac{1}{2}\langle 1|i(a^\dagger-a)|1\rangle + \frac{ie^{-i\omega t}}{2}\langle 0|i(a^\dagger-a)|1\rangle - \frac{ie^{i\omega t}}{2}\langle 1|i(a^\dagger-a)|0\rangle\Big)$$

$$\boxed{\langle p\rangle(t) = \sqrt{\frac{m\hbar\omega}{2}}\cos(\omega t)}$$

Like the classical equations of motion: $\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m}$

$$\frac{d\langle p\rangle}{dt} = -m\omega^2\langle x\rangle.$$

(c) Compute the expectation values of x^2 and p^2 as as a function of time. How are these related to the energy?

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\Big(\tfrac{1}{2}\langle 0|(a^\dagger+a)^2|0\rangle + \tfrac{1}{2}\langle 1|(a^\dagger+a)^2|1\rangle + \frac{ie^{-i\omega t}}{2}\langle 0|(a^\dagger+a)^2|1\rangle - \frac{ie^{i\omega t}}{2}\langle 1|(a^\dagger+a)^2|0\rangle\Big)$$

$$= \frac{\hbar}{2m\omega}\Big(\tfrac{1}{2}\langle 0|a^\dagger a + aa^\dagger|0\rangle + \tfrac{1}{2}\langle 1|a^\dagger a + aa^\dagger|1\rangle\Big)$$

$$= \frac{\hbar}{2m\omega}\left(\frac{2\cdot 0+1}{2} + \frac{2\cdot 1+1}{2}\right) = \boxed{\frac{\hbar}{m\omega} = \langle x^2\rangle}$$

$$\langle p^2\rangle = \frac{m\hbar\omega}{2}\Big(\tfrac{1}{2}\langle 0|-(a^\dagger-a)^2|0\rangle + \tfrac{1}{2}\langle 1|-(a^\dagger-a)^2|1\rangle + \frac{ie^{-i\omega t}}{2}\langle 0|-(a^\dagger-a)^2|1\rangle - \frac{ie^{i\omega t}}{2}\langle 1|-(a^\dagger-a)^2|0\rangle\Big)$$

$$= \frac{m\hbar\omega}{2}\Big(\tfrac{1}{2}\langle 0|a^\dagger a + aa^\dagger|0\rangle + \tfrac{1}{2}\langle 1|a^\dagger a + aa^\dagger|1\rangle\Big) = \boxed{m\hbar\omega = \langle p^2\rangle}$$

The expectation value of the energy is $\frac{\langle p^2\rangle}{2m} + \frac{1}{2}m\omega^2\langle x^2\rangle = \hbar\omega$. ☆

(d) Finally, using the results from (b) and (c) above, check the uncertainty principle for this wave function.

$$(\Delta x)^2 = \langle\psi|(x-\langle x\rangle)^2|\psi\rangle = \langle\psi|x^2|\psi\rangle - 2\langle x\rangle\langle\psi|x|\psi\rangle + \langle x\rangle^2$$

$$= \langle x^2\rangle - \langle x\rangle^2$$

$$= \frac{\hbar}{m\omega} - \frac{\hbar}{2m\omega}\sin^2(\omega t)$$

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = m\hbar\omega - \frac{m\hbar\omega}{2}\cos^2(\omega t)$$

$$\rightarrow \Delta x = \sqrt{\frac{\hbar}{m\omega}}\sqrt{1-\tfrac{1}{2}\sin^2(\omega t)}$$

$$\Delta p = \sqrt{m\hbar\omega}\sqrt{1-\tfrac{1}{2}\cos^2(\omega t)}$$

$$\rightarrow \Delta x\Delta p = \hbar\left[\left(1-\frac{\sin^2(\omega t)}{2}\right)\left(1-\frac{\cos^2(\omega t)}{2}\right)\right]^{1/2}$$

$$= \hbar\left[1-\frac{1}{2}+\frac{\sin^2(\omega t)}{2}\frac{\cos^2(\omega t)}{2}\right]^{1/2} \geq \hbar\left[\tfrac{1}{2}\right]^{1/2} > \frac{\hbar}{2}. \checkmark$$

☆ Note: $\hbar\omega$ is in between $E_0 = \frac{\hbar\omega}{2}$ and $E_1 = \frac{3\hbar\omega}{2}$.

5. Let $E_n = \frac{\hbar^2}{2m}\frac{\pi^2 n^2}{a^2}$ and $\varphi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right)$.

(a) $E = E_1 + E_2 = \frac{\hbar^2\pi^2}{2ma^2}(1+4)$

(b) $\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\varphi_1(x_1)\varphi_2(x_2) - \varphi_2(x_1)\varphi_1(x_2)\right)$

(c) $E = E_1 + E_3 = \frac{\hbar^2\pi^2}{2ma^2}(1+9)$

$\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\varphi_1(x_1)\varphi_3(x_2) - \varphi_3(x_1)\varphi_1(x_2)\right)$

(d) ground state: $E = 2E_1$

(first excited state: $E = E_1 + E_2$)

ground state:

$\psi(x_1, s_1, x_2, s_2) = \varphi_1(x_1)\varphi_1(x_2)\frac{1}{\sqrt{2}}\left(\delta_{s_1\uparrow}\delta_{s_2\downarrow} - \delta_{s_1\downarrow}\delta_{s_2\uparrow}\right)$

(e) **Boson case:**

$E = E_1 + E_1 = 2E_1$

$\psi(x_1, x_2) = \varphi_1(x_1)\varphi_1(x_2)$